ON THE LUCAS MATRIX OF ORDER 2^k SEQUENCE $\{L_n^{(2^k)}\}$

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1. Introduction

In this paper, we construct the Lucas matrix of order 2^k sequence $\{L_n^{(2^k)}\}$ by use of the Lucas matrixes, and make a study of its properties. Owing to its recurrence relation similar to the recurrence relation of the Lucas sequence, its properties similar to some properties of the Lucas sequence too.

It is well known that the Lucas sequence $\{L_n\}$ is defined for all $n \geqslant 0$ by the recurrence relation

$$L_{n+1} = L_n + L_{n-1} \qquad (\text{where } L_0 = 2,\ L_1 = 1) \qquad (1)$$

Rule (1) can be used to extend the sequence backwards, thus

$$L_{-1} = L_1 - L_0, \quad L_{-2} = L_0 - L_{-1}, \ \ldots\ldots$$

and so that

$$L_{-(n+1)} = L_{-(n-1)} - L_{-n} \qquad (2)$$

This produces (see[1])

n	0	1	2	3	4	5	......
L_{-n}	2	-1	3	-4	7	-11	......

and generally

$$L_{-n} = (-1)^n L_n \qquad (3)$$

2. To Construct the Lucas matrix of Order 2^k Sequence $\{L_n^{(2^k)}\}$

Now we construct the Lucas matrix of order 2^k sequence $\{L_n^{(2^k)}\}$ by use of the Lucas matrixes. We let

$$L_n^{(2)} = \begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix} \qquad (\text{where } n \geqslant 0) \qquad (4)$$

Then

$$L_n^{(2)} + L_{n-1}^{(2)} = \begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix} + \begin{pmatrix} L_n & L_{n-1} \\ L_{n-1} & L_{n-2} \end{pmatrix} = \begin{pmatrix} L_{n+2} & L_{n+1} \\ L_{n+1} & L_n \end{pmatrix} = L_{n+1}^{(2)}$$

Hence, we obtain the Lucas matrix of order 2 sequence$\{L_n^{(2)}\}$. It is defined for all $n \geqslant 1$ by the recurrence relation as follows:

$$L_{n+1}^{(2)} = L_n^{(2)} + L_{n-1}^{(2)} \qquad (n \geqslant 1) \qquad (5)$$

where $L_0^{(2)} = \begin{pmatrix} L_1 & L_0 \\ L_0 & L_1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$, $L_1^{(2)} = \begin{pmatrix} L_2 & L_1 \\ L_1 & L_0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$.

Rule (5) can be used to extend the sequence backwards , thus

$$L_1^{(2)} = L_1^{(2)} - L_0^{(2)} \ , \quad L_{-2}^{(2)} = L_0^{(2)} - L_{-1}^{(2)} \ , \ \ldots\ldots$$

and so that

$$L_{-(n+1)}^{(2)} = L_{-(n-1)}^{(2)} - L_{-n}^{(2)} \qquad (n \geqslant 0) \qquad (6)$$

This produces

$$L_0^{(2)} = \begin{pmatrix} L_1 & L_0 \\ L_0 & L_{-1} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} , \quad L_{-1}^{(2)} = \begin{pmatrix} L_0 & L_{-1} \\ L_{-1} & L_{-2} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} ,$$

$$L_{-2}^{(2)} = \begin{pmatrix} L_{-1} & L_{-2} \\ L_{-2} & L_{-3} \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 3 & -4 \end{pmatrix} , \quad L_{-3}^{(2)} = \begin{pmatrix} L_{-2} & L_{-3} \\ L_{-3} & L_{-4} \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ -4 & 7 \end{pmatrix} , \ldots\ldots$$

and generally

$$L_{-n}^{(2)} = \begin{pmatrix} L_{-(n-1)} & L_{-n} \\ L_{-n} & L_{-(n+1)} \end{pmatrix} \qquad (n \geqslant 0) \qquad (7)$$

Again, let the Lucas matrix of order 4 $L_n^{(4)}$ be equal to a partitioned matrix:

$$L_n^{(4)} = \begin{pmatrix} L_{n+1}^{(2)} & L_n^{(2)} \\ L_n^{(2)} & L_{n-1}^{(2)} \end{pmatrix} \qquad (n \geqslant 0) \qquad (8)$$

