ON EXACT n-STEP DOMINATION

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ABSTRACT. A graph G with vertex set $V(G)$ is an *exact n-step domination graph* if there is some subset $S \subseteq V(G)$ such that each vertex in G is distance n from exactly one vertex in S. Given a set $A \subset \mathbb{N}$, we characterize cycles C_t with sets $S \subseteq V(C_t)$ that are simultaneously a-step dominating for precisely those $a \in A$. Using Polya's method, we compute the number of n-step dominating sets for a cycle C_t that are distinct up to automorphisms of C_t. Finally, we generalize the notion of exact n-step domination.

1. INTRODUCTION

The topic of domination in graphs has recently been the subject of much research. Indeed, two books on this subject ([4] and [5]) appeared in December of 1997. In this paper we study exact n-step domination, the generalization of exact 2-step domination, which was introduced by Chartrand *et al.* [2].

A vertex u in a graph G is said to *n-step dominate* a vertex v if $d(u, v) = n$. If there exists a subset $S \subseteq V(G)$ such that each $v \in V(G)$ is n-step dominated by exactly one vertex in S, then G is an *exact n-step domination graph* and S is called an *exact n-step dominating set.*

Figure 1 has examples of exact 4-step, 5-step, and 6-step domination graphs with $|S| = 4$. These graphs support Hersh's [6] conjecture that for each $n \geq 4$ there is an exact n-step domination graph G with dominating set $S \subseteq V(G)$ such that $|S| \leq n$.

In Section 2 we prove the following theorem, generalizing Hersh's result [6, Proposition 4] characterizing cycles that are n-step dominated for some $n \in \mathbb{N}$. We denote a cycle with t vertices by C_t.

Theorem 2.8. *Let A be a nonempty set of natural numbers and let $t = 2^i t'$ where $i > 0$ and t' is odd. There exists $S \subseteq V(C_t)$ such that S simultaneously a-step dominates C_t for precisely those numbers $a \in A$ if and only if A is the set $\{t/2\}$, or A is of the form $\{c, 3c, \ldots, (2m-1)c\}$, where $c|t$, $2^{i-1} \nmid c$ and $(2m-1)c$ is the greatest odd multiple of c such that $(2m-1)c < t/2$.*

In Section 3 we prove the theorem below, where we call two sets *equivalent* if there is an automorphism that carries one to the other.

ARS COMBINATORIA 58(2001), pp. 13-22

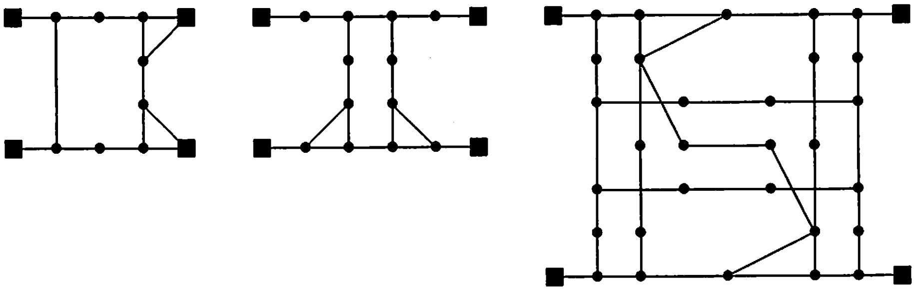

FIGURE 1. 4-Step, 5-Step, and 6-Step Domination Graphs with $|S| = 4$

Theorem 3.5. *Let C_t be n-step dominated, and let $d = \gcd(n, t)$, where $d = 2^i d'$ for some odd d'. Then the number of inequivalent n-step dominating sets for C_t is 1 if $t = 2n$, and otherwise is*

$$2^{d-2} + \frac{1}{8d} \sum_{l=1}^{d'} 2^{\gcd(2d, 2^{i+2} l)}.$$

In Section 4 we introduce k-exact n-step domination, and generalize Hersh's results [6, Propositions 1, 2] about the orders of n-step dominating sets of a graph and the product of two domination graphs. In addition, we prove that there exists a k-exact n-step domination graph for every pair of positive integers k and n.

