A combinatorial structure of affine $(\alpha_1, ..., \alpha_t)$-resolvable (r, λ)-designs

Sanpei Kageyama
Department of Mathematics
Faculty of School Education
Hiroshima University
Higashi-Hiroshima 739-8524
Japan

Tsutomu Shimata
Department of Mathematics
Shudo Gakuen
Hiroshima 730-0055
Japan

Abstract. Some constructions of affine $(\alpha_1, ..., \alpha_t)$-resolvable (r, λ)-designs are discussed, by use of affine α-resolvable balanced incomplete block designs or semi-regular group divisible designs. A structural property is also indicated.

1. Introduction

A group divisible (GD) design with parameters $v = mn, b, r, k, \lambda_1, \lambda_2$ is a block design such that the mn treatments are divided into m groups of n treatments each and any two treatments in the same group occur together in exactly λ_1 blocks, while any two treatments in different groups occur together in exactly λ_2 blocks. Furthermore, if $rk - v\lambda_2 = 0$, the GD design is said to be semi-regular. In a semi-regular GD design every block contains k/m treatments from each group (see [5: Theorem 8.5.6]). A GD design with $\lambda_1 = \lambda_2$ $(= \lambda$, say) is called a balanced incomplete block (BIB) design with parameters v, b, r, k, λ.

An (r, λ)-design is a block design with parameters v, b, r, k_j, $j = 1, ..., b$, such that every pair of treatments occurs exactly λ blocks. Note that an (r, λ)-design with constant block size is a BIB design.

A block design with parameters $v, b, r, k_j, j = 1, ..., b$, is said to be $(\alpha_1, ..., \alpha_t)$-resolvable if b blocks are separated into $t(\geq 2)$ sets S_ℓ of $\beta_\ell(\geq 1)$ blocks such that S_ℓ contains every treatment $\alpha_\ell(\geq 1)$ times, $\ell = 1, ..., t$. Here $b = \sum_{\ell=1}^{t} \beta_\ell$ and $r = \sum_{\ell=1}^{t} \alpha_\ell$. When $\alpha_1 = \cdots = \alpha_t(= \alpha$, say), the design is said to be α-resolvable. In particular, when $\alpha = 1$, it is called simply resolvable. Note that S_ℓ is also called an α_ℓ-resolution set.

In order to introduce the affine $(\alpha_1, ..., \alpha_t)$-resolvability, attention will be restricted to only those $(\alpha_1, ..., \alpha_t)$-resolvable block designs which have a constant block size, denoted by k_ℓ^*, within each set S_ℓ for $\ell = 1, ..., t$ (see [4]).

An $(\alpha_1, ..., \alpha_t)$-resolvable block design with a constant block size k_ℓ^* in each S_ℓ is said to be affine $(\alpha_1, ..., \alpha_t)$-resolvable if:

(i) for $\ell = 1, ..., t$, every two distinct blocks from S_ℓ intersect in the same number, say $q_{\ell\ell}$, of treatments;

(ii) for $\ell \neq \ell' = 1, ..., t$, every block from S_ℓ intersects every block of $S_{\ell'}$ in the same number, say $q_{\ell\ell'}$, of treatments.

It is evident that for an affine $(\alpha_1, ..., \alpha_t)$-resolvable block design

$$q_{\ell\ell}(\beta_\ell - 1) = k_\ell^*(\alpha_\ell - 1), \quad q_{\ell\ell'}\beta_{\ell'} = k_\ell^*\alpha_{\ell'}, \quad (\ell \neq \ell' = 1, ..., t).$$

Some constructions of affine $(\alpha_1, ..., \alpha_t)$-resolvable (r, λ)-designs are given by Kageyama and Sastry [3]. In this paper, we consider generalizations of two of their methods and present an inner structure property of affine α-resolvability.

In an affine α-resolvable BIB design with parameters v, $b = \beta t(= v + t - 1)$, $r = \alpha t, k, \lambda$, it holds (cf. [2]) that $q_{\ell\ell} = k(\alpha - 1)/(\beta - 1) = k + \lambda - r$ $(= q_1$, say) and $q_{\ell\ell'} = k\alpha/\beta = k^2/v$ $(= q_2$, say), for $\ell, \ell'(\ell \neq \ell') \in \{1, ..., t\}$, where $\alpha = \alpha_1 = \cdots = \alpha_t$ and $\beta = \beta_1 = \cdots = \beta_t$.