Then

$$L_n^{(4)} + L_{n-1}^{(4)} = \begin{pmatrix} L_{n+1}^{(2)} & L_n^{(2)} \\ L_n^{(2)} & L_{n-1}^{(2)} \end{pmatrix} + \begin{pmatrix} L_n^{(2)} & L_{n-1}^{(2)} \\ L_{n-1}^{(2)} & L_{n-2}^{(2)} \end{pmatrix} = \begin{pmatrix} L_{n+2}^{(2)} & L_{n+1}^{(2)} \\ L_{n+1}^{(2)} & L_n^{(2)} \end{pmatrix} = L_{n+1}^{(4)}$$

Hence, we obtain the Lucas matrix of order 4 sequence $\{L_n^{(4)}\}$. It is defined for all $n \geqslant 1$ by the recurrence relation as follows:

$$L_{n+1}^{(4)} = L_n^{(4)} + L_{n-1}^{(4)} \qquad (n \geqslant 1) \qquad (9)$$

where $L_0^{(4)} = \begin{pmatrix} L_1^{(2)} & L_0^{(2)} \\ L_0^{(2)} & L_{-1}^{(2)} \end{pmatrix}$, $L_1^{(4)} = \begin{pmatrix} L_2^{(2)} & L_1^{(2)} \\ L_1^{(2)} & L_0^{(2)} \end{pmatrix}$.

Rule (9) can be used to extend the sequence backwards, thus

$$L_{-1}^{(4)} = L_1^{(4)} - L_0^{(4)} , \quad L_{-2}^{(4)} = L_0^{(4)} - L_{-1}^{(4)} , \ldots\ldots$$

and so that

$$L_{-(n+1)}^{(4)} = L_{-(n-1)}^{(4)} - L_{-n}^{(4)} \qquad (n \geqslant 0) \qquad (10)$$

This produces

$$L_0^{(4)} = \begin{pmatrix} L_1^{(2)} & L_0^{(2)} \\ L_0^{(2)} & L_{-1}^{(2)} \end{pmatrix} , \quad L_{-1}^{(4)} = \begin{pmatrix} L_0^{(2)} & L_{-1}^{(2)} \\ L_{-1}^{(2)} & L_{-2}^{(2)} \end{pmatrix} , \quad L_{-2}^{(4)} = \begin{pmatrix} L_{-1}^{(2)} & L_{-2}^{(2)} \\ L_{-2}^{(2)} & L_{-3}^{(2)} \end{pmatrix} , \ldots\ldots$$

and generally

$$L_{-n}^{(4)} = \begin{pmatrix} L_{-(n-1)}^{(2)} & L_{-n}^{(2)} \\ L_{-n}^{(2)} & L_{-(n+1)}^{(2)} \end{pmatrix} \qquad (n \geqslant 0) \qquad (11)$$

Thus, and so on and so forth, let the Lucas matrix of order 2^k $L_n^{(2^k)}$ be equal to a partitioned matrix:

$$L_n^{(2^k)} = \begin{pmatrix} L_{n+1}^{(2^{k-1})} & L_n^{(2^{k-1})} \\ L_n^{(2^{k-1})} & L_{n-1}^{(2^{k-1})} \end{pmatrix} \qquad (n \geqslant 0, k \geqslant 1) \qquad (12)$$

Then

$$L_n^{(2^k)} + L_{n-1}^{(2^k)} = \begin{pmatrix} L_{n+1}^{(2^{k-1})} & L_n^{(2^{k-1})} \\ L_n^{(2^{k-1})} & L_{n-1}^{(2^{k-1})} \end{pmatrix} + \begin{pmatrix} L_n^{(2^{k-1})} & L_{n-1}^{(2^{k-1})} \\ L_{n-1}^{(2^{k-1})} & L_{n-2}^{(2^{k-1})} \end{pmatrix} = \begin{pmatrix} L_{n+2}^{(2^{k-1})} & L_{n+1}^{(2^{k-1})} \\ L_{n+1}^{(2^{k-1})} & L_n^{(2^{k-1})} \end{pmatrix} = L_{n+1}^{(2^k)}$$

Hence we obtain the Lucas matrix of order 2^k sequence $\{ L_n^{(2^k)} \}$. It is defined for all $n \geqslant 1$ by the recurrence relation as follows :

$$L_{n+1}^{(2^k)} = L_n^{(2^k)} + L_{n-1}^{(2^k)} \qquad (n \geqslant 1, k \geqslant 1) \qquad (13)$$

where

$$L_0^{(2^k)} = \begin{pmatrix} L_1^{(2^{k-1})} & L_0^{(2^{k-1})} \\ L_0^{(2^{k-1})} & L_{-1}^{(2^{k-1})} \end{pmatrix}, \quad L_1^{(2^k)} = \begin{pmatrix} L_2^{(2^{k-1})} & L_1^{(2^{k-1})} \\ L_1^{(2^{k-1})} & L_0^{(2^{k-1})} \end{pmatrix}$$