2. Domination in Cycles

If the distance between two vertices in a cycle C_t is m, we refer to the vertices as being both m and $t - m$ steps apart. We use $\bar{S}$ to denote the complement of S. For the sake of brevity, we may omit the word *exact* when we speak of exact n-step domination.

In this section we extend the following result.

Proposition 2.1 (Hersh [6, Proposition 4]). *A cycle C_t is an exact n-step domination graph if and only if either $t = 2n$, or $t > 2n$ and $2^{i+2} | t$ where 2^i is the largest power of 2 that divides n.*

The following results will be useful to us.

Proposition 2.2. *If $t > 2n$, a cycle C_t is n-step dominated by $S \subseteq V(C_t)$ if and only if exactly one of any two vertices that are $2n$ steps apart is in S.*

Proof. Suppose C_t is n-step dominated by S, and consider any two vertices $u, w \in V(C_t)$ that are $2n$ steps apart. There is a vertex $v \in V(C_t)$ such that the only two vertices that are distance n from v are u and w. Thus, exactly one of u and w is in S.

Conversely, it is easy to see that if exactly one of two vertices that are $2n$ steps apart is in S, then C_t is n-step dominated by S. □

Corollary 2.3. *Suppose $a \in \mathbb{N}$ and we have a cycle C_t that is n-step dominated by $S \subseteq V(C_t)$ where $t > 2n$. If a is even then two vertices that are $2na$ steps apart are either both in S or both in $\bar{S}$, and if a is odd then exactly one of two vertices that are $2na$ steps apart is in S.*

Corollary 2.4. *If C_t is n-step dominated by $S \subseteq V(C_t)$ then it is nl-step dominated by S for every odd positive integer l such that $2nl < t$.*

Proposition 2.5. *If a cycle C_t can be both m-step and n-step dominated, then it can be simultaneously m-step and n-step dominated by a set $S \subseteq V(C_t)$ if and only if 2 divides m and n with the same multiplicity.*

Proof. ($\Rightarrow$) Write $m = 2^i m'$ and $n = 2^j n'$ for m' and n' odd, and assume that $i < j$. Clearly we can find a solution to $2ma = 2nb$ for some even a and odd b. Note that S cannot be $V(C_t)$ so $t > 2n$ and $t > 2m$. By Corollary 2.3, two vertices that are $2ma$ steps apart are both in S or both in $\bar{S}$, and exactly one of two vertices that are $2nb$ steps apart is in S. Since $2ma = 2nb$, this is a contradiction. We conclude that $i = j$.

($\Leftarrow$) Let $m = 2^i m'$ and $n = 2^i n'$ for odd m' and n'. Since C_t can be m-step and n-step dominated, Proposition 2.1 implies that $t > 2n$, $t > 2m$, and $2^{i+2} | t$. Label the consecutive vertices of C_t with $v_1, v_2, \ldots, v_t$. Let $S = \{v_j | j \equiv 1, 2, 3, \ldots, 2^{i+1} \pmod{2^{i+2}}\}$. This set includes one vertex out of every pair that are either $2m$ or $2n$ steps apart. By Proposition 2.2, C_t is both m-step and n-step dominated by S. □

Proposition 2.6. *If C_t is m-step and n-step dominated by $S \subseteq V(C_t)$, and $d = \gcd(m, n)$, then C_t is d-step dominated by S.*

Proof. Proposition 2.5 implies that 2 divides m and n with the same multiplicity, so 2 divides d with that multiplicity. Write $d = ma + nb$ where a and b are integers. Now a and b must have different parity, or 2 will divide the right side of the equation with higher multiplicity than the left side. Without loss of generality, assume a is even and b is odd. By Corollary 2.3, two vertices that are $|2ma|$ steps apart are either both in S or both in $\bar{S}$; also, exactly one of two vertices that are $|2nb|$ steps apart is in S. Thus exactly one of two vertices that are $2ma + 2nb = 2d$ steps apart is in S. But this implies that C_t is d-step dominated by S. □

