Cycles Containing Given Subsets in 1-Tough Graphs

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Abstract

For a graph $G = (V, E)$ and $X \subseteq V(G)$, let $dist_G(u, v)$ be the distance between the vertices u and v in G and $\sigma_3(X)$ denote the minimum value of the degree sum (in G) of any three pairwise non-adjacent vertices of X. We obtain main result: If G is a 1-tough graph of order n and $X \subseteq V(G)$ such that $\sigma_3(X) \geq n$ and, for all $x, y \in X$, $dist_G(x, y) = 2$ implies $\max\{d(x), d(y)\} \geq \frac{n-4}{2}$, then G has a cycle C containing all vertices of X. This result generalizes a result of Bauer, Broersma and Veldman.

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1. Results

We use [3] for terminology and notations not defined here and consider finite, simple graphs only.

Throughout this paper, let G be a graph of order n and $X \subseteq V(G)$. A graph G is called *1-tough* if $\omega(G - S) \leq |S|$ for every set S of some vertices

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of G satisfying $\omega(G-S) > 1$, where $\omega(G-S)$ is denoted the number of components of $G-S$. Let $dist_G(u,v)$ be the distance between two vertices u and v as the number of edges in a shortest uv-path in G and $\sigma_k(X)$ denote the minimum value of the degree sum (in G) of any k pairwise nonadjacent vertices of X. A cycle C is called *X-longest* if no cycle of G contains more vertices of X than C. We say that G is *X-cyclable* if G has an X-cycle, i.e., a cycle containing all vertices of X. If $X = V(G)$, then we use the common terminology *circumference* (denoted by $c(G)$) to mean the length of an X-longest cycle in G. In particular, G is Hamiltonian if G is $V(G)$-cyclable.

Jung got the following result in 1978.

Theorem 1. [5] If G is a 1-tough graph of order $n \geq 11$ such that $\sigma_2(G) \geq n-4$, then G is Hamiltonian.

In 1988, Bauer, Broersma and Veldman generalized Theorem 1 as follows.

Theorem 2. [1] If G be a 1-tough graph of order $n \geq 3$ such that $\sigma_3(G) \geq n$ and, for all vertices x, y, $dist_G(x,y) = 2$ implies $\max\{d(x), d(y)\} \geq \frac{n-4}{2}$, then G is Hamiltonian.

In 1993, we obtained the following result, which completely solved the conjecture proposed by Bauer, G. Fan and Veldman in [2].

Theorem 3. [7] If G be a 1-tough graph of order $n \geq 3$ such that $\sigma_3(G) \geq n$, then $c(G) \geq \min\{n, 2\rho_2^*(G)+4\}$, where $\rho_2^*(G) = \min\{|N_G(u) \cup N_G(v)| \mid dist_G(u,v) = 2\}$.

Recently, Broersma, H. Li, J.P. Li, F. Tian and Veldman considered some problems involving some cycles through given sets of some vertices in 2-connected graphs. The details could be found in [4].

Motivated by the above facts, we can obtain the following result that extends Theorem 2, whose proof will be postponed to section 3.

Theorem 4. If G is a 1-tough graph of order n and $X \subseteq V(G)$ such that $\sigma_3(X) \geq n$ and, for all vertices $x, y \in X$, $dist_G(x,y) = 2$ implies $\max\{d(x), d(y)\} \geq \frac{n-4}{2}$, then G is X-cyclable.

As a remark, we could obtain the following strong result, whose proof is almost modeled along the proof of Theorem 4, whenever a contradiction is obtained in the proof of Theorem 4, we could either obtain a contradiction or construct the exceptional graph I_n in the proof of Theorem 5. We omit its details here.

Theorem 5. If G is a 1-tough graph of order n and $X \subseteq V(G)$ such that $\sigma_3(X) \geq n$ and, for all vertices $x, y \in X$, $dist_G(x,y) = 2$ implies

$\max\{d(x), d(y)\} \geq \frac{n-5}{2}$, then either G is X-cyclable or else n is odd and G is a spanning subgraph of the exceptional graph I_n.

The exceptional graph I_n is obtained from $\overline{K}_{\frac{n-1}{2}} \cup K_{\frac{n-5}{2}} \cup K_3$ by joining every vertex in $K_{\frac{n-5}{2}}$ to all other vertices and adding a matching between the vertices of K_3 and three vertices of $\overline{K}_{\frac{n-1}{2}}$.

Theorem 5 admits the following corollaries.

Corollary 6. If G is a 1-tough graph of order n ($n \geq 15$) and $X \subseteq V(G)$ such that $\sigma_2(X) \geq n - 5$, then either G is X-cyclable or else n is odd and G is a spanning subgraph of the exceptional graph I_n.

Corollary 7. If G is a 1-tough graph of order n such that $\sigma_3(G) \geq n$ and, for all vertices $x, y \in X$, $dist_G(x, y) = 2$ implies $\max\{d(x), d(y)\} \geq \frac{n-5}{2}$, then either G is Hamiltonian or else n is odd and G is a spanning subgraph of the exceptional graph I_n.