Rule (13) can be used to extend the sequence backwards, thus

$$L_{-1}^{(2^k)} = L_1^{(2^k)} - L_0^{(2^k)}, \quad L_{-2}^{(2^k)} = L_0^{(2^k)} - L_{-1}^{(2^k)}, \quad \ldots\ldots$$

and so that

$$L_{-(n+1)}^{(2^k)} = L_{-(n-1)}^{(2^k)} - L_{-n}^{(2^k)} \qquad (n \geqslant 0, k \geqslant 1) \qquad (14)$$

This produces

$$L_0^{(2^k)} = \begin{pmatrix} L_1^{(2^{k-1})} & L_0^{(2^{k-1})} \\ L_0^{(2^{k-1})} & L_{-1}^{(2^{k-1})} \end{pmatrix}, \; L_{-1}^{(2^k)} = \begin{pmatrix} L_0^{(2^{k-1})} & L_{-1}^{(2^{k-1})} \\ L_{-1}^{(2^{k-1})} & L_{-2}^{(2^{k-1})} \end{pmatrix}, \; L_{-2}^{(2^k)} = \begin{pmatrix} L_{-1}^{(2^{k-1})} & L_{-2}^{(2^{k-1})} \\ L_{-2}^{(2^{k-1})} & L_{-3}^{(2^{k-1})} \end{pmatrix}$$

...... and generally

$$L_{-n}^{(2^k)} = \begin{pmatrix} L_{-(n-1)}^{(2^{k-1})} & L_{-n}^{(2^{k-1})} \\ L_{-n}^{(2^{k-1})} & L_{-(n+1)}^{(2^{k-1})} \end{pmatrix} \qquad (n \geqslant 1, k \geqslant 1) \qquad (15)$$

Now we obtain a basic property of the Lucas matrix of order 2^k sequence $\{ L_n^{(2^k)} \}$ by the equation (15)

Theorem1: The Lucas matrix of order 2^k sequence $\{ L_n^{(2^k)} \}$ is satisfied with

$$L_{-n}^{(2^k)} = (-1)^n E_{2^k} L_n^{(2^k)} E_{2^k} \qquad (n \geqslant 1, k \geqslant 1) \qquad (16)$$

where E_{2^k} is equal to a partitioned matrix

$$E_{2^k} = \begin{pmatrix} O_{2^{k-1}} & E_{2^{k-1}} \\ -E_{2^{k-1}} & O_{2^{k-1}} \end{pmatrix}, \quad \text{when } k=1 \; E_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad O_{2^{k-1}} \text{ is zero}$$

matrix of order 2^{k-1}.

Proof : This is easily proved by induction . When k=1, we have

$$L_{-n}^{(2)} = \begin{pmatrix} L_{-(n-1)} & L_{-n} \\ L_{-n} & L_{-(n+1)} \end{pmatrix} = \begin{pmatrix} (-1)^{n-1}L_{n-1} & (-1)^{n}L_{n} \\ (-1)^{n}L_{n} & (-1)^{n+1}L_{n+1} \end{pmatrix} = (-1)^{n} \begin{pmatrix} -L_{n-1} & L_{n} \\ L_{n} & -L_{n-1} \end{pmatrix}$$

$$= (-1)^{n} \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} L_{n+1} & L_{n} \\ L_{n} & L_{n-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] = (-1)^{n} E_2 L_{n}^{(2)} E_2$$

Then , when k=1, the formula (16) is true . When k=2 , we have

$$L_{-n}^{(4)} = \begin{pmatrix} L_{-(n-1)}^{(2)} & L_{-n}^{(2)} \\ L_{-n}^{(2)} & L_{-(n+1)}^{(2)} \end{pmatrix} = (-1)^{n} \begin{pmatrix} -E_2 L_{n-1}^{(2)} E_2 & E_2 L_{n}^{(2)} E_2 \\ E_2 L_{n}^{(2)} E_2 & -E_2 L_{n+1}^{(2)} E_2 \end{pmatrix}$$

$$= (-1)^{n} \left[\begin{pmatrix} 0_2 & E_2 \\ -E_2 & 0_2 \end{pmatrix} \begin{pmatrix} L_{n+1}^{(2)} & L_{n}^{(2)} \\ L_{n}^{(2)} & L_{n-1}^{(2)} \end{pmatrix} \begin{pmatrix} 0_2 & E_2 \\ -E_2 & 0_2 \end{pmatrix} \right]$$

$$= (-1)^{n} E_4 L_{n}^{(4)} E_4 , \qquad \text{where } E_4 = \begin{pmatrix} 0_2 & E_2 \\ -E_2 & 0_2 \end{pmatrix} .$$

