Minimum Average Broadcast Time in a Graph of Bounded Degree

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Abstract

Broadcasting in a network, is the process whereby information, initially held by one node is disseminated to all nodes in the network. It is assumed that, in each unit of time, every vertex that has the information can send it to at most one of its neighbours that does not yet have the information. Furthermore, the networks considered here are of bounded (maximum) degree Δ, meaning that each node has at most Δ neighbours. In this article, a new parameter, the average broadcast time, defined as the minimum mean time at which a node in the network first receives the information, is introduced. It is found that when the broadcast time is much greater than the maximum degree, the average broadcast time is (approximately) between one and two time units less than the total broadcast time if the maximum degree is at least three.

1 Introduction

Broadcasting is the process whereby a message, initially present at one node of a network, is sent to all the other nodes of the network, by locally sending the

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message from informed nodes to uninformed neighbours. Further constrictions are that a node may paritcipate in only one call per unit time and a node can send the message only to an adjacent node in the network. We further assume all the nodes are synchronized and the message takes exactly one unit of time to be sent from one node to its neighbour. Two entire volumes of the journal Discrete Applied Mathematics, volumes 53 (1994) and 83 (1998), have been devoted to the issue of broadcasting in networks.

We model the communication network by a graph, the nodes by vertices, and communications links by edges of the graph. The *broadcast time* of a graph is the maximum amount of time it takes to broadcast to all vertices of the graph from any vertex, assuming this is accomplished by the most efficient method (ie: in the least amount of time.) Instead of minimizing the total broadcast time, Koh and Tcha[3] described an algorithm that minimized the average time at which a vertex first receives the information, for the special case of trees. Liestman and Pržulj[5] initiated the study of minimum average time broadcast graphs - those graphs on n vertices with the fewest edges in which every vertex can broadcast in minimum average time. In this article, we are interested in finding the minimum average time at which a vertex first receives the information in graphs of bounded degree. We do not suggest an algorithm which determines the optimal sequence that orders the time at which vertices receive the information, but instead we simply assume that the number of vertices informed in each unit of time is maximized under the constraint of bounded degree in a best possible graph.

Bounded degree graphs as a model of a communications network are particularly useful because the number of connections made to any particular node in the network is limited by both cost and technology. For an introductory discussion of broadcasting in bounded degree graphs, the reader is referred to Bermond, Hell, Liestman, and Peters[1], or Liestman and Peters[4].

When broadcasting in a network, not all the nodes receive the information at the same time. Let $f_t^\Delta, t \geq 0$ represent the maximum number of vertices which can possibly receive the information at time t in a network of maximum degree Δ. We now define the mean, or average, broadcast time, $\bar{t}_k^\Delta$ by:

$$\bar{t}_k^\Delta = \frac{\sum_{t=0}^{k} t f_t^\Delta}{\sum_{t=0}^{k} f_t^\Delta}$$

where the denominator is just the total number of vertices that have the information after k time units: i.e., $\bar{t}_k^\Delta$ is the average time at which a vertex receives the information, amongst all those vertices that have been informed by k time units if f_t^Δ new vertices are informed in time unit t.

The primary motivation for the introduction of this new parameter comes from the observation that, although all vertices will have the information after broadcasting is complete, a substantial subset of them will receive the information earlier. These vertices will be able to start processing locally. The arithmetic difference between our "mean time" and total time to complete broadcasting is therefore a good estimate of the amount of time overlap possible between the broadcasting and local processing phases.

