On the connectivity of generalized p-cycles

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Abstract

A generalized p-cycle is a digraph whose set of vertices is partitioned in p parts that are cyclically ordered in such a way that the vertices in one part are adjacent only to vertices in the next part. In this work, we mainly show the two following types of conditions in order to find generalized p-cycles with maximum connectivity:

1. For a new given parameter ℓ, related to the number of short paths in G, the diameter is small enough.
2. Given the diameter and the maximum degree, the number of vertices is large enough.

For the first problem it is shown that if $D \leq 2\ell + p - 2$, then the connectivity is maximum. Similarly, if $D \leq 2\ell + p - 1$, then the edge-connectivity is also maximum. For problem two an appropriate lower bound on the order, in terms of the maximum and minimum degree, the parameter ℓ and the diameter is deduced to guarantee maximum connectivity.

1 Introduction

The study of connectivity properties in graphs and digraphs has some applications to the design of reliable communication or interconnection networks. In particular, it is interesting to have sufficient conditions for a (di)graph to be maximally connected; see, for instance, the survey of Bermond et al.

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[5]. One particular aspect of this problem involves (di)graphs with small diameter; and another one large (di)graphs, that is, those with large number of vertices for a given degree and diameter. Both of them have been widely studied as good models for interconnection networks with small transmission delay.

Next, we define some graph concepts and present the notation we are going to use in this paper. A (finite) simple digraph G consists of a set of vertices $V = V(G)$ and a set of (directed) edges $E = E(G)$ without loops or multiple edges. For any pair of vertices $x, y \in V$, a path from x to y is called an *$x \to y$ path.* A digraph G is said to be *(strongly) connected* when for any pair of vertices $x, y \in V$, an $x \to y$ path always exists. The *distance* from x to y is denoted by $d(x, y)$, and $D = \max_{x,y\in V}\{d(x, y)\}$ stands for the *diameter* of G. The distance from x to $U \subset V$, denoted by $d(x, U)$, is the minimum over all the distances $d(x, u)$, $u \in U$. The distance from U to x, $d(U, x)$, is defined analogously. Given a set of edges $A \subset E$, we define $d(x, A) = \min_{(u,v)\in A} d(x, u)$ and $d(A, x) = \min_{(u,v)\in A} d(v, x)$. Let $\Gamma^-(x)$ and $\Gamma^+(x)$ denote respectively the sets of vertices adjacent to and from x. Their cardinalities are respectively the *in-degree* of x, $\delta^-(x)$, and *out-degree* of x, $\delta^+(x)$. The *minimum degree* δ [*maximum degree* Δ] of G is the minimum [maximum] over all the in-degrees and out-degrees of the vertices of G. We will always assume that our digraphs are connected, hence $\delta \geq 1$.

Given a subset of vertices F, let $\Gamma^+(F) = \bigcup_{x\in F}\Gamma^+(x)$ and $\Gamma^-(F) = \bigcup_{x\in F}\Gamma^-(x)$. The *positive* and *negative boundaries* of F are $\partial^+F = \Gamma^+(F)\setminus F$ and $\partial^-F = \Gamma^-(F)\setminus F$, respectively. The corresponding concepts for edges are the *positive* and *negative edge-boundaries*, $\omega^+F = \{(x, y) \in E : x \in F \text{ and } y \in V \setminus F\}$ and $\omega^-F = \{(x, y) \in E : x \in V \setminus F \text{ and } y \in F\}$. Moreover, note that $\omega^+F = \omega^-(V \setminus F)$.

Clearly, if $F \cup \partial^+F \neq V$ [$F \cup \partial^-F \neq V$] then ∂^+F [∂^-F] is a cutset of G. Similarly, if F is a proper (non-empty) subset of V, then ω^+F [ω^-F] is an edge cutset. Hence, by using these concepts, the *(vertex) connectivity* and *edge-connectivity* of G can be respectively defined as

$$\kappa = \min\{|\partial^+F| : F \subset V, F \cup \partial^+F \neq V \text{ or } |F| = 1\};$$
$$\lambda = \min\{|\omega^+F| : F \subset V, F \neq \emptyset, V\}.$$

It is well-known that, for any digraph G, $\kappa \leq \lambda \leq \delta$, (see [12]). Hence, G is said to be *maximally connected* when $\kappa = \lambda = \delta$, and *maximally edge-connected* if $\lambda = \delta$.

Following Hamidoune [14, 15], a subset F of vertices of a connected digraph G is a *positive fragment* of G if $|\partial^+F| = \kappa$ and $\overline{F} \neq \emptyset$, where $\overline{F} = V \setminus (F \cup \partial^+F)$. Analogously, F is a *negative fragment* if $|\partial^-F| = \kappa$ and $\overline{F} \neq \emptyset$, where now $\overline{F} = V \setminus (\partial^-F \cup F)$. Note that F is a positive fragment if and only if $\overline{F}$ is a negative one, and $\partial^+ F = \partial^- \overline{F}$. The set of vertices F is called a *positive α-fragment* of G if $|\omega^+ F| = \lambda$ and, similarly, F is a *negative α-fragment* if $|\omega^- F| = \lambda$.

In order to study the connectivity of digraphs, a new parameter related to the number of shortest paths was used in [8] (see also [10]):

Definition 1.1 *For a given digraph G with diameter D, let $\ell = \ell(G)$, $1 \le \ell \le D$, be the greatest integer so that, for any $x, y \in V$,*

(a) if $d(x,y) < \ell$, the shortest $x \to y$ path is unique and there are no $x \to y$ paths of length $d(x,y)+1$;

(b) if $d(x,y) = \ell$, there is only one shortest $x \to y$ path.

Note that ℓ is well defined if G has no loops.

In recent years, several results relating the connectivity of a (di)graph with the aforementioned parameters, n, Δ, δ, ℓ and D, have been given. For instance, Imase, Soneka and Okada [17] proved that when the minimum and maximum degrees, δ, Δ and diameter D of a connected digraph are fixed, the maximum connectivity is attained if the order is big enough. See the survey of Bermond, Homobono and Peyrat [5] for more details.