Then, when k=2, the formula (16) is true. Assume the formula (16) to be true for k=m-1. In similar manner, we can prove that the formula (16) is true for k=m .

To sum up, the formula (16) is proved .

3. The Sum Formula of { $L_n^{(2^k)}$ }

Theorem 2 : The sum formula of { $L_n^{(2^k)}$ } is as follows :

$$\sum_{i=1}^{n} L_i^{(2^k)} = L_{n+2}^{(2^k)} - L_2^{(2^k)} \qquad (n \geqslant 1, k \geqslant 1) \qquad (17)$$

Proof :

$$\sum_{i=1}^{n} L_i^{(2^k)} = \sum_{i=3}^{n+2} L_i^{(2^k)} - \sum_{i=2}^{n+1} L_i^{(2^k)}$$

$$= \left(\sum_{i=1}^{n+2} L_i^{(2^k)} - L_1^{(2^k)} - L_2^{(2^k)} \right) - \left(\sum_{i=1}^{n+2} L_i^{(2^k)} - L_1^{(2^k)} - L_{n+2}^{(2^k)} \right)$$

$$= L_{n+2}^{(2^k)} - L_2^{(2^k)}$$

4. Other properties of $\{ L_n^{(2^k)} \}$

Theorem 3 :

$$L_n^{(2^{k+1})} = L_1^{(2^{k+1})} \begin{pmatrix} I_{2^k} & I_{2^k} \\ I_{2^k} & 0_{2^k} \end{pmatrix}^{n-1} \qquad (n \geqslant 1, k \geqslant 0) \qquad (18)$$

where I_{2^k} is unit matrix of order 2^k, O_{2^k} is zero matrix of order 2^k.

Proof :

$$L_n^{(2^{k+1})} = \begin{pmatrix} L_{n+1}^{(2^k)} & L_n^{(2^k)} \\ L_n^{(2^k)} & L_{n-1}^{(2^k)} \end{pmatrix} = \begin{pmatrix} L_n^{(2^k)} & L_{n-1}^{(2^k)} \\ L_{n-1}^{(2^k)} & L_{n-2}^{(2^k)} \end{pmatrix} \begin{pmatrix} I_{2^k} & I_{2^k} \\ I_{2^k} & 0_{2^k} \end{pmatrix} = \ldots\ldots$$

$$= \begin{pmatrix} L_3^{(2^k)} & L_2^{(2^k)} \\ L_2^{(2^k)} & L_1^{(2^k)} \end{pmatrix} \begin{pmatrix} I_{2^k} & I_{2^k} \\ I_{2^k} & 0_{2^k} \end{pmatrix}^{n-2}$$

$$= L_1^{(2^{k+1})} \begin{pmatrix} I_{2^k} & I_{2^k} \\ I_{2^k} & 0_{2^k} \end{pmatrix}^{n-1}$$

Theorem 4 :

$$\sum_{i=1}^{n} L_{2i-1}^{(2^k)} = L_{2n}^{(2^k)} - L_0^{(2^k)} \qquad (n \geqslant 1, k \geqslant 1) \qquad (19)$$

Proof :

$$\sum_{i=1}^{n} L_{2i-1}^{(2^k)} = \sum_{i=1}^{n} L_{2i}^{(2^k)} - \sum_{i=1}^{n} L_{2i-2}^{(2^k)}$$

$$= \sum_{i=1}^{n} L_{2i}^{(2^k)} - \sum_{i=0}^{n-1} L_{2i}^{(2^k)}$$

$$= L_{2n}^{(2^k)} - L_0^{(2^k)}$$

Theorem 5 :

$$\sum_{i=1}^{n} L_{2i}^{(2^k)} = L_{2n+1}^{(2^k)} - L_1^{(2^k)} \qquad (n \geqslant 1, k \geqslant 1) \qquad (20)$$