2. Notations and Preliminary Lemmas

In order to prove our main result, we introduce some additional terminology and notations.

Let C be a cycle of G and $X \subseteq V(G)$. A cycle C is called X-dominating if all neighbors of each vertex of $X - V(C)$ are on C. We denote by $\overrightarrow{C}$ the cycle C with a given orientation and by $\overleftarrow{C}$ the same cycle with the reverse orientation. If $u, v \in V(C)$, then $u\overrightarrow{C}v$ denotes the set of consecutive vertices or the subpath of C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices or the subpath, in reverse order, are given by $v\overleftarrow{C}u$. We consider $u\overrightarrow{C}v$ and $v\overleftarrow{C}u$ both as paths and vertices sets. We use u^+ to denote the successor of u along $\overrightarrow{C}$ and u^- its predecessor. We use u^{+k} and u^{-k} to denote $(u^{+(k-1)})^+$ and $(u^{-(k-1)})^-$ for an integer $k \geq 2$, respectively.

Our proof of Theorem 4 heavily relies on the following two lemmas.

Lemma A. [6] Let G be a graph of order n and $X \subseteq V(G)$ such that $\delta(X) \geq 2$ and $\sigma_3(X) \geq n$. Suppose that G contains an X-longest cycle C that is X-dominating. If $x_0 \in X - V(C)$ and $N(x_0) = \{v_1, v_2, \ldots, v_m\}$, then $(X - V(C)) \cup \{x_1, x_2, \ldots, x_m\}$ is an independent set of vertices, where x_i is the first vertex of X on $v_i^+\overrightarrow{C}v_{i+1}^-$ for any $i \in \{1, 2, \ldots, m\}$.

Lemma B. [6] If G is a 1-tough graph of order $n \geq 3$ and $X \subseteq V(G)$ satisfies $\sigma_3(X) \geq n$, then G contains an X-longest cycle C that is X-dominating. Furthermore, if G is not X-cyclable, then $\max\{d(x) | x \in X - V(C)\} \geq \frac{\sigma_3(X)}{3}$.

3. Proof of Theorem 4

Throughout this section, we may assume that G satisfies the assumptions of Theorem 4, but G is not X-cyclable. By Lemma B, we choose an X-longest cycle C that is X-dominating and a vertex $x_0 \in X - V(C)$ such that $d(x_0) = \max\{d(x)|x \in X - V(C)\}$ among the set of all X-longest cycles that are X-dominating. Hence $d(x_0) \geq \frac{\sigma_3(X)}{3}$.

Let $A = N(x_0)$ and $v_1, v_2, \ldots, v_{|A|}$ be the vertices of A, occuring on $\overrightarrow{C}$ in consecutive order. Since C is X-longest, we have $X \cap (v_i^+ \overrightarrow{C} v_{i+1}^-) \neq \emptyset$ for each $i \in \{1, 2, \ldots, |A|\}$. For any $i \in \{1, 2, \ldots, |A|\}$, let x_i be the first vertex of X on $v_i^+ \overrightarrow{C} v_{i+1}^-$ and y_i the last vertex of X on $v_i^+ \overrightarrow{C} v_{i+1}^-$. Then $(X - V(C)) \cup \{x_1, x_2, \ldots, x_{|A|}\}$ $((X - V(C)) \cup \{y_1, y_2, \ldots, y_{|A|}\}$, respectively) is an independent set of vertices by Lemma A. A segment $v_i^+ \overrightarrow{C} v_{i+1}^-$ is called t-segment if $|v_i^+ \overrightarrow{C} v_{i+1}^-| = t$. A 1-segment $v_i^+ \overrightarrow{C} v_{i+1}^-$ is called a *proper* 1-segment if this 1-segment (vertex) has no neighbor out of C. Let S denote the set of 1-segments and S' denote the set of proper 1-segments, respectively. Put $s = |S|$ and $s' = |S'|$.

Claim 1. If $i \neq j$, then $ab \notin E(G)$, where either $a \in v_i^+ \overrightarrow{C} x_i$, $b \in v_j^+ \overrightarrow{C} x_j$ or $a \in y_i \overrightarrow{C} v_{i+1}^-$, $b \in y_j \overrightarrow{C} v_{j+1}^-$.

Proof of Claim 1. Suppose that there exist two vertices $a \in v_i^+ \overrightarrow{C} x_i$ and $b \in v_j^+ \overrightarrow{C} x_j$ such that $ab \in E(G)$, then the cycle $x_0 v_j \overleftarrow{C} x_i \overleftarrow{C} ab \overrightarrow{C} x_j \overrightarrow{C} v_i x_0$ contains more vertices of X than C, a contradiction. Similarly, if there exist two vertices $a \in y_i \overrightarrow{C} v_{i+1}^-$ and $b \in y_j \overrightarrow{C} v_{j+1}^-$ such that $ab \in E(G)$, then the cycle $x_0 v_{i+1} \overrightarrow{C} y_j \overrightarrow{C} ba \overleftarrow{C} y_i \overleftarrow{C} v_{j+1} x_0$ contains more vertices of X than C, a contradiction.