Proposition 2.7. *A cycle C_t where $t \geq 2n$ is n-step dominated by $S \subseteq V(C_t)$ if and only if C_t is d-step dominated by S, where $d = \gcd(n, t)$.*

Proof. ($\Rightarrow$) Write $2d = 2na + 2tb$ for $a, b \in \mathbb{Z}$. Since n is an odd multiple of d, the integer a must be odd. Corollary 2.3 implies that exactly one of two vertices that are $2na$ steps apart is in S. Clearly two vertices that are $2tb$ steps apart are the same vertex. But then exactly one of two vertices that are $2na + 2tb = 2d$ steps apart is in S. Therefore C_t is d-step dominated by S.

($\Leftarrow$) Note that n is an odd multiple of d, because Proposition 2.1 implies that 2 has a greater multiplicity in t than in n. But then Corollary 2.4 implies that C_t is n-step dominated by S. □

We now have the tools to prove the following.

Theorem 2.8. *Let A be a nonempty set of natural numbers and let $t = 2^i t'$ where $i > 0$ and t' is odd. There exists $S \subseteq V(C_t)$ such that S simultaneously a-step dominates C_t for precisely those numbers $a \in A$ if and only if A is the set $\{t/2\}$, or A is of the form $\{c, 3c, \ldots, (2m-1)c\}$, where $c|t$, $2^{i-1} \nmid c$ and $(2m-1)c$ is the greatest odd multiple of c such that $(2m-1)c < t/2$.*

Proof. ($\Rightarrow$) Consider a dominating set S and let $A \subset \mathbb{N}$ be the set of numbers such that C_t is a-step dominated by S for precisely those numbers in A. If $(t/2) \in A$ then $S = V(C_t)$ and clearly A can have no other elements. If $(t/2) \notin A$, then $a < t/2$ for all $a \in A$ because $(t/2)$ is the diameter of C_t. Let d be the greatest common divisor of all elements of A, and let $c = \gcd(d, t)$. By Proposition 2.6, C_t is d-step dominated by S, and then Proposition 2.7 implies that C_t is c-step dominated by S. But now Corollary 2.4 implies that C_t is cl-step dominated by S for all odd l such that $2cl < t$. By Proposition 2.5 the set A can contain no other numbers and has the form given in the theorem.

($\Leftarrow$) Consider a set $A \subset \mathbb{N}$ of the form given in the theorem. If $A = \{t/2\}$, then $S = V(C_t)$ is the required subset. Now suppose $A = \{c, 3c, \ldots, (2m-1)c\}$ where $c|t$. Label the consecutive vertices of C_t with $v_1, v_2, \ldots, v_t$. Consider the set $S = \{v_j | j \equiv 1, 2, \ldots, 2c \pmod{4c}\}$. Note that $4c|t$ and that S is not n-step dominating for any $n < c$. For any $a \in A$ we have $a = cl$ for some odd l, so by Corollary 2.4, C_t is a-step dominated by S for all $a \in A$. Since c is the greatest common divisor of all the elements of A, and there are no other multiples of c that have the same multiplicity of 2 as c, we find that C_t is a-step dominated by S for precisely those $a \in A$. □

3. The Number of Inequivalent Sets Dominating a Cycle

Given a cycle C_t, we define two subsets S_1 and S_2 of $V(C_t)$ to be *equivalent* if there is an automorphism ϕ of C_t such that $\phi(S_1) = S_2$. We now use Polya's method to compute the number of inequivalent sets that n-step dominate C_t.

Polya's method utilizes Burnside's Lemma, which states that the number of orbits of a group action is equal to the average number of fixed points of the elements. (For more detail, see [3, p. 437]). It allows us to count inequivalent dominating sets by counting dominating sets that are fixed by the elements of the dihedral group.

Burnside's Lemma. *Let G be a finite group acting on a finite set X. Then the action has $\sum_{g \in G} |X^g|/|G|$ orbits, where X^g is the set of elements of X invariant under g.*

First we examine the reflections of the dihedral group. Since we are considering only even cycles, there are two types of reflections: those that fix no vertices, and those that fix two vertices. We need consider only one reflection in each conjugacy class.