Proof :

$$\sum_{i=1}^{n} L_{2i}^{(2^k)} = \sum_{i=1}^{n} L_{2i+1}^{(2^k)} - \sum_{i=1}^{n} L_{2i-1}^{(2^k)}$$

$$= \sum_{i=1}^{n} L_{2i+1}^{(2^k)} - \sum_{i=0}^{n-1} L_{2i+1}^{(2^k)}$$

$$= L_{2n+1}^{(2^k)} - L_{1}^{(2^k)}$$

Theorem 6 :

$$\sum_{i=1}^{2n} (-1)^{i} L_{i}^{(2^k)} = L_{2n-1}^{(2^k)} + L_{0}^{(2^k)} - L_{1}^{(2^k)} \quad (n \geqslant 1, k \geqslant 1) \qquad (21)$$

Proof : Subtraction of (19) from (20) produces (21)

Theorem 7 :

$$\sum_{i=1}^{n} L_{i-1}^{(2^k)}/2^{i} = \frac{1}{2}(L_{0}^{(2^k)} + L_{3}^{(2^k)}) - L_{n+2}^{(2^k)}/2^{n} \quad (n \geqslant 1, k \geqslant 1) \qquad (22)$$

Proof : This is easily proved by induction. Obviously, the formula (22) is true for n=1. Assume it to be true for 2, 3, ..., n-1. Add $L_{n-1}^{(2^k)}/2^{n}$ on both sides. The right-hand side becomes

$$\frac{1}{2}(L_{0}^{(2^k)} + L_{3}^{(2^k)}) - (2L_{n+1}^{(2^k)} - L_{n-1}^{(2^k)})/2^{n}$$

$$= \frac{1}{2}(L_{0}^{(2^k)} + L_{3}^{(2^k)}) - (L_{n+1}^{(2^k)} + L_{n}^{(2^k)})/2^{n}$$

$$= \frac{1}{2}(L_{0}^{(2^k)} + L_{3}^{(2^k)}) - L_{n+2}^{(2^k)}/2^{n} \qquad \text{QED.}$$

Theorem 8 :

$$L_{n+p}^{(2^k)} = \sum_{i=0}^{p} \binom{p}{i} L_{n-i}^{(2^k)} \qquad (n \geqslant 1, k \geqslant 1) \qquad (23)$$

Proof : Observe that the formula (23) holds for all integer n when p=1. We shall prove, by induction, that it holds for every positive integer p.

Let the formula (23) be true up to some value of p. Then

$$L_{m+p}^{(2^k)} = \sum_{i=0}^{p}\binom{p}{i}L_{m-i}^{(2^k)} = \sum_{i=0}^{p}\binom{p}{i}(L_{m-i-1}^{(2^k)} + L_{m-i-2}^{(2^k)})$$

$$= L_{m-1}^{(2^k)} + \sum_{i=1}^{p}\binom{p}{i}L_{m-i-1}^{(2^k)} + \sum_{i=0}^{p-1}\binom{p}{i}L_{m-i-2}^{(2^k)} + L_{m-p-2}^{(2^k)}$$

$$= L_{m-1}^{(2^k)} + \sum_{i=1}^{p}\binom{p}{i}L_{m-i-1}^{(2^k)} + \sum_{i=1}^{p}\binom{p}{i-1}L_{m-i-1}^{(2^k)} + L_{m-p-2}^{(2^k)}$$

$$= L_{m-1}^{(2^k)} + \sum_{i=1}^{p}\binom{p+1}{i}L_{m-i-1}^{(2^k)} + L_{m-p-2}^{(2^k)}$$

$$= \sum_{i=0}^{p+1}\binom{p+1}{i}L_{m-i-1}^{(2^k)}$$

Let m=n+1, then

$$L_{n+p+1}^{(2^k)} = \sum_{i=0}^{p+1}\binom{p+1}{i}L_{n-i}^{(2^k)}$$

This is again the formula (23), but p is replaced by p+1. Thus the formula (23) holds for all p.