□

Claim 2. If $i \neq j$, then there exits no vertex $z \in x_i^+ \overrightarrow{C} v_j^-$ such that $az^+, bz \in E(G)$, where $a \in v_i^+ \overrightarrow{C} x_i$ and $b \in v_j^+ \overrightarrow{C} x_j$.

Proof of Claim 2. Suppose that there exists a vertex $z \in x_i^+ \overrightarrow{C} v_j^-$ such that $az^+, bz \in E(G)$, where $i \neq j$, $a \in v_i^+ \overrightarrow{C} x_i$ and $b \in v_j^+ \overrightarrow{C} x_j$, then the cycle $x_0 v_j \overleftarrow{C} z^+ a \overrightarrow{C} x_i \overrightarrow{C} zb \overrightarrow{C} x_j \overrightarrow{C} v_i x_0$ contains more vertices of X than C, a contradiction.

□

With the similar arguments to the proof of Claim 2, we obtain the following result.

Claim 3. If $i \neq j$, then there exits no vertex $z \in v_j \overrightarrow{C} v_i^-$ such that

$az^+, bz \in E(G)$, or $az, bz^+ \in E(G)$, where $a \in v_i^+ \overrightarrow{C} x_i$ and $b \in y_{j-1} \overrightarrow{C} v_j^-$.

□

Claim 4. $|S| \geq |S'| \geq 2$. Moreover, for each $u \in S'$, $d(u) \leq d(x_0)$.

Proof of Claim 4. Put $U = \{x_1, x_2, \ldots, x_{|A|}\} - S$ and $W = \{y_1, y_2, \ldots, y_{|A|}\} - S$. Let $O(U) = N_G(U) - V(C)$ and $O(W) = N_G(W) - V(C)$. Since any two vertices of $\{x_1, x_2, \ldots, x_{|A|}\}$ ($\{y_1, y_2, \ldots, y_{|A|}\}$, respectively) have no common neighbors in $V(G) - V(C)$ and any vertex of $\{x_1, x_2, \ldots, x_{|A|}\}$ ($\{y_1, y_2, \ldots, y_{|A|}\}$, respectively) has no neighbors in $X - V(C)$ by Lemma A, we have

$$n - 1 - (s - s') - \max\{|O(U)|, |O(W)|\} \geq |V(C)| \geq 3(d(x_0) - s) + 2s$$

whence

$$\begin{aligned} s' &\geq 3d(x_0) - n + 1 + \max\{|O(U)|, |O(W)|\} \\ &\geq \sigma_3(X) - n + 1 + \max\{|O(U)|, |O(W)|\} \\ &\geq 1. \end{aligned}$$

Suppose $s' = 1$, without loss of generality, we assume that $S' = \{x_1\}$, then the above inequalities imply that $d(x_0) = \frac{1}{3}n$, $\max\{|O(U)|, |O(W)|\} = 0$ and $|V(C)| = 3d(x_0) - s$; Moreover we get that C contains only 1-segments and 2-segments. Suppose $v_i^+ v_{j+1}^- \notin E(G)$ for all $i \neq j$, then all distinct segments are not connected by an edge or a path whose internal vertices are in $V(G) - V(C)$, hence $\omega(G - A) \geq |A| + 1$, which contradicts the fact that G 1-tough. Thus, there exists $v_i^+ v_{j+1}^- \in E(G)$ for some $i \neq j$.

Since the X-longest cycle C contains only 1-segments and 2-segments, we get $v_{i+1}^-, v_j^+ \in X$ and $i, j \neq 1$. We choose a minimal integer i such that $v_i^+ v_{j+1}^- \in E(G)$. Then $v_t^+ v_{j'+1}^- \notin E(G)$ for each $t \in \{1, 2, \ldots, i-1\}$. By the fact $\max\{|O(U)|, |O(W)|\} = 0$, the cycle $C' = x_0 v_{i+1} \overrightarrow{C} v_{j+1}^- v_i^+ \overleftarrow{C} v_{j+1} x_0$ is another X-longest cycle that is X-dominating and satisfying $v_{i+1}^- \notin C'$. By the choice of C and the fact $v_{i+1}^- \in X$, we have $d(v_{i+1}^-) \leq d(x_0)$. Since x_0, x_1 and v_{i+1}^- are three nonadjacent vertices of X, we have $n \leq d(x_0) + d(x_1) + d(v_{i+1}^-) \leq 3d(x_0) = n$. So $d(x_0) = d(x_1) = d(v_{i+1}^-) = \frac{\sigma_3(X)}{3} = \frac{n}{3}$. Hence, we have $N(x_1) = N(x_0)$.