The line digraph technique is a good general method for obtaining large digraphs with fixed degree and diameter. In the line digraph LG of a digraph G, each vertex represents an edge of G. Thus, $V(LG) = \{uv : (u,v) \in E(G)\}$; and a vertex uv is adjacent to a vertex wz if and only if $v = w$, that is, when the edge (u,v) is adjacent to the edge (w,z) in G. For any $k > 1$ the *k-iterated line digraph*, L^kG, is defined recursively by $L^kG = LL^{k-1}G$. From the definition it is evident that the order of LG equals the size of G, $|V(LG)| = |E(G)|$, and that their maximum and minimum degrees coincide, $\Delta(LG) = \Delta(G) = \Delta$, $\delta(LG) = \delta(G) = \delta$. Moreover, if G is d-regular ($\delta^-(x) = \delta^+(x) = d$, for any $x \in V$), $d > 1$, and has order n and diameter D, then L^kG is also d-regular and has $d^k n$ vertices and diameter

$$D(L^kG) = D(G) + k. \tag{1}$$

See, for instance, Fiol, Yebra and Alegre [11] and Reddy, Kuhl, Hosseini and Lee [18]. In fact, (1) still holds for any (strongly) connected digraph other than a directed cycle (see Aigner [2]). In [8], it is shown that for any digraph G with $\delta \ge 2$ the parameter ℓ also satisfies an equality like (1). Namely,

$$\ell(L^kG) = \ell(G) + k. \tag{2}$$

In addition, as the vertices of LG represent the edges of G, it can be easily shown that $\kappa(LG) = \lambda(G)$.

Fiol and Fabrega in [8] showed for any digraph with parameter ℓ and diameter D that

$$\begin{aligned} \kappa = \delta \quad & \text{if } D \le 2\ell - 1; \\ \lambda = \delta \quad & \text{if } D \le 2\ell. \end{aligned} \tag{3}$$

For bipartite digraphs similar results involving this parameter and the diameter are given in [9]:

$$\begin{aligned} \kappa = \delta \quad & \text{if } D \le 2\ell; \\ \lambda = \delta \quad & \text{if } D \le 2\ell + 1. \end{aligned} \tag{4}$$

For both general digraphs and bipartite digraphs, some recent work has shown that, roughly speaking, the larger the order the larger the connectivities. For instance, the following conditions for general digraphs are shown in [7]:

$$\begin{aligned} \kappa = \delta \quad & \text{if } n > (\delta - 1)\{n(\Delta, \ell - 1) + n(\Delta, D - \ell) - 2\} + \Delta^\ell + 1; \\ \lambda = \delta \quad & \text{if } n > (\delta - 1)\{n(\Delta, \ell - 1) + n(\Delta, D - \ell - 1)\} + \Delta^\ell. \end{aligned} \tag{5}$$

where $n(\Delta, \ell) = \sum_{i=0}^{\ell} \Delta^i$.
Concerning to bipartite digraphs the following results were proved in [4]:

$$\begin{aligned} \kappa = \delta \quad & \text{if } n > (\delta - 1)\{n(\Delta, \ell) + n(\Delta, D - \ell - 1) - 2\} + 2,\ \delta \ge 3; \\ \lambda = \delta \quad & \text{if } n > (\delta - 1)\{n(\Delta, \ell) + n(\Delta, D - \ell - 2)\}. \end{aligned} \tag{6}$$

The purpose of this paper is to state sufficient conditions to assure maximum connectivity in a special kind of digraphs: generalized cycles. A *generalized p-cycle* is a digraph G in which its set of vertices can be partitioned in p parts,

$$V = \bigcup_{\alpha \in \mathbb{Z}_p} V_\alpha,$$

in such a way that the vertices in the partite set V_α are only adjacent to vertices in $V_{\alpha+1}$, where the sum is in $\mathbb{Z}_p$. Observe that any digraph can be shown as a p-cycle with $p = 1$. Moreover, bipartite digraphs are generalized p-cycles with $p = 2$.

Gómez, Padró and Perennes showed in [13] that a digraph is a generalized p-cycle if and only if its line digraph is a generalized p-cycle as well. In the same paper it is proved that a digraph G is a generalized p-cycle if and only if, for any pair of vertices x, y, the lengths of all paths from x to y are congruent modulo p. Therefore, when $p \ge 2$ the definition of parameter ℓ can be simplified by saying that it is the greatest integer such that, for any pair of vertices $x, y \in V$ at distance $d(x, y) \le \ell$, the shortest $x \to y$ path is unique.

In this paper we are concerned with the two following types of conditions in order to find generalized p-cycles with maximum connectivity:

1. For a given parameter ℓ the diameter is small enough.

2. Given the diameter, the maximum and minimum degrees and parameter ℓ, the number of vertices is large enough.

We offer some solutions to these problems which generalize the above mentioned conditions (3), (4), (5) and (6), given for the cases $p = 1$ (general digraphs), and $p = 2$ (bipartite digraphs).

For all definitions not given here we refer the reader to the book by Chartrand and Lesniak [6].

2 Generalized cycles with small diameter

In this section we give results that generalize the above mentioned (3) and (4) for all generalized p-cycles. With this aim, we present the following concepts which were introduced in [3]. The *deepness* of a positive fragment F is $\mu(F) = \max_{x \in F} d(x, \partial^+ F)$. Similarly, the deepness of a negative fragment F is $\mu(F) = \max_{x \in F} d(\partial^- F, x)$. With respect to α-fragments, the *deepness* of a positive α-fragment F is $\nu(F) = \max_{x \in F} d(x, \omega^+ F)$. The *deepness* of a negative α-fragment F is defined analogously, $\nu(F) = \max_{x \in F} d(\omega^- F, x)$.

In [8] it was implicitly shown that the parameter ℓ is related to the deepness of any fragment or α-fragment. For the sake of convenience we repeat the proof of this useful fact in the following lemma.

Lemma 2.1 *Let G be a digraph with parameter ℓ, minimum degree δ, diameter D and connectivities κ and λ. Let F denote a positive fragment or α-fragment of G. Then,*

(a) if $\kappa < \delta$, then $\mu(F) \geq \ell$ and $\mu(\overline{F}) \geq \ell$;

(b) if $\lambda < \delta$, then $\nu(F) \geq \ell$ and $\nu(V \setminus F) \geq \ell$.

Proof. Let F be a positive fragment, that is, $|\partial^+ F| = \kappa \leq \delta - 1$, and assume that $\mu(F) \leq \ell - 1$. Let x be a vertex of F such that $d(x, \partial^+ F) = \mu(F)$ and consider δ of its out-neighbors, $x_1, x_2, \ldots, x_\delta$. For each x_i let f_i be a vertex in $\partial^+(F)$ at minimum distance from x_i. Hence, $f_i = f_j$ for some $i \neq j$, and then there would be two different $x \to f_i$ paths of length $\ell - 1$ or ℓ, which is a contradiction with the definition of parameter ℓ. Considering the converse digraph of G, we can also prove $\mu(\overline{F}) \geq \ell$.