Proposition 3.1. *For any $c \in \mathbb{N}$ and any fixed-point free reflection, there are 2^n n-step dominating sets for C_{4nc} that are fixed by that reflection.*

Proof. Label consecutive vertices of C_{4nc} with $\{1, 2, \ldots, 4nc\}$, and consider the reflection r defined by $r(i) = 4nc - i + 1$. We show that there is a bijection between the collection of all subsets of $\{1, 2, \ldots, n\}$ and the collection of all dominating sets for C_{4nc}. Given $S' \subset \{1, 2, \ldots, n\}$, let $S'' = S' \cup r(S')$, where $r(S') = \{r(s') \mid s' \in S'\}$. Note that S'' is a subset of the $2n$ consecutive vertices $T = \{1, 2, \ldots, n\} \cup \{4nc - n + 1, \ldots, 4nc\}$. Proposition 2.2 implies that there is a unique dominating set S such that $S \cap T = S''$. S is the set of all $w \in V(C_{4nc})$ such that $w = v + 2na$ for some $v \in T$ where either $v \in S''$ and a is even, or $v \notin S''$ and a is odd. By construction, S is fixed by the reflection r, and conversely every dominating set fixed by r can be obtained by this construction. Since there are 2^n subsets of $\{1, 2, \ldots, n\}$, there are 2^n n-step dominating sets for C_{4nc} that are fixed by each fixed-point free reflection. □

Proposition 3.2. *For any $c \in \mathbb{N}$ and any reflection fixing two points, there are no n-step dominating sets for C_{4nc} that are fixed by that reflection.*

Proof. Label consecutive vertices of C_{4nc} with $\{0, 1, \ldots, 4nc - 1\}$, and consider the reflection r defined by $r(i) = 4nc - i$, that fixes vertex 0 and vertex $2nc$. Suppose that an n-step dominating set S is fixed by r. Since $r(n) = 4nc - n$, either both vertices are in S or both are in $\bar{S}$. But vertex $4nc - n$ and vertex n are also $2n$ steps apart, so Proposition 2.2 implies that exactly one of them is in S. This contradicts the previous statement. □

Now we examine the rotations of the dihedral group. We will use $2^i || n$ to mean that 2^i is the largest power of 2 which divides n.

Proposition 3.3. *Suppose we have an n-step dominated cycle C_{4nc} where $c \in \mathbb{N}$ and $2^i || n$ for some non-negative integer i. Then there are no n-step dominating sets that are fixed by a rotation of r steps unless $2^{i+2} | r$.*

Proof. Suppose that an n-step dominating set S is fixed by a rotation of r steps, where $2^{i+2} \nmid r$. Write $c = 2^j c'$ where c' is odd, and let $o(r)$ denote the order of r in the additive group $\mathbb{Z}_{4nc}$. Then $r \cdot o(r) = 4nca$ for some $a \in \mathbb{N}$. Note that a is odd because otherwise $r\cdot(o(r)/2) = 4nc\cdot(a/2)$, which implies that $o(r)$ is not the order of r. Since $r \cdot o(r) = 4nca = 2^{i+j+2}n'c'a$, where $n'c'a$ is odd, we must have $2^{j+1}|o(r)$. So we may write $o(r) = 2^{j+1}r'$ where $r' \in \mathbb{N}$. Now $r \cdot 2^{j+1}r' = r \cdot o(r) = 4nca = 2^{j+2}nc'a$, which simplifies to $r \cdot r' = 2nc'a$ where $c'a$ is odd. But two vertices that are a multiple of r steps apart must either both be in S or both be in $\bar{S}$, while exactly one of two vertices that are an odd multiple of $2n$ steps apart must be in S. This is a contradiction. □

Proposition 3.4. *Suppose we have an n-step dominated cycle C_{4nc} where $c \in \mathbb{N}$ and $2^i \| n$ for some non-negative integer i. Then for each $r \in \mathbb{N}$ such that $2^{i+2}|r$, exactly $2^{\gcd(2n,r)}$ n-step dominating sets for C_{4nc} are fixed by the rotation of r steps.*

Proof. Label the consecutive vertices of C_{4nc} with $\{1,2,\ldots,4nc\}$. Let $d = \gcd(2n,r)$. We will show that there is a bijection between the collection of all subsets of $\{1,2,\ldots,d\}$ and the collection of all exact n-step dominating sets for C_{4nc} that are fixed by the rotation of r steps. Then since there are 2^d subsets of $\{1,2,\ldots,d\}$, we find that exactly $2^{\gcd(2n,r)}$ n-step dominating sets for C_{4nc} are fixed by the rotation of r steps.