Suppose $i > j$, since the fact $N(x_1) = N(x_0)$ leads $v_{j+1} \in N(x_1)$, then the cycle $C'' = x_0 x_1^+ \overrightarrow{C} v_{j+1}^- v_i^+ \overrightarrow{C} x_1 v_{j+1} \overrightarrow{C} v_i x_0$ contains more vertices of X than C, a contradiction. This shows that $i < j$. By the choice of i, we have $v_{i+1}^- v_t^+ \notin E(G)$, where $t \in \{1, 2, \ldots, i-1\}$. If $v_{i+1}^- v_s^+ \in E$ for $i + 1 \leq s \leq j$, we get the cycle $C''' = x_0 v_{j+1} \overrightarrow{C} v_i^+ v_{j+1}^- \overleftarrow{C} v_s^+ v_{i+1}^- \overrightarrow{C} v_s x_0$ containing more vertices of X than C. If $v_{i+1}^- v_s^+ \in E$ for $j + 1 \leq s \leq |A|$, we also get the cycle $C'''' = x_0 v_2 \overrightarrow{C} v_i^+ v_{j+1}^- \overleftarrow{C} v_{i+1}^- v_s^+ \overrightarrow{C} x_1 v_s \overleftarrow{C} v_{j+1} x_0$ containing more vertices of X than C (note $N(x_1) = N(x_0)$). This shows $N(v_{i+1}^-) \subseteq N(x_0) \cup \{v_i^+\}$ (since $|O(W)| = 0$). Since C is X-longest and the subpath $v_{j+1}\overrightarrow{C}v_i$ contains the 1-segment x_1, we can easily obtain that $v_{i+1}^- v_i \notin E$, $v_{i+1}^- v_{j+1} \notin E$ and $v_i \neq v_{j+1}$, so $N(v_{i+1}^-) \subseteq (A \cup \{v_i^+\}) - \{v_i, v_{j+1}\}$. Hence $d(v_{i+1}^-) \leq d(x_0) + 1 - 2 = d(x_0) - 1$, contradicting $d(v_{i+1}^-) = d(x_0)$. Therefore, $|S| \geq |S'| \geq 2$.

Thus, for each $u \in S'$, $d(u) \leq d(x_0)$, else we can replace u in C by x_0.

□

Claim 5. If G is 1-tough, then $|V(C)| \geq 2|A| + 2$ and the equality holds only if C contains two 2-segments and all other segments are 1-segments.

Proof of Claim 5. Put $Z = V(C) - (A \cup A^+)$. Since C is an X-longer cycle, we have $A \cap A^+ = \emptyset$, then $|V(C)| = |Z| + 2|A| \geq 2|A|$. Suppose $|V(C)| \leq 2|A| + 1$, then $|Z| \leq 1$, so all segments of C are 1-segments except only a 2-segment. Since C is X-longest and each segment does not connect to the others by an edge or a path whose internal vertices are in $V(G) - V(C)$, we get $\omega(G - A) \geq |A| + 1$, which contradicts the fact G is 1-tough. Hence $|V(C)| \geq 2|A| + 2$.

Suppose $|V(C)| = 2|A| + 2$, we get $|Z| = 2$, then all segments of C are 1-segments except that C contains either two 2-segments or a 3-segment. If $|V(C)| = 2|A| + 2$ and C contains a 3-segment, say $v_i^+ v_i^{+2} v_{i+1}^-$, then all other segments are 1-segments. By Claim 1, we have $v_i^+ v_j^+ \notin E(G)$ and $v_{i+1}^- v_j^+ \notin E(G)$ for any $j \in \{1, 2, \ldots, |A|\} - \{i\}$. Suppose $v_l^+ v_i^{+2} \in E(G)$ for some $l \in \{1, 2, \ldots, |A|\} - \{i\}$, then $N(v_i^+) \cap V(C) \subseteq A \cup \{v_i^{+2}\}$ and $N(v_{i+1}^-) \cap V(C) \subseteq A \cup \{v_i^{+2}\}$ by Claims 1-3. By the fact that any two vertices of $A^+ \cup \{v_{i+1}^-\}$ do not connect to each other by an edge or a path whose internal vertices are in $V(G) - V(C)$, we obtain $\omega(G - (A \cup \{v_i^{+2}\})) > |A| + 1$, which contradicts the fact G is 1-tough. So $v_j^+ v_i^{+2} \notin E(G)$ for any $j \in \{1, 2, \ldots, |A|\} - \{i\}$. Moreover, we get $\omega(G - A) > |A|$, which also contradicts the fact G is 1-tough. Hence G contains no 3-segment.

Thus, the equality $|V(C)| = 2|A| + 2$ leads that C contains two 2-segments and all other segments are 1-segments.

□

Claim 6. At most one vertex of $\{x_1, x_2, \ldots, x_{|A|}\}$ ($\{y_1, y_2, \ldots, y_{|A|}\}$, respectively) has degree smaller than $\frac{n-4}{2}$.