(b) The edge case is proved in a similar way if $\nu(F) \geq 1$. Let us see that the assumption $\lambda = |\omega^+ F| < \delta$, implies $\nu(F) > 0$. Indeed, clearly, $|F| > 1$. If $\nu(F) = 0$ then $|F| < \delta$ and the number of edges, β, which have their initial and final vertices in F satisfies $|F|(|F| - 1) \geq \beta = \sum_{x \in F} \delta^+(x) - |\omega^+ F| \geq |F|\delta - \delta$. Then $|F| \geq \delta$, which is a contradiction. ■

Our main result of this section is the following theorem.

Theorem 2.1 *Let G be a connected generalized p-cycle, $p \geq 2$, with parameter ℓ, minimum degree δ, diameter D and connectivities κ and λ. Then,*

(a) $\kappa = \delta$ *if* $D \leq 2\ell + p - 2$;

(b) $\lambda = \delta$ *if* $D \leq 2\ell + p - 1$.

Proof. To prove (a) let us assume that $\kappa < \delta$ and let F be a positive fragment. As any path from F to $\overline{F}$ goes through $\partial^+ F$ we can consider a vertex $x \in F$ and a vertex $y \in \overline{F}$ so that $d(x,y) \geq d(x,\partial^+ F) + d(\partial^+ F, y) \geq \mu(F) + \mu(\overline{F})$, where $\mu(F)$ and $\mu(\overline{F})$ are the deepness of the positive and negative fragment F and $\overline{F}$, respectively. Hence $D \geq \mu(F) + \mu(\overline{F})$; without loss of generality, suppose $\mu(F) \leq \mu(\overline{F})$ (if not use the converse digraph of G.) Therefore, from Lemma 2.1 it follows $\ell \leq \mu(F) \leq \mu(\overline{F})$, and hence, we can consider the non-empty sets:

$$F(\ell) = \{x \in F, d(x, \partial^+ F) \geq \ell\}, \quad \overline{F}(\ell) = \{y \in \overline{F}, d(\partial^+ F, y) \geq \ell\}.$$

As G is a p-cycle, its set of vertices can be partitioned in p parts, $V = \bigcup_{\alpha \in \mathbb{Z}_p} V_\alpha$, in such a way that the vertices in the partite set V_α are only adjacent to vertices in $V_{\alpha+1}$, where the sum is in $\mathbb{Z}_p$. We claim that for each $0 \leq \alpha \leq p-1$, $F(\ell) \cap V_\alpha \neq \emptyset$. Indeed, we can take a vertex $x_0 \in F(\ell) \cap V_\alpha$, for some α, such that $d(x_0, \partial^+ F) = \ell$. Then $\Gamma^+(x_0) \cap F(\ell) \neq \emptyset$, since otherwise, there would be two distinct paths from x_0 to some vertex of $\partial^+ F$ of length ℓ, which contradicts the definition of this parameter. Hence, we can consider a vertex $x_1 \in F(\ell) \cap V_{\alpha+1}$ so that $d(x_1, \partial^+ F) = d(x_1, x_r) + d(x_r, \partial^+ F) = d(x_1, x_r) + \ell$, where obviously $x_r \in V_{\alpha+r}$ for some $1 \leq r$ and so $F(\ell) \cap V_{\alpha+j} \neq \emptyset$, for each $0 \leq j \leq r$. Since $d(x_r, \partial^+ F) = \ell$ we find again that $\Gamma^+(x_r) \cap F(\ell) \neq \emptyset$, and thus $F(\ell) \cap V_{\alpha+r+1} \neq \emptyset$. This shows that in $F(\ell)$ vertices of every partite set must exist. Similarly, it is proved that for each $0 \leq \alpha \leq p-1$, $\overline{F}(\ell) \cap V_\alpha \neq \emptyset$, that is, $\overline{F}(\ell)$ contains vertices of every partite set.

Now, let us consider the integer r, $0 \leq r \leq p-1$, so that $D + 1 \equiv r \pmod{p}$. If $x \in F(\ell) \cap V_\alpha$ then for all vertices $y \in \overline{F}(\ell) \cap V_{\alpha+r}$ we find for some integer $h \geq 1$ that $2\ell \leq d(x,y) = D + 1 - hp \leq D - p + 1$, because the lengths of all paths from V_α to $V_{\alpha+r}$ are congruent with r modulo p. This means that, $D \geq 2\ell + p - 1$, which contradicts the hypothesis.

Case (b) can be shown to be a corollary of (a). Indeed, assume that (b) does not hold. Then, there would be a generalized p-cycle with $\delta > 1$, parameter ℓ, edge-connectivity $\lambda < \delta$, and $D \leq 2\ell + p - 1$. Thus, according to the results (1) and (2), its line digraph LG would be a generalized p-cycle (see [13]), and would have the same minimum degree, parameter $\ell' = \ell + 1$, vertex-connectivity $\kappa' = \lambda < \delta$ and diameter $D' = D + 1 < 2\ell' + p - 2$ contradicting (a). ■

As a consequence of the above proof, the deepness of the fragments increases at least in one unit if the minimum degree is small enough. This is shown in the following corollary.

Corollary 2.1 *Let G be a connected generalized p-cycle with parameter ℓ, minimum degree δ so that $p \geq \delta$, diameter D and connectivities κ and λ. Let F be a positive fragment or α-fragment. Then,*

(a) if $\kappa < \delta$, then $\mu(F) \geq \ell + 1$ and $\mu(\overline{F}) \geq \ell + 1$;

(b) if $\lambda < \delta$, then $\nu(F) \geq \ell + 1$ and $\nu(V \setminus F) \geq \ell + 1$.

Proof. (a) Assume that $\kappa < \delta$ and let F be a positive fragment. Suppose that $\mu(F) = \ell$ and let $F(\ell)$, $\overline{F}(\ell)$ be as in the proof of Theorem 2.1. We know that for each $0 \leq \alpha \leq p-1$, $F(\ell) \cap V_\alpha \neq \emptyset$, $\overline{F}(\ell) \cap V_\alpha \neq \emptyset$. As $\kappa < \delta$ and $p \geq \delta$, there exists some α so that $V_\alpha \cap \partial^+ F = \emptyset$. Now we consider a vertex $x \in F(\ell) \cap V_{\alpha-\ell}$; the vertices of $\partial^+ F$ at distance ℓ from x must belong to V_α, and then $V_\alpha \cap \partial^+ F \neq \emptyset$, which is a contradiction, and hence, $\ell + 1 \leq \mu(F)$. Considering the converse digraph of G, we can also prove $\mu(\overline{F}) \geq \ell + 1$.

