A New Perspective on the Union-Closed Sets Conjecture

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Abstract. We establish a connection between the principle of inclusion-exclusion and the union-closed sets conjecture. In particular, it is shown that every counterexample to the union-closed sets conjecture must satisfy an improved inclusion-exclusion identity.

The following conjecture, known as the *union-closed sets conjecture*, was proposed by P. Frankl in 1979:

Conjecture (Frankl). *Let $\mathcal{X}$ be a non-empty finite union-closed set of non-empty sets. Then, there exists an element $x \in \bigcup \mathcal{X}$ which is contained in at least half of the sets in $\mathcal{X}$.*

Here, *union-closed* means that $X \cup Y \in \mathcal{X}$ for any $X, Y \in \mathcal{X}$, and $\bigcup \mathcal{X}$ is a shorthand for $\bigcup_{X \in \mathcal{X}} X$.

Although the union-closed sets conjecture is easily stated and comprehensible even to non-mathematicians, it still remains unsolved for twenty years now. Proofs only exist for some very special cases [2, 3, 5–7].

Our main result, stated subsequently, gives a necessary condition on counterexamples to the conjecture.

We use the following notations and terminology: For any set A, $I(A)$ denotes the indicator function (or characteristic function) of A, that is, $I(A)(x) = 1$ if $x \in A$ and $I(A)(x) = 0$ if $x \notin A$. $|\cdot|$ is used to denote cardinality. A set-system $\mathcal{X}$ is a *chain* if $X \subseteq Y$ or $Y \subseteq X$ for any $X, Y \in \mathcal{X}$.

Theorem 1. *Any counterexample $\mathcal{X}$ to the union-closed sets conjecture satisfies the following inclusion-exclusion identity:*

$$I\left(\bigcup \mathcal{X}\right) = \sum_{\substack{\mathcal{Y} \subseteq \mathcal{X} \\ 0 < |\mathcal{Y}| < |\mathcal{X}|/2 \\ \mathcal{Y} \text{ is a chain}}} (-1)^{|\mathcal{Y}|-1} I\left(\bigcap \mathcal{Y}\right).$$

Remarks. Note that the usual inclusion-exclusion identity for $\bigcup \mathcal{X}$ is

$$I\left(\bigcup \mathcal{X}\right) = \sum_{\substack{\mathcal{Y} \subseteq \mathcal{X} \\ \mathcal{Y} \neq \emptyset}} (-1)^{|\mathcal{Y}|-1} I\left(\bigcap \mathcal{Y}\right),$$

valid for every finite set-system $\mathcal{X}$. If $\mathcal{X}$ is a counterexample to the union-closed sets conjecture, then Theorem 1 says that the sum in the identity can be restricted to *chains of cardinality less than* $|\mathcal{X}|/2$.

Also note that $\bigcap \mathcal{Y} \in \mathcal{Y}$ for any non-empty chain $\mathcal{Y}$. In fact, $\bigcap \mathcal{Y}$ is the minimum of $\mathcal{Y}$ with respect to set inclusion. Thus, the identity in Theorem 1 can equivalently be stated as

$$I\left(\bigcup \mathcal{X}\right) = \sum_{\substack{\mathcal{Y} \subseteq \mathcal{X} \\ 0 < |\mathcal{Y}| < |\mathcal{X}|/2 \\ \mathcal{Y} \text{ is a chain}}} (-1)^{|\mathcal{Y}|-1} I(\min \mathcal{Y}) .$$

This, of course, does not apply to the usual inclusion-exclusion identity.

We further remark that the identity in Theorem 1 can be integrated with respect to any measure (e.g., the counting measure) on the Boolean algebra generated by $\mathcal{X}$, thus resulting in a statement involving measures rather than characteristic functions.

The proof of Theorem 1 relies on an improvement of the inclusion-exclusion principle, which goes back to Narushima [4] (see also [1]). In order to state this improvement, some further definitions are needed.

A partially ordered set P is a *join-semilattice* if any $x, y \in P$ have a least common upper bound $x \vee y$ in P, which is called the *join* of x and y. A *chain* in a partially ordered set P is a subset Q of P such that any two elements of Q are comparable with respect to the ordering relation on P.

Proposition (Narushima). *Let $(A_p)_{p \in P}$ be a finite family of sets, indexed by some join-semilattice P such that $A_x \cap A_y \subseteq A_{x \vee y}$ for any $x, y \in P$. Then,*

$$I\left(\bigcup_{p \in P} A_p\right) = \sum_{\substack{Q \subseteq P, Q \neq \emptyset \\ Q \text{ is a chain}}} (-1)^{|Q|-1} I\left(\bigcap_{q \in Q} A_q\right).$$

We now apply this proposition to prove Theorem 1.

Proof of Theorem 1: Define $P := \mathcal{X}$ and $A_X := X$ for any $X \in \mathcal{X}$. Since $\mathcal{X}$ is union-closed, the proposition gives

$$I\left(\bigcup \mathcal{X}\right) = \sum_{\substack{\mathcal{Y} \subseteq \mathcal{X}, \mathcal{Y} \neq \emptyset \\ \mathcal{Y} \text{ is a chain}}} (-1)^{|\mathcal{Y}|-1} I\left(\bigcap \mathcal{Y}\right).$$

Since $\mathcal{X}$ is a counterexample to the union-closed sets conjecture, $\bigcap \mathcal{Y} = \emptyset$ for any $\mathcal{Y} \subseteq \mathcal{X}$ satisfying $|\mathcal{Y}| \geq |\mathcal{X}|/2$, whence Theorem 1 is proved. □

Theorem 1 can equivalently be stated in terms of the Euler characteristic. Recall that an ***abstract simplicial complex*** $\mathcal{S}$ is a set of non-empty subsets of some finite set such that $S \in \mathcal{S}$ and $T \subset S$ imply $T \in \mathcal{S}$. The ***Euler characteristic*** $\chi(\mathcal{S})$ of an abstract simplicial complex $\mathcal{S}$ is defined by

$$\chi(\mathcal{S}) := \sum_{S \in \mathcal{S}} (-1)^{|S|-1} .$$

The equivalent formulation of Theorem 1 follows.

Theorem 1'. *Let $\mathcal{X}$ be a counterexample to the union-closed sets conjecture. Then, for any $x \in \bigcup \mathcal{X}$ the abstract simplicial complex*

$$\mathcal{S}(x) := \{\mathcal{Y} \mid \mathcal{Y} \textit{ is a chain}, 0 < |\mathcal{Y}| < |\mathcal{X}|/2, x \in \min \mathcal{Y}\}$$

has Euler characteristic 1.

Proof: The left-hand side and the right-hand side of the identity in Theorem 1, evaluated at x, give 1 resp. $\chi(\mathcal{S}(x))$, whence the result. □

References

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