Proof of Claim 6. Suppose that there exist two vertices x_i, x_j ($i \neq j$) of $\{x_1, x_2, \ldots, x_{|A|}\}$ such that $d(x_i) < \frac{n-4}{2}$ and $d(x_j) < \frac{n-4}{2}$. Since $x_i x_j \notin E(G)$ by Claim 1, we get $dist_G(x_i, x_j) \geq 2$. If $N(x_i) \cap N(x_j) \neq \emptyset$, we have $dist_G(x_i, x_j) = 2$. By the second degree condition of Theorem 4, we have $\max\{d(x_i), d(x_j)\} \geq \frac{n-4}{2}$, which contradicts our assumptions $d(x_i) < \frac{n-4}{2}$ and $d(x_j) < \frac{n-4}{2}$. This shows that $N(x_i) \cap N(x_j) = \emptyset$. By Lemma A and the claims above, we have $(N(x_i) \cup N(x_j)) \cap V(C) \subseteq V(C) - \{x_1, x_2, \ldots, x_{|A|}\}$. It follows that

$$\begin{aligned}
d(x_i) + d(x_j) &= |N(x_i) \cup N(x_j)| + |N(x_i) \cap N(x_j)| \\
&= |(N(x_i) \cup N(x_j)) \cap V(C)| \\
&\quad + |(N(x_i) \cup N(x_j)) - V(C)| \\
&\leq |V(C) - \{x_1, x_2, \ldots, x_{|A|}\}| \\
&\quad + |(N(x_i) \cup N(x_j)) - V(C)| \\
&= |V(C)| - |A| + |(N(x_i) \cup N(x_j)) - V(C)| \\
&= |N(x_i) \cup N(x_j) \cup V(C)| - |A| \\
&\leq n - 1 - d(x_0)
\end{aligned}$$

Hence $\sigma_3(X) \leq d(x_i) + d(x_j) + d(x_0) \leq n - 1$, a contradiction.

Similarly, at most one vertex of $\{y_1, y_2, \ldots, y_{|A|}\}$ has degree smaller than $\frac{n-4}{2}$.

□

By Claim 4, we have some vertex $u \in S'$ satisfying $d(u) \leq d(x_0)$. Since $dist_G(u, x_0) = 2$, we get $d(x_0) = |A| \geq \frac{n-4}{2}$. By Claim 5 and the fact $|A| \geq \frac{n-4}{2}$, we obtain

$$n - 2 \leq 2|A| + 2 \leq |V(C)| \leq n - 1 \leq 2|A| + 3$$

Note that $|A| \leq \frac{n-3}{2}$. By the fact $n - 1 \leq |V(C) \cup \{x_0\}| \leq n$, we get $|V(G) - (V(C) \cup \{x_0\})| \leq 1$. Below, we will distinguish the two cases, in each of which we obtain a contradiction. For convenience, we set $\epsilon(q) = |N_G(q) - V(C)|$ for any $q \in V(G)$.

Case 1. $|V(C)| = 2|A| + 2$

By Claim 5, C contains two 2-segments, say $v_1^+ \overrightarrow{C} v_2^-$ and $v_i^+ \overrightarrow{C} v_{i+1}^-$, and all other segments are 1-segments. By Claims 1-3 and the fact that G is 1-tough, we obtain either $v_1^+ v_{i+1}^- \in E(G)$ or $v_2^- v_i^+ \in E(G)$. Without loss of generality, we may assume $v_1^+ v_{i+1}^- \in E(G)$, then $v_2^-, v_i^+ \in X$ by Claim 1. We note that $|V(C)| \geq n - 2$, i.e., $|V(G) - (V(C) \cup \{x_0\})| \leq 1$.

Case 1.1 $i \geq 5$

Since x_2, x_3 and x_4 are all 1-segments and $\epsilon(x_2) + \epsilon(x_3) + \epsilon(x_4) \leq 1$, then at least two vertices of $\{x_2, x_3, x_4\}$ are proper 1-segments. By

Claim 6, we can choose a proper 1-segment $x_j \in \{x_2, x_3, x_4\}$ satisfying $d(x_j) \geq \frac{n-4}{2}$. Note that $|A| \leq \frac{n-3}{2}$, we obtain at least one vertex of $\{v_1, v_{i+1}\}$, say v_1, such that v_1 is adjacent to x_j. So we can construct a cycle $C' = x_0 v_{i+1} \overrightarrow{C} v_1 x_j \overrightarrow{C} v_{i+1}^- v_1^+ \overrightarrow{C} v_j x_0$ containing more vertices of X than C, a contradiction.

Case 1.2 $\quad i = 4$

In this case, suppose that $v_1 \neq v_5$, by the facts v_1, $v_5 \notin N(x_2) \cup N(x_3)$, we get $(N(x_2) \cup N(x_3)) \cap V(C) \subseteq A - \{v_1, v_5\}$, then we obtain

$$\begin{aligned}\min\{d(x_2), d(x_3)\} &\leq |A| - 2 + \min\{\epsilon(x_2), \epsilon(x_3)\} \\ &\leq \tfrac{|V(C)|-2}{2} - 2 + \tfrac{\epsilon(x_2)+\epsilon(x_3)}{2} \\ &= \tfrac{|V(C)|+\epsilon(x_2)+\epsilon(x_3)}{2} - 3 \\ &< \tfrac{n-4}{2}\end{aligned}$$