(b) Let us assume that $\lambda < \delta$ and let F be a positive α-fragment. From Lemma 2.1 it follows that $\ell \leq \nu(F)$. Let us consider the non-empty sets $F_0 = \{x \in F : (x, y) \in \omega^+ F\}$, $\overline{F}_0 = \{y \in V \setminus F : (x, y) \in \omega^+ F\}$. Now we define the following sets:

$$F(\ell) = \{x \in F : d(x, F_0) \geq \ell\}, \quad \overline{F}(\ell) = \{y \in V \setminus F : d(\overline{F}_0, y) \geq \ell\}.$$

As in the proof of Theorem 2.1 we find that for each $0 \leq \alpha \leq p-1$, $F(\ell) \cap V_\alpha \neq \emptyset$, and $\overline{F}(\ell) \cap V_\alpha \neq \emptyset$. Hence, reasoning as in case (a) we reach the desired result. ■

In fact this result was implicitly proved for bipartite digraphs with minimum degree $\delta = 2$ in [4]. The following corollary gives an improvement of Theorem 2.1 for p-cycles with $p \geq \delta$.

Corollary 2.2 *Let G be a connected generalized p-cycle with parameter ℓ, minimum degree δ, so that $p \geq \delta$, diameter D and connectivities κ and λ. Then,*

(a) $\kappa = \delta$ if $D \leq 2\ell + p - 1$;

(b) $\lambda = \delta$ if $D \leq 2\ell + p$.

Proof. (a) Assume that $\kappa < \delta$ and let F be a positive fragment. By Corollary 2.1 we can take a vertex $x \in F$, so that $d(x, \partial^+ F) = \mu(F) \geq \ell + 1$. Suppose that $x \in V_\alpha$. Then, we consider a vertex $y \in \overline{F}(\ell) \cap V_{\alpha+D+1}$, in such a way that $2\ell + 1 \leq d(x,y) \leq D - p + 1$. This leads to $D \geq 2\ell + p$, which contradicts the hypothesis.

Case (b) is analogous to case (a) by considering α-fragments. ■

From the above results (1) and (2) together with Theorem 2.1 and Corollary 2.2 we can deduce the following sufficient condition for the k-iterated line digraph to be maximally connected.

Corollary 2.3 *Let G be a connected generalized p-cycle, $p \geq 2$, with parameter ℓ, minimum degree δ, diameter D and connectivities κ and λ. Then,*

(a)
$$\kappa(L^k G) = \delta \quad \text{if } k \geq D - 2\ell - p + 2;$$
$$\lambda(L^k G) = \delta \quad \text{if } k \geq D - 2\ell - p + 1.$$

(b) If $p \geq \delta$,
$$\kappa(L^k G) = \delta \quad \text{if } k \geq D - 2\ell - p + 1;$$
$$\lambda(L^k G) = \delta \quad \text{if } k \geq D - 2\ell - p. \quad ■$$

3 Large generalized cycles

This section is devoted to deducing sufficient conditions which guarantee that generalized p-cycles are maximally connected if they have enough vertices for a given maximum degree Δ and diameter D. These conditions extend the previously known results (5) and (6). To begin with, let us consider the case of (vertex) connectivity. First, we compute the minimum and the maximum deepness of any positive or negative fragment.

Lemma 3.1 *Let G be a generalized p-cycle with minimum degree δ, connectivity $\kappa < \delta$, diameter D and parameter ℓ. Then, for all positive or negative fragments F,*

(i) $\mu(F) \geq \ell$ and $\mu(\overline{F}) \leq D - \ell - p + 1$;

(ii) $\mu(F) \geq \ell + 1$ and $\mu(\overline{F}) \leq D - \ell - p$ if $p \geq \delta$.

Proof. To prove (i) let F be a positive fragment of a generalized p-cycle G, hence $|\partial^+ F| = \kappa$ and $D \geq 2\ell + p - 1$, for otherwise Theorem 2.1 holds and G is maximally connected. From now on, we denote by $\mu = \mu(F)$ and $\mu' = \mu(\overline{F})$. From Lemma 2.1 $\mu, \mu' \geq \ell$. Besides, the sets F and $\overline{F}$ can be partitioned into subsets F_i, $0 \leq i \leq \mu$, and $\overline{F}_j$, $0 \leq j \leq \mu'$, according to their distance to and from $\partial^+ F$, that is, $F_i = \{x \in F;\ d(x, \partial^+ F) = i\}$ and $\overline{F}_j = \{y \in \overline{F};\ d(\partial^+ F, y) = j\}$ ($F_0 = \overline{F}_0 = \partial^+ F$.) As any path from F to $\overline{F}$ goes through $\partial^+ F$, the distance from a vertex in F_μ to one in $\overline{F}_{\mu'}$ is at least $\mu + \mu' \leq D$. Now, we are going to see that $\mu' \leq D - \ell - p + 1$. As in the proof of Theorem 2.1 we have that $F(\ell) \cap V_\alpha \neq \emptyset$. Then, let us consider a vertex x_α in each partite set V_α, $0 \leq \alpha \leq p-1$, so that $x_\alpha \in F(\ell)$, and $d(x_\alpha, x_{\alpha+1}) = 1$. Suppose that $\mu' \geq D - \ell - p + 2$ and consider a vertex $y \in \overline{F}_{D-\ell-p+2}$. Hence, $d(x_\alpha, y) \geq \ell + D - \ell - p + 2 \geq D - p + 2$, for $0 \leq \alpha \leq p-1$. In particular, $d(x_{p-1}, y) = D - p + 2 + k$, for some $0 \leq k \leq p-2$, since $d(x_{p-1}, y) \leq D$. Hence, from vertex x_k to vertex y we have two paths, namely, a shortest path of length $d(x_k, y)$, and the path $x_k, x_{k+1}, \ldots, x_{p-1} \to y$ of length $d(x_{p-1}, y) + p - 1 - k = D + 1$. Since the length of these two paths are congruent modulo p and $d(x_k, y) \leq D$, we find that $d(x_k, y) = D + 1 - hp$ for some positive integer h. But this contradicts the fact that $d(x_k, y) \geq D - p + 2$. Therefore, $\mu' \leq D - \ell - p + 1$.