Let S' be a subset of $\{1,2,\ldots,d\}$. We can write each $w \in V(C_{4nc})$ in the form $w = v + (2na + rb)$ for $v \in \{1,2,\ldots,d\}$ and $a, b \in \mathbb{Z}$. Then we define S to be the set of all such w where

- $v \in S'$ and a is even, or
- $v \notin S'$ and a is odd

It suffices to show that every w can be written in the form $v+(2na+rb)$, and that furthermore, if $w = v + (2na + rb)$ and $w = v' + (2na' + rb')$, then $v = v'$, and a and a' have the same parity. These conditions are sufficient because, combined with the definition of S, they show that S is n-step dominating and is fixed by the rotation of r steps. Conversely, every n-step dominating set that is fixed by the rotation of r steps must be of this form.

Note that each $w \in V(C_{4nc})$ can be written uniquely in the form $w = v + de$ where $v \in \{1,2,\ldots,d\}$, and $e \in \mathbb{Z}$. But the vertices $v + (2na + rb)$ for $a, b \in \mathbb{Z}$ are precisely those vertices of the form $v + de$, because $d = \gcd(2n,r)$. Thus every w can be written in the form $v + (2na + rb)$ where $a, b \in \mathbb{Z}$ and v is unique.

Now suppose that $w = v + (2na + rb)$ and $w = v + (2na' + rb')$, where a and a' have different parity. Then $0 = 2n(a - a') + r(b - b')$, which implies that $2ng = rh$ for odd g and $h \in \mathbb{Z}$. Write $r = 2^{i+2}r'$ where $r' \in \mathbb{N}$, and $n = 2^i n'$ where n' is odd. Then $2^{i+1}n'g = 2^{i+2}r'h$ where $n'g$ is odd.

Now $2^{i+2} | 2^{i+2}r'g$ but $2^{i+2} \nmid 2^{i+1}n'g$. This is a contradiction. Therefore the parity of a is a well-defined function of w, and we have a bijection between the collection of all subsets of $\{1, 2, \ldots, d\}$ and the collection of all dominating sets for C_{4nc} that are fixed by the rotation of r steps. □

Now we prove the following theorem.

Theorem 3.5. *Let C_t be n-step dominated, and let $d = \gcd(n, t)$, where $d = 2^i d'$ for some odd d'. Then the number of inequivalent n-step dominating sets for C_t is 1 if $t = 2n$, and otherwise is*

$$2^{d-2} + \frac{1}{8d} \sum_{l=1}^{d'} 2^{\gcd(2d, 2^{i+2}l)}.$$

Proof. Clearly if $t = 2n$ the unique set that n-step dominates C_t is $S = V(C_t)$.

Now consider the case $t = 4nc$ where $c \in \mathbb{N}$ and $n = 2^i n'$ for odd n. Proposition 2.1 implies that the cycles C_{4nc} comprise *all* n-step dominated cycles with length a multiple of n. Burnside's Lemma tells us that the average number of elements that are fixed by the action of the dihedral group of $8nc$ elements is the number of orbits of the group action, that is, the number of inequivalent n-step dominating sets for C_{4nc}. Using Propositions 3.1, 3.2, and 3.3, we find that the only elements of the dihedral group that fix n-step dominating sets for C_{4nc} are the fixed-point free reflections and the rotations of r steps such that $2^{i+2}|r$. Each of the $2nc$ fixed-point free reflections fixes 2^n dominating sets, and by Proposition 3.4, a rotation of $2^{i+2}l$ steps fixes $2^{\gcd(2n, 2^{i+2}l)}$ dominating sets. Thus the number of inequivalent n-step dominating sets for C_{4nc}, where $c \in \mathbb{N}$ and $2^i || n$, is

$$\frac{1}{8nc}\left(2^{n+1}nc + c\sum_{l=1}^{n'} 2^{\gcd(2n, 2^{i+2}l)}\right) = 2^{n-2} + \frac{1}{8n}\sum_{l=1}^{n'} 2^{\gcd(2n, 2^{i+2}l)}.$$

