A New Distance Measure between Graphs

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Abstract

Let G_1 and G_2 be two graphs of the same size such that $V(G_1) = V(G_2)$, and let H be a connected graph of order at least 3. The graphs G_1 and G_2 are H-adjacent if G_1 and G_2 contain copies H_1 and H_2 of H, respectively, such that H_1 and H_2 share some but not all edges and $G_2 = G_1 - E(H_1) + E(H_2)$. The graphs G_1 and G_2 are H-connected if G_1 can be obtained from G_2 by a sequence of H-adjacencies. The relation H-connectedness is an equivalence relation on the set of all graphs of a fixed order and fixed size. The resulting equivalence classes are investigated for various choices of the graph H.

1 Introduction

A basic problem in drug design consists of finding a compound that satisfies a spectrum of biological and chemical properties. Although drug design problems are central to pharmaceutical research, statisticians have yet to become involved in this area. A major reason for this is that these problems are viewed statistically as optimization problems, and standard statistical optimization methods are based on Euclidean space or vector representation. Here the formal representations are labeled graphs and/or three-dimensional atomic configurations. Hence before statistical optimization procedures can be defined on these spaces whose points are structures and not vectors, very basic mathematical notions of distance between labeled graphs or graphs in general must be defined and studied.

There is a commonly recognized principle in chemistry that similar compounds generally have similar properties (see [12]). This principle implies that we have a metric, or at least a pseudometric, on the set of chemical graphs. One problem, of course, is choosing the appropriate metric. The Dugundji-Ugi principle of minimum chemical distance provides an interesting illustration of choosing the appropriate metric. In this formalism, the reactants and products of a chemical reaction are represented as graphs with the possible inclusion of loops and multiple edges [8, 11]. Two distances are involved. The first, the "experimental distance", is the sum over the individual steps of the reaction of the number of valence electrons that participate in each step. The second, called the "chemical distance", can be shown to correspond to the move distance between the graph of the reactants and the graph of the products. (The *move distance* is the minimum number of edges that must be "moved" to transform one graph into the other.) The principle asserts the equality of the experimental and chemical distances. Counterexamples are known to exist, however. Such discrepancies are often attributed to the formation of a bond in a reaction intermediate that is subsequently cleaved in the course of the reaction [8]. It may be that some of the counterexamples to the principle of chemical distance would cease to exist if the principle of minimum chemical distance was reformulated in terms of the appropriate graphical distance. One such possibility is explored in [11]. Other applications of graphical metrics to problems in chemistry may be found in [1, 9] and recently a survey was completed on graphical metrics [4].

In this paper we will explore another graphical metric, first defined in [7], with the idea that fixed parts of the molecule can be moved in one step. It is not our intention to show that the metric we will define satisfies the Dugundji-Ugi principle of minimum chemical distance but rather to provide yet another measure of the distance between two graphs that is, in some sense, more restrictive than the previously defined metrics.

Let G_1 and G_2 be two graphs of the same order and same size such that $V(G_1) = V(G_2)$, and let H be a connected graph of order at least 3.

1.1 Definition. Two subgraphs H_1 and H_2 of G_1 and G_2, respectively, are *H-adjacent* if $H_1 \cong H_2 \cong H$ and H_1 and H_2 share some but not all edges, that is, $E(H_1) \cap E(H_2) \neq \emptyset$ and $E(H_2) - E(H_1) \neq \emptyset$ (so also $E(H_1) - E(H_2) \neq \emptyset$). The graphs G_1 and G_2 are themselves *H-adjacent* if G_1 and G_2 contain H-adjacent subgraphs H_1 and H_2, respectively, such that $E(H_2) - E(H_1) \subseteq E(\overline{G_1})$ and $G_2 = G_1 - E(H_1) + E(H_2)$. (For graphs G_1 and G_2, we will often write $G_1 = G_2$ to indicate that the graphs G_1 and G_2 are isomorphic.)

1.2 Definition. A $G_1 - G_2$ *H-path* is a sequence $G_1 = F_0, F_1, \ldots, F_k = G_2$ of graphs of the same order and same size such that F_i is H-adjacent to F_{i+1} for $i = 0, 1, \ldots, k-1$. The graphs G_1 and G_2 are *H-connected* if there exists a $G_1 - G_2$ H-path. For H-connected graphs G_1 and G_2, the *H-distance* $d_H(G_1, G_2)$ from G_1 to G_2 is the minimum number of H-adjacencies required to transform G_1 into G_2

Let $H = P_3$. In Figure 1, the path $H_1 : u, v, w$ of G_1 is H-adjacent to the path $H_2 : v, w, u$ of G_2 and, in fact, since $G_2 = G_1 - E(H_1) + E(H_2)$, the graphs G_1 and G_2 are H-adjacent. The path $H_2' : w, x, y$ of the graph G_2 is H-adjacent to the path $H_3 : x, w, y$ of the graph G_3, shown in Figure 1, and thus since $G_3 = G_2 - E(H_2') + E(H_3)$, the graphs G_2 and G_3 are H-adjacent. Clearly, G_1 is not H-adjacent to G_3 but since G_1, G_2, G_3 is an H-path from G_1 to G_3, the graph G_1 is H-connected to G_3.