Without loss of generality, we may assume that $d(x_2) \leq d(x_3)$, then $d(x_2) < \frac{n-4}{2}$. By the fact $|V(G) - (V(C) \cup \{x_0\})| \leq 1$, we get $\epsilon(v_2^-) + \epsilon(x_3) + \epsilon(x_4) \leq 1$, so at least two vertices of $\{v_2^-, x_3, x_4\}$ have no neighbors out of C. By the facts $N(v_2^-) \cap V(C) \subseteq A \cup \{v_1^+\} - \{v_1, v_5\}$, $N(x_3) \cap V(C) \subseteq A - \{v_1, v_5\}$ and $N(x_4) \cap V(C) \subseteq A \cup \{v_5^-\} - \{v_1, v_5\}$, we get

$$\min\{d(v_2^-), d(x_3), d(x_4)\} \leq (|A| + 1) - 2 \leq \frac{n-3}{2} - 1 < \frac{n-4}{2}$$

Since $d(x_2) < \frac{n-4}{2}$ and $v_2^-, x_2, x_3, x_4 \in \{y_1, y_2, \ldots, y_{|A|}\}$, so at least two vertices of $\{y_1, y_2, \ldots, y_{|A|}\}$ have degree smaller than $\frac{n-4}{2}$, which contradicts Claim 6. Thus, $v_1 = v_5$, i.e., $|V(C)| = 10$.

For the case $i = 4$ and $v_1 = v_5$ (and $|V(C)| = 10$), we obtain

$$\begin{aligned}\sigma_3(X) &\leq d(x_2) + d(x_3) + d(x_0) \\ &\leq (3 + \epsilon(x_2)) + (3 + \epsilon(x_3)) + 4 \\ &= |V(C)| + \epsilon(x_2) + \epsilon(x_3) \\ &\leq n - 1\end{aligned}$$

a contradiction.

Case 1.3 $\quad i \leq 3$

Claim 4 implies $v_1 \neq v_{i+1}$. It is easy to see that v_2^-, x_2, $v_i^+ \in X$ and $dist_G(v_2^-, x_2) = 2$, so we get $\max\{d(v_2^-), d(x_2)\} \geq \frac{n-4}{2}$. On the other hand, since $v_2^- v_1$, $v_2^- v_{i+1} \notin E(G)$, we get $N(v_2^-) \subseteq A \cup \{v_1^+\} - \{v_1, v_{i+1}\}$, and by the fact $|A| \leq \frac{n-3}{2}$, we obtain

$$d(v_2^-) \leq |A| + 1 - 2 \leq \frac{n-3}{2} - 1 < \frac{n-4}{2}.$$

Similarly, since $x_2v_1, x_2v_{i+1} \notin E(G)$, we get $N(x_2) \subseteq A \cup \{v_3^-\} - \{v_1, v_{i+1}\}$ (here, $x_2 = v_i^+$ when $i = 2$), and by the fact $|A| \leq \frac{n-3}{2}$, we obtain

$$d(x_2) \leq |A| + 1 - 2 \leq \frac{n-3}{2} - 1 < \frac{n-4}{2}.$$

Hence, $\max\{d(v_2^-), d(x_2)\} < \frac{n-4}{2}$, which contradicts our result $\max\{d(v_2^-), d(x_2)\} \geq \frac{n-4}{2}$.

Case 2. $|V(C)| = 2|A| + 3$

Put $Z = V(C) - (A \cup A^+)$. By the fact $|A| \geq \frac{n-4}{2}$, we obtain $|V(C)| = n - 1$, $|A| = \frac{n-4}{2}$ and $|Z| = 3$. Note that $\epsilon(q) = 0$ for any $q \in V(G) - A$. We consider the following three possibilities.

Case 2.1 C contains a 4-segment

Without loss of generality, we assume that $v_1^+, v_1^{+2}, v_2^{-2}, v_2^-$ are the vertices of the 4-segment. Suppose that neither v_1^{+2} nor v_2^{-2} is adjacent to any 1-segment, then $\omega(G - A) > |A|$ by Claim 1, a contradiction. This shows that at least one vertex of $\{v_1^{+2}, v_2^{-2}\}$ is adjacent to some 1-segments. Without loss of generality, we may assume that $v_1^{+2}x_i \in E(G)$ for some 1-segment x_i. Claims 2-3 imply that v_2^{-2} is not adjacent to any 1-segment, while the same is true for v_2^- by Claims 1-2. So we obtain $\omega(G - (A \cup \{v_1^{+2}\})) > |A \cup \{v_1^{+2}\}|$, a contradiction.

Case 2.2 C contains a 3-segment and a 2-segment

In this case, we assume that v_1^+, v_1^{+2}, v_2^- are the vertices of the 3-segment and that v_i^+, v_{i+1}^- are the vertices of the 2-segment.