To prove (ii) notice that from Corollary 2.1 it follows that $\mu = \mu(F) \geq \ell + 1$, since $p \geq \delta$. Reasoning again as in case (i) we find that $\mu' = \mu(\overline{F}) \leq D - \ell - p$. ■

As an inmediate consequence of Lemma 3.1 (ii) it follows that (6) also holds for bipartite digraphs with $\delta = 2$, and hence, the hypothesis $\delta \geq 3$ of (6) can be eliminated. The next theorem contains a similar result to (6) for $p \geq 3$.

Theorem 3.1 *Let G be a generalized p-cycle with $p \geq 3$, connectivity κ, order n, maximum and minimum degrees Δ and δ, respectively, diameter D and parameter ℓ. Then,*

(i) $\kappa < \delta \Rightarrow n \leq \kappa\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p + 1) - 1\} - 2\delta + 2.$

(ii) If $p \geq \delta$, $\kappa < \delta \Rightarrow n \leq \kappa\{n(\Delta, \ell+p-2) + n(\Delta, D-\ell-p+1) - 1 - \Delta^{\ell+2}\}.$

Proof. We use the same notation as in Lemma 3.1. Note that $|F_i| \leq \Delta|F_{i-1}|$, $1 \leq i \leq \mu$, and $|\overline{F}_j| \leq \Delta|\overline{F}_{j-1}|$, $1 \leq j \leq \mu'$. Without loss of generality suppose $\mu \leq \mu'$. By means of Lemma 3.1, we have to study the following cases:

(a) $\mu' < D - \ell - p + 1$ (Note that this is the only case in which $p \geq \delta$.)

(a.1) If $\ell \leq \mu \leq \ell + p - 2$ the order of G must satisfy that

$$\begin{aligned} n = \sum_{i=0}^{\mu} |F_i| + \sum_{j=0}^{\mu'} |\overline{F}_j| - |\partial^+ F| &\leq \kappa\{n(\Delta, \mu) + n(\Delta, D - \ell - p) - 1\} \\ &\leq \kappa\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p + 1) - 1\} - \kappa\Delta^{D-\ell-p+1} \\ &\leq \kappa\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p + 1) - 1\} - 2\delta + 2, \end{aligned}$$

because in this case $\ell \le \mu \le \mu' \le D-\ell-p$, hence $D \ge 2\ell+p$ and then $\kappa\Delta^{D-\ell-p+1} \ge \Delta^{\ell+1} \ge \Delta^2 \ge 2(\delta-1)$.

Note that if $p \ge \delta$ from Lemma 3.1 (ii) it follows $\ell+1 \le \mu \le \mu' \le D-\ell-p$, and hence $\kappa\Delta^{D-\ell-p+1} \ge \kappa\Delta^{\ell+2}$, obtaining the desired result (ii) of theorem as well.

(a.2) If $\mu \ge \ell+p-1$ the order of G must satisfy that

$$n = \sum_{i=0}^{\mu} |F_i| + \sum_{j=0}^{\mu'} |\overline{F}_j| - |\partial^+ F| \le \kappa\{n(\Delta,\mu) + n(\Delta, D-\ell-p) - 1\}$$

$$= \kappa\{n(\Delta,\ell+p-2) + n(\Delta,D-\ell-p+1) - 1\} + \kappa\{\sum_{i=\ell+p-1}^{\mu} \Delta^i - \Delta^{D-\ell-p+1}\}$$

$$\le \kappa\{n(\Delta,\ell+p-2) + n(\Delta,D-\ell-p+1) - 1\} - \kappa\Delta^{\ell+2}$$

$$\le \kappa\{n(\Delta,\ell+p-2) + n(\Delta,D-\ell-p+1) - 1\} - 2\delta + 2,$$

since $\kappa\left(\frac{\Delta^{\mu+1}-\Delta^{\ell+p-1}}{\Delta-1} - \Delta^{D-\ell-p+1}\right) \le \kappa\left(\frac{\Delta^{D-\ell-p+1}-\Delta^{\ell+p-1}}{\Delta-1} - \Delta^{D-\ell-p+1}\right) = \frac{\kappa}{\Delta-1}\left((2-\Delta)\Delta^{D-\ell-p+1} - \Delta^{\ell+p-1}\right) \le \frac{\kappa}{\Delta-1}(1-\Delta)\Delta^{\ell+p-1} \le -\kappa\Delta^{\ell+2} \le -2\delta+2$, because $\ell+p-1 \le \mu \le \mu' \le D-\ell-p$ and $p \ge 3$.

(b) $\mu' = D-\ell-p+1$. From Lemma 3.1 it follows $p < \delta$ and so $\delta \ge 3$. Besides, $\ell \le \mu \le \ell+p-1$. Indeed, since all the paths from $x \in F_\mu$ to $y \in \overline{F}_{\mu'}$ go through $\partial^+ F$, it must be that $D \ge d(x,y) \ge d(x,\partial^+F) + d(\partial^+F, y) = \mu+\mu' = \mu + D-\ell-p+1$. Therefore, $\mu \le \ell+p-1$.

(b.1) If $\mu = \ell+p-1$ we can consider a vertex $x \in F_\mu$ and $y \in \overline{F}_{\mu'}$. Therefore, $d(x,y) \ge \mu+\mu' = \ell+p-1+D-\ell-p+1 = D$. Moreover, for all $x \in F_{\ell+p-1}$, $\Gamma^+(x) \subset F_{\ell+p-2}$; otherwise, let $x' \in \Gamma^+(x) \cap F_{\ell+p-1}$. As before, all the paths from x' to y go through ∂^+F, and also $d(x',y) = D$. Then, we would have two different paths from x to y, one of length D and the other, $xx' \to y$, of length $D+1$, which is impossible in a generalized p-cycle with $p \ge 2$. Hence, for all $x \in F_{\ell+p-1}$, $\Gamma^+(x) \subset F_{\ell+p-2}$, which implies that $|F_{\ell+p-1}| \le \frac{\Delta}{\delta}|F_{\ell+p-2}|$. In a similar way, we prove that for any vertex $y \in \overline{F}_{D-\ell-p+1}$, $\Gamma^-(y) \subset \overline{F}_{D-\ell-p}$, and therefore $|\overline{F}_{D-\ell-p+1}| \le \frac{\Delta}{\delta}|\overline{F}_{D-\ell-p}|$. In this way we obtain that