2 Computing the Average Broadcast Time

f_t^Δ satisfies the following constraints:

$$\begin{aligned}
f_0^\Delta &= 1 \\
f_1^\Delta &= 1 \\
f_t^\Delta &= 2f_{t-1}^\Delta = 2^{t-1}, && 2 \leq t \leq \Delta \\
f_{\Delta+1}^\Delta &= 2^\Delta - 2 \\
f_t^\Delta &= \textstyle\sum_{i=t-\Delta+1}^{t-1} f_i^\Delta = 2f_{t-1}^\Delta - f_{t-\Delta}^\Delta, && t \geq \Delta + 2
\end{aligned}$$

where the last recurrence is just the one for the generalized Fibonacci sequences, as discussed in Knuth[2, p. 269]. Our initial conditions are not identical to Knuth's, however. Our initial conditions for the last recurrence are the values of $f_2^\Delta, f_3^\Delta, \ldots, f_{\Delta+1}^\Delta$. For $\Delta = 3$, we have the usual Fibonnacci recurrence, but with slightly modified initial conditions.

For $t < \Delta$, the number of vertices that have the information doubles in each time unit. Therefore half of the vertices receive the information in the last round. The mean time to receive the information is then approximately one time unit less than the total broadcasting time (in fact $\bar{t}_k^\Delta = k - 1 + \frac{1}{2^k}$.) We will find the mean time for the other extreme, $t \gg \Delta$, below:

It is easy to verify that the generating function for the above recurrence relationships with our initial conditions is:

$$\begin{aligned}
F^\Delta(x) &= \sum_{t\geq 0} f_t^\Delta x^t = \frac{1 - x + x^\Delta - x^{\Delta+1}}{1 - 2x + x^\Delta} \\
&= \frac{(1-x)(1+x^\Delta)}{(1-x)(1 - x - x^2 - \cdots - x^{\Delta-2} - x^{\Delta-1})}
\end{aligned}$$

Using this function, the expression for the mean time becomes:

$$\bar{t}_k^{\Delta} = \frac{[x^{k-1}](1-x)^{-1}\frac{d}{dx}F^{\Delta}(x)}{[x^k](1-x)^{-1}F^{\Delta}(x)}$$

where $[x^k]$ is the operator that extracts the coefficient of x^k in the power series expansion. Substituting for $F^{\Delta}(x)$, the denominator of this expression becomes:

$$[x^k]\frac{1+x^{\Delta}}{1-2x+x^{\Delta}} = [x^k]\left(\frac{A}{1-x} + \frac{B}{1-rx} + R_1(x)\right), \Delta \geq 3$$

where the right-hand-side is just the partial fraction expansion of the left. If $1/r$ is the dominant root of $1-2x+x^{\Delta}=0$ then $r \approx 2-2^{-\Delta+1}$, for large Δ, and $r = 1.61803\ldots$, the golden ratio, for $\Delta = 3$. The term $R_1(x)$ "absorbs" the roots whose reciprocals lie within the unit circle in the complex plane. Thus $R_1(1/r)$, $R_1(1)$ are defined and finite. Moreover , $\lim_{k\to\infty}[x^k]R_1(x) = 0$.

If we ignore the $R_1(x)$ term and extract the coefficients we get that the denominator is approximately:

$$A + Br^k$$

which is valid for large k.

Since $1-2(1/r)+(1/r)^{\Delta}=0$, $(1/r)^{\Delta-1} = 2-r$. This useful relationship will be applied later on.

Evaluating the constant A:

$$A = \lim_{x\to 1}(1-x)\left\{\frac{1+x^{\Delta}}{1-2x+x^{\Delta}}\right\}$$

and upon using L'Hôpital's Rule we get:

$$A = \lim_{x\to 1}\frac{\frac{d}{dx}(1-x)(1+x^{\Delta})}{\frac{d}{dx}(1-2x+x^{\Delta})} = \frac{-2}{\Delta-2}$$

Similarly, evaluating the constant B:

$$B = \lim_{x\to 1/r}(1-rx)\left\{\frac{1+x^{\Delta}}{1-2x+x^{\Delta}}\right\} = \lim_{x\to 1/r}\frac{\frac{d}{dx}(1-rx)(1+x^{\Delta})}{\frac{d}{dx}(1-2x+x^{\Delta})} = \frac{2}{2-\Delta(2-r)}$$

Where we have both applied L'Hôpital's Rule and our useful relationship.