Suppose that $v_1^+v_{i+1}^- \notin E(G)$ and $v_2^-v_i^+ \notin E(G)$. If $v_1^+v_2^- \notin E(G)$, then we get $\omega(G - (A \cup \{v_1^{+2}\})) > |A \cup \{v_1^{+2}\}|$ by Claim 1, a contradiction. If $v_1^+v_2^- \in E(G)$, then v_1^{+2} is not adjacent to any vertex in $(A^+ \cup A^-) - \{v_1^+, v_2^-\}$ by Claim 2 or Claim 3, so we also get $\omega(G - A) > |A|$, a contradiction. This shows that either $v_1^+v_{i+1}^- \in E(G)$ or $v_2^-v_i^+ \in E(G)$. Below, we only consider the case $v_1^+v_{i+1}^- \in E(G)$. (For the case $v_2^-v_i^+ \in E(G)$, we consider the cycle C on the reverse orientation $\overleftarrow{C}$, we also obtain a contradiction). Claim 1 implies that $v_i^+ \in X$, i.e., $v_i^+ = x_i$.

Case 2.2.1 $i \geq 4$

We consider the two vertices x_2 and x_3. By the facts $\epsilon(x_2) = 0$, $\epsilon(x_3) = 0$ and Claims 1-3, it is very easy to obtain that $N(x_2) \subseteq A - \{v_1, v_{i+1}\}$ and $N(x_3) \subseteq A - \{v_1, v_{i+1}\}$. So we get $\max\{d(x_2), d(x_3)\} \leq |A| - 1 = \frac{n-6}{2} < \frac{n-4}{2}$. On the other hand, $dist_G(x_2, x_3) = 2$ implies $\max\{d(x_2), d(x_3)\} \geq \frac{n-4}{2}$, a contradiction.

Case 2.2.2 $i = 3$

In this case, the subpath $v_{i+1}\overrightarrow{C}v_1$ contains at least one proper 1-segment by Claim 4, so we get $v_1 \neq v_4$. By the facts $\epsilon(x_2) = 0$, $\epsilon(x_3) = 0$ and Claims 1-3, it is very easy to obtain that $N(x_2) \subseteq A - \{v_1, v_4\}$ and $N(v_3^+) \subseteq (A \cup \{v_4^-\}) - \{v_1, v_4\}$, hence we get $\max\{d(x_2), d(x_3)\} \leq |A| + 1 - 2 = \frac{n-6}{2} < \frac{n-4}{2}$. On the other hand, $dist_G(x_2, x_3) = 2$ implies $\max\{d(x_2), d(v_4^-)\} \geq \frac{n-4}{2}$, a contradiction.

Case 2.2.3 $i = 2$

In this case, by Claims 1-4, it is very easy to obtain that $N(x_2) \subseteq (A \cup \{v_3^-\}) - \{v_1, v_3\}$ and $v_1 \neq v_3$, so we get $d(x_2) \leq |A| + 1 - 2 < \frac{n-4}{2}$.

By Claims 4 and 6, the subpath $v_3\overrightarrow{C}v_1$ contains at least two (proper) 1-segments and we get $d(x_k) \geq \frac{n-4}{2}$ for any (proper) 1-segment $x_k \in v_3\overrightarrow{C}v_1$. Again by Claims 1-3, we get the fact $N(x_k) \subseteq A$. Moreover we get $N(x_k) = A$ by the facts $d(x_k) \geq \frac{n-4}{2}$ and $|A| = \frac{n-4}{2}$. This follows that $x_k v_2 \in E(G)$ for any 1-segment $x_k \in v_3\overrightarrow{C}v_1$. Below, we consider the last vertex y_1 of X on $v_1^+\overrightarrow{C}v_2^-$.

- If there exists a 1-segment $x_k \in v_3\overrightarrow{C}v_1$ such that $y_1 x_k^- \in E(G)$, we can construct a cycle $C' = x_0 v_3 \overrightarrow{C} x_k^- y_1 \overleftarrow{C} x_1 v_3^- \overleftarrow{C} v_2 x_k \overrightarrow{C} v_1 x_0$ containing more vertices of X than C, a contradiction.

- If $y_1 x_k^- \notin E(G)$ for any 1-segment $x_k \in v_3\overrightarrow{C}v_1$, we get $N(y_1) \subseteq \{v_1^+, v_1^{+2}, v_2^-, v_2\} - \{y_1\}$. So we obtain that

$$\sigma_3(X) \leq d(y_1) + d(x_2) + d(x_0) < 3 + \frac{n-4}{2} + \frac{n-4}{2} < n,$$

a contradiction, too.

Case 2.3 C contains three 2-segments

In this case, we may assume that $v_1^+\overrightarrow{C}v_2^-$, $v_i^+\overrightarrow{C}v_{i+1}^-$ and $v_j^+\overrightarrow{C}v_{j+1}^-$ $(1 < i < j)$ are the three 2-segments. If no vertex in any 2-segment is adjacent to any vertex in a different 2-segment, then we get $\omega(G - A) > |A|$, a contradiction. Hence, we only consider that $v_1^+ v_{i+1}^- \in E(G)$ or $v_1^+ v_{j+1}^- \in E(G)$ or $v_i^+ v_{j+1}^- \in E(G)$. (For the other cases, we consider the cycle C on the reverse orientation $\overleftarrow{C}$, we also obtain a contradiction).