$$n = \sum_{i=0}^{\ell+p-2} |F_i| + \sum_{j=0}^{D-\ell-p} |\overline{F}_j| - |\partial^+F| + |F_{\ell+p-1}| + |\overline{F}_{D-\ell-p+1}|$$

$$\le \kappa\{n(\Delta,\ell+p-2) + n(\Delta,D-\ell-p) - 1\} + \tfrac{\kappa}{\delta}\{\Delta^{\ell+p-1} + \Delta^{D-\ell-p+1}\}$$

$$= \kappa\{n(\Delta,\ell+p-2) + n(\Delta,D-\ell-p+1) - 1\} + \tfrac{\kappa}{\delta}\{\Delta^{\ell+p-1} + \Delta^{D-\ell-p+1}\} - \kappa\Delta^{D-\ell-p+1} \le \kappa\{n(\Delta,\ell+p-2) + n(\Delta,D-\ell-p+1) - 1\} - 2\delta + 2,$$

since $\kappa(\frac{1}{\delta}\{\Delta^\mu + \Delta^{\mu'}\} - \Delta^{\mu'}) \le \kappa(\frac{2}{\delta}-1)\Delta^{\mu'} \le -2\delta+2$, because $3 \le \ell+2 \le \ell+p-1 = \mu \le \mu'$ and $\delta \ge 3$.

(b.2) If $\mu = \ell + p - 2$, we can consider vertex $x \in F_\mu$ and $y \in \overline{F}_{\mu'}$. As all the paths from $x \in F_\mu$ to y go through $\partial^+ F$, it must be that $d(x, y) \geq d(x, \partial^+ F) + d(\partial^+ F, y) = \mu + \mu' = \ell + p - 2 + D - \ell - p + 1 = D - 1$. In addition, either for all $x \in F_{\ell+p-2}$, $\Gamma^+(x) \subset F_{\ell+p-3}$ or there exists a vertex $x \in F_{\ell+p-2}$ with some outneighbor $x' \in \Gamma^+(x) \cap F_{\ell+p-2}$. In the first case we would find that $|F_{\ell+p-2}| \leq \frac{\Delta}{\delta} |F_{\ell+p-3}|$. In the second case we obtain that $d(x', y) \geq D - 1$. In this case, notice that if $d(x, y) = D - 1$ then the length of the path $xx' \to y$ must be $1 + d(x', y) = D - 1 + hp$ for some integer $h \geq 0$. From $d(x', y) \geq D - 1$ follows $h \geq 1$, so $d(x', y) \geq D - 2 + p \geq D + 1$ because $p \geq 3$, which is a contradiction. Hence, the only possibility is that $d(x, y) = D$, $d(x', y) = D - 1$ and $\Gamma^+(x') \subset F_{\ell+p-3}$. Thus we have proved that $|F_{\ell+p-2}| \leq \Delta |F_{\ell+p-3}| - (\delta - 1)$. In a similar way we obtain that $|F_{D-\ell-p+1}| \leq \Delta |F_{D-\ell-p}| - (\delta - 1)$. Therefore, the order of G must satisfy that

$$\begin{aligned} n &= \sum_{i=0}^{\ell+p-3} |F_i| + \sum_{j=0}^{D-\ell-p} |\overline{F}_j| - |\partial^+ F| + |F_{\ell+p-2}| + |\overline{F}_{D-\ell-p+1}| \\ &\leq \kappa\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p + 1) - 1\} - 2\delta + 2. \end{aligned}$$

(b.3) Finally, if $\ell \leq \mu \leq \ell + p - 3$ then,

$$\begin{aligned} n &= \sum_{i=0}^{\mu} |F_i| + \sum_{j=0}^{D-\ell-p+1} |\overline{F}_j| - |\partial^+ F| \\ &\leq \kappa\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p + 1) - 1\} - \kappa \Delta^{\ell+p-2} \\ &\leq \kappa\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p + 1) - 1\} - 2\delta + 2. \qquad \blacksquare \end{aligned}$$

The following corollary, which is just a restatement of the above theorem, gives a sufficient condition on the number of vertices for any generalized p-cycle with $p \geq 3$ to have maximum connectivity.

Corollary 3.1 *Let G be a generalized p-cycle with $p \geq 3$, connectivity κ, order n, maximum and minimum degrees Δ and δ, respectively, diameter D and parameter ℓ. Then, $\kappa = \delta$ if*

(i) $n > (\delta - 1)\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p + 1) - 3\}$.

(ii) If $p \geq \delta$, $n > (\delta - 1)\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p + 1) - 1 - \Delta^{\ell+2}\}$.

Recalling that a digraph is a generalized p-cycle if and only if its line digraph is, we can apply the line digraph technique to Theorem 3.1. So, we obtain a sufficient condition on the number of edges for G to have maximum edge-connectivity.

Corollary 3.2 *Let G be a generalized p-cycle with $p \geq 3$, connectivity λ, size m, maximum and minimum degrees Δ and δ, respectively, diameter D and parameter ℓ. Then,* *(i)* $\lambda < \delta \Rightarrow m \le \lambda\{n(\Delta, \ell+p-1) + n(\Delta, D-\ell-p+1) - 1\} - 2\delta + 2$.

(ii) If $p \ge \delta$, $\lambda < \delta \Rightarrow m \le \lambda\{n(\Delta, \ell+p-1) + n(\Delta, D-\ell-p+1) - 1 - \Delta^{\ell+3}\}$.

Proof. Suppose that the result is not true. Then there would be a generalized p-cycle with $p \ge 3$, m edges, parameters δ, Δ, ℓ, D and edge-connectivity $\lambda < \delta$ so that

$$m > \lambda\{n(\Delta, \ell+p-1) + n(\Delta, D-\ell-p+1) - 1\} - 2\delta + 2.$$

Then, the line digraph of G, LG, would have $n' = m$ vertices, minimum and maximum degree δ and Δ, diameter $D' = D+1$, parameter $\ell' = \ell+1$, and connectivity $\kappa' = \lambda < \delta$, satisfying

$$n' > \kappa'\{n(\Delta, \ell'+p-2) + n(\Delta, D'-\ell'-p+1) - 1\} - 2\delta + 2,$$

which contradicts Theorem 3.1. The case $p \ge \delta$ is proved similarly. ■

When the digraph G is d-regular, it has $m = dn$ edges and we get the following corollary.