For convenience, $\boldsymbol{I}_s$ denotes the identity matrix of order s, $\boldsymbol{1}_t$ denotes the $t \times 1$ matrix all of whose elements are unity, and $\boldsymbol{A} \otimes \boldsymbol{B}$

denotes the Kronecker product of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$.

2. Constructions

Two methods (i.e. II and III) of construction of affine $(\alpha_1, ..., \alpha_t)$-resolvable (r, λ)-designs provided by Kageyama and Sastry [3] are generalized as in the following. Their proofs are straightforward and hence omitted.

Method 2.1. Let $\boldsymbol{N}$ be the $v \times b$ incidence matrix of an affine resolvable BIB design with parameters $v = s^2, b = s(s+1), r = s+1, k = s, \lambda = 1$. Then $[\boldsymbol{N}' : \boldsymbol{I}_{s+1} \otimes \mathbf{1}_s : \mathbf{1}_{s(s+1)}\mathbf{1}'_p]$ shows an affine $(s, 1, ..., 1)$-resolvable $(s+p+1, p+1)$-design with $q_{\ell\ell} = 1$ or 0, $q_{\ell\ell'} = 1$, $s, s+1$, or $s(s+1)$, for $\ell, \ell' \in \{1, 2, ..., p+2\}$ and any non-negative integer p.

Method 2.2. Let $\boldsymbol{N}$ be the $v \times b$ incidence matrix of an affine resolvable BIB design with parameters $v = s^2, b = s(s+1), r = s+1, k = s, \lambda = 1$. Then $[\boldsymbol{N}' : (\mathbf{1}_{s+1}\mathbf{1}'_{s+1} - \boldsymbol{I}_{s+1}) \otimes \mathbf{1}_s : \mathbf{1}_{s(s+1)}\mathbf{1}'_p]$ shows an affine $(s, s, 1, ..., 1)$-resolvable $(2s+p, s+p)$-design with $q_{\ell\ell} = 1$ or $s(s-1)$, $q_{\ell\ell'} = s$, $s+1$, or $s(s+1)$, for $\ell, \ell' \in \{1, 2, ..., p+2\}$ and any non-negative integer p.

Note that the affine resolvable BIB design used in Methods 2.1 and 2.2 exists if s is a prime or a prime power.

Method 2.3. Let $\boldsymbol{N}$ be the $v \times b$ incidence matrix of an affine resolvable semi-regular GD design with parameters $v = mn, b = v + r - m, r, k, \lambda_1, \lambda_2 = \lambda_1 + 1$. Then $[\boldsymbol{N} : \boldsymbol{I}_m \otimes \mathbf{1}_n : \mathbf{1}_v\mathbf{1}'_p]$ shows an affine resolvable $(r+p+1, p+\lambda_2)$-design with $q_{\ell\ell} = 0$, $q_{\ell\ell'} = k$, $n, v, k^2/v$, or k/m, for $\ell, \ell' \in \{1, 2, ..., r+1+p\}$ and any non-negative integer p.

Method 2.4. Let $\boldsymbol{N}$ be the $v \times b$ incidence matrix of an affine resolvable semi-regular GD design with parameters $v = mn, b = v + r - m, r, k, \lambda_1, \lambda_2 = \lambda_1 + 1$. Then $[\boldsymbol{N} : (\mathbf{1}_m\mathbf{1}'_m - \boldsymbol{I}_m) \otimes \mathbf{1}_n : \mathbf{1}_v\mathbf{1}'_p]$ shows an affine $(1, m-1, 1, ..., 1)$-resolvable $(r+p+m-1, p+\lambda_2+m-2)$-design with $q_{\ell\ell} = 0$ or $n(m-2)$, $q_{\ell\ell'} = k$, $v, n(m-1), k^2/v$, or $k(m-1)/m$.

for $\ell, \ell' \in \{1, 2, \ldots, r+1+p\}$ and any non-negative integer p. Here the number of 1-resolution sets is $p+1$.

Note that there are affine resolvable semi-regular GD designs with $\lambda_2 = \lambda_1 + 1$ can be found in Clatworthy [1], i.e., SR1, SR23, SR38, SR44, SR60, SR71, SR87, SR97, SR105 within the scope of $r, k \le 10$.

Finally, an inner structure property of affine α-resolvability is given.