Case 2.3.1 $v_1^+ v_{i+1}^- \in E(G)$

In this case, we get that $v_2^-, v_2^+ \in X$ (by Claim 1) and $dist_G(v_2^-, v_2^+) = 2$, so $\max\{d(v_2^-), d(v_2^+)\} \geq \frac{n-4}{2}$. On the other hand, it is easy to see that $v_1 \neq v_{i+1}$ (since the subpath $v_{i+1}\overrightarrow{C}v_1$ contains the 2-segment $v_j^+\overrightarrow{C}v_{j+1}^-$)

and that $N(v_2^-) \subseteq (A \cup \{v_1^+\}) - \{v_1, v_{i+1}\}$ and $N(v_2^+) \subseteq (A \cup \{v_{i+1}^-\}) - \{v_1, v_{i+1}\}$ (here, $N(v_2^+) \subseteq A - \{v_1, v_{i+}\}$ for the 1-segment v_2^+), then we get $\max\{d(v_2^-), d(v_2^+)\} \leq (|A|+1) - 2 < \frac{n-4}{2}$, a contradiction.

Case 2.3.2 $v_i^+ v_{j+1}^- \in E(G)$

In this case, with the similar arguments in Case 2.3.1, we can easily obtain a contradiction.

Case 2.3.3 $v_1^+ v_{j+1}^- \in E(G)$

In this case, we only consider the facts $v_1^+ v_{i+1}^- \notin E(G)$, $v_i^+ v_{j+1}^- \notin E(G)$ and $v_1^+ v_{j+1}^- \in E(G)$, otherwise we get a contradiction by the similar arguments in Case 2.3.1.

– If the subpath $v_2 \overrightarrow{C} v_j$ contains (at least) two proper 1-segments, say $x_{i'}$ and $x_{j'}$, it is easy to see that $N(x_{i'}) \subseteq A - \{v_1, v_{j+1}\}$ and $N(x_{j'}) \subseteq A - \{v_1, v_{j+1}\}$, then we get $\max\{d(x_{i'}), d(x_{j'})\} \leq |A| - 1 < \frac{n-4}{2}$, which contradicts Claim 6.

– If the subpath $v_2 \overrightarrow{C} v_j$ contains only one proper 1-segments, say $x_{i'}$, then $v_1 \neq v_{j+1}$ by Claim 4. Without loss of generality, we may assume that $x_{i'} \in v_2 \overrightarrow{C} v_i$. So $x_{i'} = v_2^+$, $i = 3$ and $j = 4$. We consider the two vertices v_2^- and x_2 $(= y_2)$. Since $v_2^- \in X$ by Claim 1 and $dist_G(v_2^-, x_2) = 2$, we get $\max\{d(v_2^-), d(v_2^+)\} \geq \frac{n-4}{2}$. On the other hand, it is easy to see that $N(v_2^-) \subseteq (A \cup \{v_1^+\}) - \{v_1, v_5\}$ and $N(x_2) \subseteq A - \{v_1, v_5\}$, then we get $\max\{d(v_2^-), d(v_2^+)\} \leq (|A|+1) - 2 < \frac{n-4}{2}$, a contradiction.

– If the subpath $v_2 \overrightarrow{C} v_j$ contains no (proper) 1-segments, then $i = 2$, $j = 3$ and $v_1 \neq v_4$ (by Claim 4). Since $v_2^+ \overrightarrow{C} v_3^- \cap X \neq \emptyset$, we get that at least one of the two vertices $\{v_2^+, v_3^-\}$ belongs to X.

When $v_2^+ \in X$, we consider the two vertices v_2^- and v_2^+. Since $dist_G(v_2^-, v_2^+) = 2$, we get $\max\{d(v_2^-), d(v_2^+)\} \geq \frac{n-4}{2}$. On the other hand, it is easy to see that $N(v_2^-) \subseteq (A \cup \{v_1^+\}) - \{v_1, v_5\}$ and $N(v_2^+) \subseteq (A \cup \{v_3^-\}) - \{v_1, v_5\}$, then we get $\max\{d(v_2^-), d(v_2^+)\} \leq (|A|+1) - 2 < \frac{n-4}{2}$, a contradiction.

When $v_3^- \in X$, we consider the two vertices v_3^- and v_3^+. Since $dist_G(v_3^-, v_3^+) = 2$, we also get $\max\{d(v_2^-), d(v_2^+)\} \geq \frac{n-4}{2}$. On the other hand, it is easy to see that $N(v_3^-) \subseteq (A \cup \{v_2^+\}) - \{v_1, v_5\}$ and $N(v_3^+) \subseteq (A \cup \{v_4^-\}) - \{v_1, v_5\}$, then we get $\max\{d(v_2^-), d(v_2^+)\} \leq (|A|+1) - 2 < \frac{n-4}{2}$, a contradiction, too.

This completes the proof of Theorem 4.

□

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