Corollary 3.3 *Let G be a d-regular generalized p-cycle with $p \ge 3$, connectivities κ, λ, order n, diameter D and parameter ℓ. Then,*

$$\textit{(i)}\ \kappa = d \textit{ if} \begin{cases} n > d^{\ell+p-1} + d^{D-\ell-p+2} - 3d + 1; \\ n > d^{\ell+p-1} + d^{D-\ell-p+2} - (d^{\ell+2}+1)(d-1) - 2, \textit{ if } p \ge d. \end{cases}$$

$$\textit{(ii)}\ \lambda = d \textit{ if} \begin{cases} n > d^{\ell+p-1} + d^{D-\ell-p+1} - 3; \\ n > d^{\ell+p-1} + d^{D-\ell-p+1} - d^{\ell+2}(d-1) - 1, \textit{ if } p \ge d. \end{cases}$$

■

From the above results we can deduce the following sufficient condition for the k-iterated line d-regular generalized p-cycle to be maximally connected.

Corollary 3.4 *Let G be a d-regular generalized p-cycle with $p \ge 3$, connectivities κ, λ, order n, diameter D and parameter ℓ. Then,*

$$\textit{(i)}\ \kappa(L^kG) = d \textit{ if } k > \log_d \frac{d^{D-\ell-p+2} - 3d + 1}{n - d^{\ell+p-1}}.$$

$$\textit{(ii)}\ \lambda(L^kG) = d \textit{ if } k > \log_d \frac{d^{D-\ell-p+1} - 3}{n - d^{\ell+p-1}}. \quad ■$$

The above corollary gives the following results to be compared with those of Corollary 2.3.

$\kappa(L^kG) = d$ if $k > D - \ell - p + 2 - \log_d(n - d^{\ell+p-1})$.
$\lambda(L^kG) = d$ if $k > D - \ell - p + 1 - \log_d(n - d^{\ell+p-1})$.

An upper bound on the number of vertices for any generalized p-cycle with edge-connectivity $\lambda < \delta$ that extends the result (6) for $p \geq 3$, can be obtained by using direct reasoning. With this aim, first we need to bound the deepness of the α-fragments.

Lemma 3.2 *Let G be a generalized p-cycle with minimum degree δ, edge-connectivity $\lambda < \delta$, diameter D and parameter ℓ. Then, for all positive or negative α-fragments F,*

(i) $\nu(F) \geq \ell$ and $\nu(V \setminus F) \leq D - \ell - p$;

(ii) $\nu(F) \geq \ell + 1$ and $\nu(V \setminus F) \leq D - \ell - p - 1$ if $p \geq \delta$.

Proof. To prove (i) let F be a positive α-fragment of a generalized p-cycle G. Then $|\omega^+ F| = \lambda$ and $D \geq 2\ell + p$, for otherwise Theorem 2.1 holds and $\lambda = \delta$. In what follows we denote by $\nu = \nu(F)$ and $\nu' = \nu(V \setminus F)$. From Lemma 2.1 it follows that $\nu, \nu' \geq \ell$. Let us consider the two non-empty disjoint sets $F_0 = \{x \in F : (x, y) \in \omega^+ F\}$ and $\overline{F}_0 = \{y \in V \setminus F : (x, y) \in \omega^+ F\}$. It is clear that $|F_0| \leq |\omega^+ F|$ and $|\overline{F}_0| \leq |\omega^+ F|$. Then, we define $F_i = \{x \in F; d(x, F_0) = i\}$, $0 \leq i \leq \nu$ and $\overline{F}_j = \{y \in V \setminus F : d(\overline{F}_0, y) = j\}$, $0 \leq j \leq \nu'$. As any path from F to $V \setminus F$ goes throught an arc of $\omega^+ F$ it follows that $\nu + 1 + \nu' \leq D$. Reasoning in the same way as in the proof of Lemma 3.1 it is obtained that $\nu' \leq D - \ell - p$. Corollary 2.1 is sufficient to prove (ii), since $p \geq \delta$. As $\nu \geq \ell + 1$ and reasoning as in case (i) we find that $\nu' \leq D - \ell - p - 1$. ■

Now, we are ready to state the above-mentioned conditions involving the order of G whose proof is similar to the proof of Theorem 3.1.

Theorem 3.2 *Let G be a generalized p-cycle with $p \geq 3$, edge-connectivity λ, order n, maximum and minimum degrees Δ and δ, respectively, diameter D and parameter ℓ. Then,*

(i) $\lambda < \delta \Rightarrow n \leq \lambda\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p)\}$.

(ii) If $p \geq \delta$, $\lambda < \delta \Rightarrow n \leq \lambda\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p) - \Delta^{\ell+2}\}$.

Proof. We use the same notation as in Lemma 3.2 and assume that $\nu \leq \nu'$. By means of Lemma 3.2, we have to study the following cases:

(a) $\nu' < D - \ell - p$ (Note that this is the only case in which $p \geq \delta$.)

(a.1) If $\ell \leq \nu \leq \ell + p - 2$ the order of G must satisfy that

$$n = \sum_{i=0}^{\nu} |F_i| + \sum_{j=0}^{\nu'} |\overline{F}_j| \leq \lambda\{n(\Delta, \nu) + n(\Delta, D - \ell - p - 1)\}$$
$$\leq \lambda\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p)\}.$$

If $p \geq \delta$ by Lemma 3.2 (ii) $\ell+1 \leq \nu \leq \nu' \leq D-\ell-p-1$, hence $D \geq 2\ell+p+2$ and then $\lambda\Delta^{D-\ell-p} \geq \lambda\Delta^{\ell+2}$, obtaining the desired result (ii) of theorem as well.