Theorem 9 :

$$L_{2n}^{(2^k)} = \sum_{i=0}^{n}\binom{n}{i}L_{i}^{(2^k)} \qquad (n \geqslant 1, k \geqslant 1) \qquad (24)$$

proof : Because $\binom{p}{i} = \binom{p}{p-i}$, the formula (23) can also be written

$$L_{n+p}^{(2^k)} = \sum_{i=0}^{p}\binom{p}{i}L_{n-p+i}^{(2^k)} \qquad (n \geqslant 1, k \geqslant 1) \qquad (25)$$

When n=p, then

$$L_{2n}^{(2^k)} = \sum_{i=0}^{n}\binom{n}{i}L_{i}^{(2^k)}$$

Corollary 1: When n=m+(t-1)p , the formula (25) can be written

$$L_{m+tp}^{(2^k)} = \sum_{i=0}^{p}\binom{p}{i}L_{m+(t-2)p+i}^{(2^k)} \qquad (k \geqslant 1) \qquad (26)$$

Corollary 2: When t=2, the formula (26) can be written

$$L_{m+2p}^{(2^k)} = \sum_{i=0}^{p}\binom{p}{i}L_{m+i}^{(2^k)} \qquad (k \geqslant 1) \qquad (27)$$

Theorem 10 : For m=1, p=n, the formula (27) can be transformed into

$$L_{2n+1}^{(2^k)} = L_{-1}^{(2^k)} + \sum_{i=0}^{n}\binom{n+1}{i+1}L_{i}^{(2^k)} \quad (n \geqslant 1, k \geqslant 1) \qquad (28)$$

Proof :

$$L_{2n+1}^{(2^k)} = \sum_{i=0}^{n}\binom{n}{i}L_{i+1}^{(2^k)}$$

$$= L_1^{(2^k)} + \sum_{i=1}^{n}\binom{n}{i}(L_i^{(2^k)} + L_{i-1}^{(2^k)})$$

$$= L_1^{(2^k)} + \sum_{i=1}^{n}\binom{n}{i}L_i^{(2^k)} + \sum_{i=0}^{n-1}\binom{n}{i+1}L_i^{(2^k)}$$

$$= L_1^{(2^k)} + L_n^{(2^k)} + \sum_{i=1}^{n-1}\binom{n}{i}L_i^{(2^k)} + \binom{n}{1}L_0^{(2^k)} + \sum_{i=1}^{n-1}\binom{n}{i+1}L_i^{(2^k)}$$

$$= L_1^{(2^k)} + L_n^{(2^k)} + \sum_{i=0}^{n-1}\binom{n+1}{i+1}L_i^{(2^k)} - L_0^{(2^k)}$$

$$= L_{-1}^{(2^k)} + \sum_{i=0}^{n}\binom{n+1}{i+1}L_i^{(2^k)}$$

Theorem 11 :

$$L_{n-p}^{(2^k)} = \sum_{i=0}^{p}(-1)^i\binom{p}{i}L_{n+p-i}^{(2^k)} \qquad (n \geqslant 1, k \geqslant 1) \qquad (29)$$

Proof : Obviously, the formula (29) holds for p=1. Write

$$L_{m-p}^{(2^k)} = \sum_{i=0}^{p}(-1)^i\binom{p}{i}(L_{m+p+2-i}^{(2^k)} - L_{m+p+1-i}^{(2^k)})$$

$$= L_{m+p+2}^{(2^k)} + \sum_{i=1}^{p}[(-1)^i\binom{p}{i}L_{m+p+2-i}^{(2^k)} + (-1)^i\binom{p}{i-1}L_{m+p+2-i}^{(2^k)}] + (-1)^{p+1}L_{m+1}^{(2^k)}$$

$$= \sum_{i=0}^{p+1}(-1)^i\binom{p+1}{i}L_{m+p+2-i}^{(2^k)}$$

Let m=n-1, then

$$L_{n-p-1}^{(2^k)} = \sum_{i=0}^{p+1}(-1)^i\binom{p+1}{i}L_{n+p+1-i}^{(2^k)}$$

which is the same as formula (29), with p+1 replacing p Hence formula (29) is proved.

Corollary 3 : The formula (29) can be written

$$L_{n-p}^{(2^k)} = \sum_{i=0}^{p}(-1)^{p-i}\binom{p}{i}L_{n+i}^{(2^k)} \qquad (k \geqslant 1) \qquad (30)$$

Corollary 4 : In the formula (30), let n=m-(t-1)p, then

$$L_{m-tp}^{(2^k)} = \sum_{i=0}^{p}(-1)^{p-i}\binom{p}{i}L_{m-(t-1)p+i}^{(2^k)} \qquad (k \geqslant 1) \qquad (31)$$

Corollary 5 : In the formula (31), let t=2, then

$$L_{m-2p}^{(2^k)} = \sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} L_{m-p+i}^{(2^k)}$$

$$= \sum_{i=0}^{p} (-1)^{i} \binom{p}{i} L_{m-i}^{(2^k)} \qquad (k \geqslant 1) \qquad (32)$$

References

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