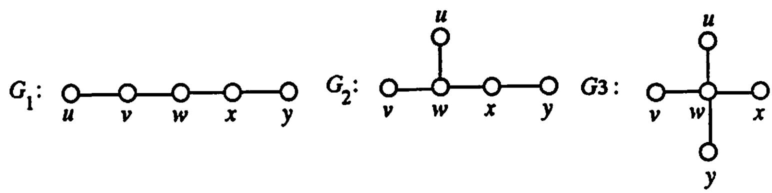

Figure 1

The relation H-connectedness is an equivalence relation on the set of all graphs of the same order and same size, and it is not clear for a given graph H what these equivalence classes are. In this paper, we investigate the equivalence classes in some specific cases, namely, when H is a path, a star of order 4, or a triangle. For a given graph H, the H-distance is a metric on the space of all graph of a fixed order and fixed size for which it is defined. However, it is not clear, for a given graph H, when this distance is defined. We will also investigate some properties of the distance induced by this transformation when H is P_3 and how it is related to some of the other known graphical distances.

2 H-Connected Graphs when H is a Path

We begin by considering H-adjacency when $H = P_3$. Let G_1 and G_2 be P_3-adjacent graphs. Then G_1 contains a copy H_1 of P_3, say $H_1 : u, v, w$, and G_2 contains a copy H_2 of P_3 with $E(H_1) \cap E(H_2) \neq \emptyset$ and $E(H_2) - E(H_1) \subseteq E(\overline{G_1})$. Since $E(H_1) \subseteq E(G_1)$ and H_1 has exactly two edges, it follows that H_1 and H_2 have exactly one edge in common, say uv, and H_2 contains exactly one edge that is not present in G_1. So $H_2 : u, v, z$ or $H_2 : z, u, v$ for some $z \in V(G_2)$. Thus (1) $G_2 = G_1 - vw + vz$ or (2) $G_2 = G_1 - vw + uz$, where u, v, w, and z are not necessarily distinct. Hence P_3-adjacency is the transfer of an edge from one graph to another. The transfer of an edge from one graph to another has been previously studied in many articles, including [2, 5, 3, 6, 10].

2.1 Definition. The graph G_2 is said to be obtained from G_1 by an *edge move* if G_1 contains (not necessarily distinct) vertices u, v, w, and x such that $uv \in E(G_1)$, $wx \notin E(G_1)$, and $G_2 = G_1 - uv + wx$.

This concept was first defined in [1, 9], and it was shown in [2] that for every two graphs of the same order and same size, each can be transformed into the other by a sequence of edge moves.

2.2 Definition. The graph G_2 is said to be obtained from G_1 by an *edge rotation* if G_1 contains distinct vertices u, v, and w such that $uv \in E(G_1)$, $uw \notin E(G_1)$ and $G_2 = G_1 - uv + uw$.

In [6] it was shown that for every two graphs of the same order and same size, each can be transformed into the other by a sequence of edge rotations.

2.3 Definition. A graph G_2 can be obtained from G_1 by an *edge slide* if G_1 contains distinct vertices u, v, and w such that $uv, vw \in E(G_1), uw \notin E(G_1)$, and $G_2 = G_1 - uv + uw$.

In [10] it was shown that the edge slide preserves connectedness, and that a graph G_1 can be obtained from a graph G_2 by a sequence of edge slides if and only if G_1 and G_2 have the same number of components and corresponding components of G_1 and G_2 have the same order and same size.

2.4 Definition. The graph G_2 is obtained from G_1 by an *edge jump* if G_1 contains four distinct vertices u, v, w, and x such that $uv \in E(G_1)$, $wx \notin E(G_2)$ and $G_2 = G_1 - uv + wx$.

Observe that an edge jump is any edge move that is not an edge rotation. In [3] it was shown that for every two graphs of the same order (at least 5) and same size, each can be transformed into the another by a sequence of edge jumps.

Of course, a P_3-adjacency is a special case of an edge move, in fact, an edge slide is always a P_3-adjacency. Clearly, if the graphs G_1 and G_2 are P_3-adjacent, then G_2 can be obtained from G_1 by an edge move. If the graphs G_1 and G_2 are P_3-adjacent and $G_2 = G_1 - vw + vz$ as in (1), then G_2 can be obtained from G_1 by an edge rotation. In fact, in this case, P_3-adjacency is more restrictive than edge-rotation since P_3-adjacency requires the presence of another edge incident to v. If w and z are also adjacent, then the P_3-adjacency is an edge slide. Next, if G_1 and G_2 are P_3-adjacent with $G_2 = G_1 - vw + uz$ as in (2) with u, v, w, and z distinct, then G_2 is obtained from G_1 by an edge jump.