(a.2) If $\nu \geq \ell+p-1$ the order of G must satisfy that

$$n = \sum_{i=0}^{\nu} |F_i| + \sum_{j=0}^{\nu'} |\overline{F}_j| \leq \lambda\{n(\Delta,\nu) + n(\Delta, D-\ell-p-1)\}$$
$$= \lambda\{n(\Delta,\ell+p-2) + n(\Delta,D-\ell-p)\} + \lambda\{\sum_{i=\ell+p-1}^{\nu} \Delta^i - \Delta^{D-\ell-p}\}$$
$$\leq \lambda\{n(\Delta,\ell+p-2) + n(\Delta,D-\ell-p)\} - \lambda\Delta^{\ell+2}$$
$$\leq \lambda\{n(\Delta,\ell+p-2) + n(\Delta,D-\ell-p)\},$$

since $\lambda\left(\frac{\Delta^{\nu+1}-\Delta^{\ell+p-1}}{\Delta-1} - \Delta^{D-\ell-p}\right) \leq \lambda\left(\frac{\Delta^{D-\ell-p}-\Delta^{\ell+p-1}}{\Delta-1} - \Delta^{D-\ell-p}\right) = \frac{\lambda}{\Delta-1}\left((2-\Delta)\Delta^{D-\ell-p} - \Delta^{\ell+p-1}\right) \leq \frac{\lambda}{\Delta-1}(1-\Delta)\Delta^{\ell+p-1} \leq -\lambda\Delta^{\ell+2}$, because $\ell+p-1 \leq \nu \leq \nu' \leq D-\ell-p-1$ and $p \geq 3$.

(b) $\nu' = D-\ell-p$. Then $\ell \leq \nu \leq \ell+p-1$ since all the paths from $x \in F_\nu$ to $y \in \overline{F}_{\nu'}$ go through ω^+F, it must be that $D \geq d(x,y) \geq d(x,\omega^+F)+1+d(\omega^+F,y) \geq \nu+1+\nu' = \nu+1+D-\ell-p$. Thus, $\nu \leq \ell+p-1$.

(b.1) If $\nu = \ell+p-1$ we can consider a vertex $x \in F_\nu$ and $y \in \overline{F}_{\nu'}$. It must be that $d(x,y) \geq \nu+1+\nu' = \ell+p-1+1+D-\ell-p = D$. Moreover, for all $x \in F_{\ell+p-1}$, $\Gamma^+(x) \subset F_{\ell+p-2}$; otherwise, let $x' \in \Gamma^+(x) \cap F_{\ell+p-1}$. As before, all the paths from x' to y go through ω^+F, and also $d(x',y) = D$. Then, we would have two different paths from x to y, one of length D and the other, $xx' \to y$, of length $D+1$, which is impossible in a generalized p-cycle with $p \geq 2$. Hence, for all $x \in F_{\ell+p-1}$, $\Gamma^+(x) \subset F_{\ell+p-2}$, which implies that $|F_{\ell+p-1}| \leq \frac{\Delta}{\delta}|F_{\ell+p-2}|$. In a similar way, we prove that for any vertex $y \in \overline{F}_{D-\ell-p}$, $\Gamma^-(y) \subset \overline{F}_{D-\ell-p-1}$, and therefore $|\overline{F}_{D-\ell-p}| \leq \frac{\Delta}{\delta}|\overline{F}_{D-\ell-p-1}|$. In this way we obtain that

$$n = \sum_{i=0}^{\ell+p-2} |F_i| + \sum_{j=0}^{D-\ell-p-1} |\overline{F}_j| + |F_{\ell+p-1}| + |\overline{F}_{D-\ell-p}|$$
$$\leq \lambda\{n(\Delta,\ell+p-2) + n(\Delta,D-\ell-p-1)\} + \frac{\lambda}{\delta}\{\Delta^{\ell+p-1} + \Delta^{D-\ell-p}\}$$
$$= \lambda\{n(\Delta,\ell+p-2)+n(\Delta,D-\ell-p)\}+\frac{\lambda}{\delta}\{\Delta^{\ell+p-1}+\Delta^{D-\ell-p}\}-\lambda\Delta^{D-\ell-p}$$
$$\leq \lambda\{n(\Delta,\ell+p-2) + n(\Delta,D-\ell-p)\},$$

since $\lambda(\frac{1}{\delta}\{\Delta^\nu + \Delta^{\nu'}\} - \Delta^{\nu'}) \leq \lambda(\frac{2}{\delta}-1)\Delta^{\nu'} \leq 0$, because $\nu \leq \nu'$.

(b.2) Finally, if $\ell \leq \nu \leq \ell+p-2$ then,

$$n = \sum_{i=0}^{\nu} |F_i| + \sum_{j=0}^{D-\ell-p} |\overline{F}_j| \leq \lambda\{n(\Delta,\ell+p-2) + n(\Delta,D-\ell-p)\}. \quad \blacksquare$$

	p	$D \leq$	$n >$
$\kappa = d$	1	$2\ell - 1$ [8]	$d^D + d^2 - d - 1$ [17] $d^{D-\ell+1} + 2d^\ell - 2d + 1$ [7]
	2	2ℓ [9], [*]	$2(d^{D-1} - 1)$ [1] $d^{D-\ell} + 2d^{\ell+1} - 2d + 2$ [4]
	≥ 3	$2\ell + p - 2$ [*]	$d^{D-\ell-p+2} + d^{\ell+p-1} - 3d + 1$ [*]
$\lambda = d$	1	2ℓ [8]	$d^{D-1} + d^2 - 2$ [17] $d^{D-\ell} + 2d^\ell - 2$ [7]
	2	$2\ell + 1$ [9], [*]	$2d^{D-2}$ [1] $d^{D-\ell-1} + d^{\ell+1} - 2$ [4]
	≥ 3	$2\ell + p - 1$ [*]	$d^{D-\ell-p+1} + d^{\ell+p-1} - 3$ [*]

Table 1: Sufficient conditions for a d-regular generalized p-cycle to have maximum connectivities. The [*] indicates this paper

It is interesting to note that, since $n \geq m/\Delta$, the above theorem also implies the result of Corollary 3.2 and hence that of Corollary 3.3 as well. The following corollary gives a sufficient condition on the number of vertices for any generalized p-cycle to have maximum edge-connectivity.

Corollary 3.5 *Let G be a generalized p-cycle with $p \geq 3$, edge-connectivity λ, order n, maximum and minimum degrees Δ and δ, respectively, diameter D and parameter ℓ. Then,*

(i) $\lambda = \delta$ *if* $n > (\delta - 1)\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p)\}$.

(ii) If $p \geq \delta$, $\lambda = \delta$ *if* $n > (\delta - 1)\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p) - \Delta^{\ell+2}\}$.

In Table 1, we give a more concise comparison between some of the results of this paper, derived as corollaries of Theorems 2.1, 3.1 and 3.2, and the corresponding earlier results. For simplicity, we have limited ourselves to the sufficient conditions for a d-regular generalized p-cycles to be maximally connected or edge-connected.

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