Now consider any n-step dominated cycle C_t. Proposition 2.7 tells us that if $d = \gcd(n, t)$, the cycle C_t is n-step dominated by $S \subsetneq V(C_t)$ if and only if C_t is d-step dominated by S. Thus, the number of n-step dominating sets for C_t is equal to the number of d-step dominating sets for C_t. But t is a multiple of d, so $t = 4dc$ for some $c \in \mathbb{N}$, and we have already calculated the number of inequivalent d-step dominating sets for C_{4dc}. □

Corollary 3.6. *An n-step dominating set for a cycle C_t is unique if and only if $t = 2n$ or $\gcd(t, n) = 1$.*

If we consider n-step domination where n is a power of 2 or n is prime, we get cleaner formulas for the number of inequivalent n-step dominating sets.

Corollary 3.7. *The number of* 2^{m-2}*-step dominating sets for a cycle* $C_{2^m c}$*, where* $c \in \mathbb{N}$*, is*

$$2^{2^{m-1}-m-1} + 2^{2^{m-2}-2}.$$

Corollary 3.8. *The number of inequivalent p-step dominating sets for a cycle* C_{4pc}*, where p is prime and* $c \in \mathbb{N}$*, is*

$$\frac{2^{2p-2} + p \cdot 2^{p-1} + p - 1}{2p}.$$

4. k-Exact n-Step Domination

We generalize the definition of exact n-step domination as follows. A graph G with vertex set $V(G)$ is a *k-exact n-step domination graph* if there is some subset $S \subseteq V(G)$ such that each vertex in G is distance n from exactly k vertices in S. Notice that exact n-step domination is equivalent to 1-exact n-step domination. Our first two results generalize some of Hersh's [6] results on exact n-step domination.

Proposition 4.1. *All k-exact n-step dominating sets of a graph have equal order.*

Proof. Suppose a graph G has two dominating sets S_1 and S_2. Let $X = \{ (x_1, x_2) \mid x_1 \in S_1, x_2 \in S_2, d(x_1, x_2) = n \}$. Every vertex in S_1 is n-step dominated by exactly k vertices in S_2, so every vertex in S_1 is distance n from exactly k vertices in S_2. Thus, $|X| = k|S_1|$. Similarly, $|X| = k|S_2|$, so $|S_1| = |S_2|$. □

Proposition 4.2. *If G has diameter m and is k-exact m-step dominated by* S_1*, and H has diameter n and is l-exact n-step dominated by* S_2*, then the cartesian product* $G \times H$ *has diameter* $m+n$ *and is kl-exact* $(m+n)$*-step dominated by* $S_1 \times S_2$*.*

Proof. Consider a vertex $(v, w) \in V(G) \times V(H)$. Note that because of the diameter conditions, the distance between any two vertices in $G \times H$ is at most $m + n$. Thus, we have $d((v, w), (v', w')) = m + n$ if and only if both v is m-step dominated by v' in G and w is n-step dominated by w' in H. For every $(v, w) \in V(G) \times V(H)$ there are exactly kl such pairs $(v', w') \in S_1 \times S_2$, so $G \times H$ has diameter $m + n$, and $S_1 \times S_2$ is a kl-exact $(m + n)$-step dominating set for $G \times H$. □

In order to prove the next result, we must define *multiplication* of vertices. Suppose we have a graph G with vertex set $V(G)$ and edge set $E(G)$, such that $v \in V(G)$. We use uv to denote an edge between vertices u and v. We can multiply the vertex v by a positive integer $m > 1$ as follows. Let $M(v) = \{v_2, v_3, \ldots, v_m\}$ be a set of vertices, and construct a graph G'

such that $V(G') = V(G) \cup M(v)$. We refer to vertex v as both v and v_1. Let $E(G') = E(G) \cup \{v_i w \mid v_i \in M(v), w \in V(G), vw \in E(G)\} \cup \{v_i v \mid v_i \in M(v)\}$. Then we say that we have *multiplied* vertex v by m, and the elements of $M(v)$ are the *multiples* of v. When $m = 2$, we say that we have *doubled* v.