The numerator of the expression for the mean time can be found similarly:

$$[x^{k-1}](1-x)^{-1}\frac{d}{dx}\left\{\frac{(1-x)(1+x^{\Delta})}{1-2x+x^{\Delta}}\right\} = [x^{k-1}]\frac{1-2\Delta x^{\Delta}+2\Delta x^{\Delta+1}-x^{2\Delta}}{(1-x)(1-2x+x^{\Delta})^2}$$

Note that $(1-x)$ is a double factor of the top of this expression and a triple factor of the bottom part.

As before, we get:

$$= [x^{k-1}]\left(\frac{a}{1-x} + \frac{b}{1-rx} + \frac{c}{(1-rx)^2} + R_2(x)\right), \Delta \geq 3$$

where r is the same as in the expression for the denominator and $R_2(x)$ satisfies the same conditions as $R_1(x)$ does. Extracting the coefficients and ignoring the $R_2(x)$ term, we get approximately:

$$a + br^{k-1} + ckr^{k-1}$$

The remaining task is to evaluate a, b, and c.

$$a = \lim_{x \to 1}(1-x)\left\{\frac{1 - 2\Delta x^{\Delta} + 2\Delta x^{\Delta+1} - x^{2\Delta}}{(1-x)(1-2x+x^{\Delta})^2}\right\}$$

Applying L'Hôpital's rule twice, taking the limit, and simplifying yields:

$$a = \frac{\Delta}{(\Delta-2)^2}$$

Next we find c:

$$\begin{aligned} c &= \lim_{x \to 1/r}(1-xr)^2\left\{\frac{1 - 2\Delta x^{\Delta} + 2\Delta x^{\Delta+1} - x^{2\Delta}}{(1-x)(1-2x+x^{\Delta})^2}\right\} \\ &= \frac{2r}{2-\Delta(2-r)} \end{aligned}$$

Where we have once again twice applied L'Hôpital's Rule and used our useful relationship. To get b we need to go a step further:

$$\begin{aligned} b &= -\frac{1}{r}\lim_{x \to 1/r}\frac{d}{dx}(1-xr)^2\left\{\frac{1 - 2\Delta x^{\Delta} + 2\Delta x^{\Delta+1} - x^{2\Delta}}{(1-x)(1-2x+x^{\Delta})^2}\right\} \\ &= \frac{-2r}{(2-\Delta(2-r))\,(r-1)} \end{aligned}$$

This time, we resorted to using MAPLE as L'Hôpital's rule had to be applied three times.

Finally, we get:

$$\bar{t}_k^{\Delta} \approx \frac{\frac{\Delta}{(\Delta-2)^2} + \left(k - \frac{1}{r-1}\right)\left(\frac{2}{2-\Delta(2-r)}\right)r^k}{\frac{-2}{\Delta-2} + \left(\frac{2}{2-\Delta(2-r)}\right)r^k}$$

For large k, we may ignore the constant terms to get:

$$\bar{t}_k^\Delta \approx k - \frac{1}{r-1}.$$

It can be shown that our derivation of the coefficients in the partial fraction expansion is valid for all roots (as long as the particular root is of multiplicity one in the denominator of the original generating function expression, $F^\Delta(x)$), not just the dominant one. We may use this fact to obtain the following exact expression of the mean time for $\Delta = 3$:

$$\bar{t}_k^3 = \frac{3 + \frac{2}{\sqrt{5}}\left((k - r_+)r_+^{k+2} - (k - r_-)r_-^{k+2}\right)}{-2 + \frac{2}{\sqrt{5}}(r_+^{k+2} - r_-^{k+2})}$$

where $r_\pm = \frac{1\pm\sqrt{5}}{2}$. In terms of the usual Fibonacci numbers, F_k, this becomes:

$$\bar{t}_k^3 = \frac{3 + 2(kF_{k+1} - F_{k+2})}{-2 + 2F_{k+1}}$$

which is valid for all $k \geq 1$.

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