Theorem 2.1. The existence of an affine α-resolvable BIB design with parameters $v, b = \beta t, r = \alpha t, k, \lambda$, $q_1 = k + \lambda - r$, $q_2 = k^2/v$, implies the existence of an affine $(\alpha, \ldots, \alpha, \beta - \alpha, \ldots, \beta - \alpha)$-resolvable (r^*, λ^*)-design with parameters $v^* = v, b^* = b, r^* = r + c(\beta - 2\alpha), k^* = k$ or $v - k$, $\lambda^* = \lambda + c(\beta - 2\alpha)$, $q_{\ell\ell} = k + \lambda - r$ or $v + \lambda - r - k$, $q_{\ell\ell'} = k^2/v$, $k - k^2/v$ or $v - 2k + k^2/v$, where c is the number of $(\beta - \alpha)$-resolution sets and $1 \le c \le t - 1$.

Proof. In the affine α-resolvable BIB design with parameters $v, b = \beta t, r = \alpha t, k, \lambda$, $q_1 = k + \lambda - r$, $q_2 = k^2/v$, assume that any c α-resolution sets are exchanged into their complementary sets, i.e. $(\beta - \alpha)$-resolution sets, for $1 \le c \le t - 1$. Then in the resulting design $\mathcal{D}$ it holds that a replication number r^* is $r^* = r - \alpha c + (\beta - \alpha)c = r + (\beta - 2\alpha)c$. For counting of coincidence numbers of any two treatments in $\mathcal{D}$, letting that these two treatments have a coincidence number s in the c sets of $\mathcal{D}$ for $0 \le s \le \lambda$, it follows that $\lambda^* = (\lambda - s) + \{\beta c - s - 2(\alpha c - s)\} = \lambda + (\beta - 2\alpha)c$. Furthermore, it can be shown that among block intersection numbers of $\mathcal{D}$,

$$q_{\ell\ell} = \begin{cases} q_1 & \text{in non-complementary sets,} \\ v - 2k + q_1 & \text{in complementary sets,} \end{cases}$$

and

$$q_{\ell\ell'} = \begin{cases} q_2 & \text{in non-complementary sets,} \\ k - q_2 & \text{between complementary and non-complementary sets,} \\ v - 2k + q_2 & \text{in complementary sets.} \end{cases}$$

Hence the proof is completed. □

Example. Consider an affine resolvable BIB design with parameters $v = 9, b = 12, r = 4, k = 3, \lambda = 1, \alpha = 1, \beta = 3, t = 4, q_1 = 0, q_2 = 1$, whose incidence matrix is given by

$$\left[\begin{array}{ccc|ccc|ccc|ccc}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0
\end{array}\right].$$

Now exchanging the first and second resolution sets into their complementary sets, we obtain an affine (2,2,1,1)-resolvable (6,3)-design with parameters $v^* = 9, b^* = 12, r^* = 6, k^* = 3$ or 6, $\lambda^* = 3$, $\alpha_1 = \alpha_2 = 2, \alpha_3 = \alpha_4 = 1, \beta = 3, t = 4, q_{11} = 3, q_{12} = 2, q_{13} = q_{14} = 1$, $q_{22} = 3, q_{23} = 2, q_{24} = 2, q_{33} = 0, q_{34} = 1, q_{44} = 0$, whose incidence matrix is given by

$$\left[\begin{array}{ccc|ccc|ccc|ccc}
0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0
\end{array}\right].$$

Note that the same design is provided by exchanging any two sets into their complementary sets.

Acknowledgement

Kageyama was supported in part by Grant-in-Aid for Scientific Research (C) 09640272, Japan.

References

1. W. H. Clatworthy, "Tables of Two-Associate-Class Partially Balanced Designs", NBS Applied Mathematics Series 63, Washington, D.C, 1973.

2. S. Kageyama, *On μ-resolvable and affine μ-resolvable balanced incomplete block designs*, Ann. Statist. **1**(1973), 195-203.

3. S. Kageyama and D. V. S. Sastry, *On affine $(\mu_1, \ldots, \mu_t)$-resolvable (r, λ)-designs*, Ars Combinatoria **36**(1993), 221-223.

4. R. Mukerjee and S. Kageyama, *On resolvable and affine resolvable variance-balanced designs*, Biometrika **72**(1988), 165-172.

5. D. Raghavarao, "Constructions and Combinatorial Problems in Design of Experiments", Dover, New York, 1988.