Proposition 4.3. *There exists a k-exact n-step domination graph of diameter n for every pair of positive integers k and n.*

Proof. For $n > 2$, we construct a k-exact n-step domination graph G' of diameter n with dominating set S' by taking an exact n-step domination graph G of diameter n such that $S = V(G)$, and multiplying each vertex of $V(G)$ by k. Then we let $S' = V(G')$. For example, we can form G' by multiplying each vertex of C_{2n} by k and letting $S' = V(G')$.

Now we show that such a graph G' is k-exact n-step dominated by S'. Consider any two vertices $u, w \in V(G')$. Note that $u = v_i$ for some $v \in V(G)$ and $i \in \{1, 2, \ldots, k\}$, and $w = x_j$ for some $x \in V(G)$ and $j \in \{1, 2, \ldots, k\}$. Because of the way we constructed G', we have $d(u, w) = d(v, x)$. This shows that the diameter of G' is equal to the diameter of G. In particular, it implies that if v is n-step dominated by w in G, then u is n-step dominated by precisely those vertices $w_1, w_2, \ldots, w_k$ in G'. Therefore G' is a k-exact n-step domination graph of diameter n. □

Another example of a k-exact n-step domination graph of diameter n can be constructed as follows. Let $T = \{1 + (2n-2)m \mid m = 0, \ldots, \lceil \frac{k}{2} \rceil\}$. The circulant graph $C = C_{2kn-2k+2}(T)$ is the graph on $2kn - 2k + 2$ nodes $v_1, \ldots, v_{2kn-2k+2}$ with vertex v_i adjacent to each vertex $v_{i \pm t_j \pmod{2kn-2k+2}}$ for all $t_j \in T$.

Proposition 4.4. *There is no upper bound on the order of the vertex set of a k-exact n-step domination graph.*

Proof. Consider any k-exact n-step domination graph G with dominating set S. We can construct a new graph G' by multiplying a vertex $v \in V(G)$ by any $m \in \mathbb{N}$. Now G' is k-exact n-step dominated by S, and $|V(G')| = |V(G)| + m - 1$. □

In [6] Hersh raised the question of whether all exact n-step domination graphs G of diameter n satisfy $S = V(G)$. We find that the answer is no. Given any exact n-step domination graph G of diameter n that is dominated by $S \subseteq V(G)$, construct G' by doubling a vertex $v \in V(G)$. Now G' is n-step dominated by S but $S \neq V(G')$. For example, in Figure [2] we have doubled one vertex of C_6. The vertices of S are denoted by squares. Note that this is an exact 3-step domination graph of diameter 3, but $S \neq V(G)$.

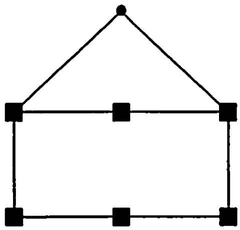

FIGURE 2. An Exact 3-Step Domination Graph of Diameter 3 with $S \neq V(G)$

There are several open questions related to exact n-step domination. These include finding the number of inequivalent dominating sets for graphs other than cycles, and finding a lower bound on the order of a k-exact n-step dominating set.

ACKNOWLEDGMENTS

This work was done under the supervision of Joseph Gallian, at the University of Minnesota, Duluth, with financial support from the National Science Foundation (DMS-9531373-001) and the National Security Agency (MDA 904-96-1-0044). The author wishes to thank Daniel Biss, Samit Dasgupta, Joseph Gallian, Tricia Hersh, Daniel Isaksen, David Moulton, and David Witte for their suggestions and encouragement.

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