Our goal is to show that almost every pair of graphs of the same order and same size are P_3-connected. We begin with the following lemma.

2.5 Lemma. *Let G_1 and G_2 be two graphs of the same order and same size with maximum degree at least 2 such that G_2 can be obtained from G_1 by an edge rotation. Then G_1 and G_2 are P_3-connected by a sequence of at most three P_3-adjacencies.*

PROOF. Since $G_2 = G_1 - uv + uw$, it follows that G_2 is obtained from G_1 by an edge rotation. We now proceed by cases.

CASE 1. SUPPOSE THAT vw IS AN EDGE OF (NECESSARILY) BOTH G_1 AND G_2. Then G_2 is obtained from G_1 by an edge slide and thus G_1 and G_2 are P_3-adjacent.

CASE 2. SUPPOSE THAT u HAS DEGREE AT LEAST 2 IN G_1 (AND HENCE IN G_2). Then the edge rotation transforming G_1 into G_2 is a P_3-adjacency.

CASE 3. SUPPOSE THAT $\deg_{G_1} u = \deg_{G_2} u = 1$ AND THAT THERE EXISTS A VERTEX z, DISTINCT FROM v AND w, WITH DEGREE AT LEAST 2 (NECESSARILY IN BOTH GRAPHS G_1 AND G_2). Let s and t be two vertices (different from u

by Case 2) adjacent to z. Next, let $F_1 = G_1 - \{sz, zt\} + \{sz, zu\}$ and $F_2 = F_1 - \{zu, uv\} + \{zu, uw\}$. Since $G_2 = F_2 - \{uz, zs\} + \{sz, zt\}$, we have that G_1, F_1, F_2, G_2 is a P_3-path so that G_1 and G_2 are P_3-connected by a sequence of three P_3-adjacencies.

CASE 4. SUPPOSE THAT EVERY VERTEX, OTHER THAN v AND w, HAS DEGREE AT MOST 1 IN BOTH G_1 AND G_2 AND THAT v AND w ARE NOT ADJACENT. Since G_1 has maximum degree at least 2, it follows that $\deg_{G_1} v \geq 2$ or $\deg_{G_1} w \geq 2$. Suppose first that $\deg_{G_1} v \geq 2$. If $\deg_{G_1} v = 2$, then $\deg_{G_2} v = 1$ so that $\deg_{G_2} w \geq 2$. Then there exists (not necessarily distinct) vertices s and t such that s is adjacent to v and t is adjacent to w. Let $F_1 = G_1 - \{uv, vs\} + \{sv, vw\}$. Since $G_2 = F_1 - \{vw, wt\} + \{uw, wt\}$, we have that G_1 and G_2 are P_3-connected by a sequence of two P_3-adjacencies. On the other hand, if $\deg_{G_1} v > 2$, then there exists distinct vertices z_1 and z_2 adjacent to v in both G_1 and G_2. Let $F_1 = G_1 - \{uv, vz_1\} + \{vu, uw\}$. Since $G_2 = F_1 - \{uv, vz_2\} + \{z_1v, vz_2\}$, we have that G_1 and G_2 are P_3-connected by a sequence of two P_3-adjacencies. Finally, suppose that $\deg_{G_1} w \geq 2$. Then, $\deg_{G_2} w \geq 3$. Reversing the process just described for v when $\deg_{G_1} v > 2$ by starting with G_2 and working with w in place of v, we have that G_1 and G_2 are P_3-connected by a sequence of two P_3-adjacencies. ■

2.6 Theorem. *Let G_1 and G_2 be two graphs of the same order and the same size with maximum degree at least 2. Then G_1 and G_2 are P_3-connected.*

PROOF. Since we may remove an equal number of isolated vertices from both G_1 and G_2, we may assume that G_1 has minimum degree at least 1. Thus, any graph obtained from G_1 by a sequence of edge rotations has maximum degree at least 2. Since G_1 and G_2 have the same order and size, we may transform G_1 into G_2 by a sequence of edge rotations. Let $G_1 = F_0, F_1, F_2, \ldots, F_k = G_2$ be a sequence of graphs such that F_{i+1} is obtained from F_i by an edge rotation for $0 \leq i \leq k-1$. Since each F_i has maximum degree at least 2 for $1 \leq i \leq k-1$, by Lemma 2.5, we have that F_i is P_3-connected to F_{i+1}. Therefore, G_1 is P_3-connected to G_2. ■

Thus, the P_3-distance is defined for every pair of graphs of the same order and same size containing P_3 as a subgraph. We now compare the rotation and jump distances with the P_3-distance. The *rotation distance* $d_r(G_1, G_2)$ between two graphs G_1 and G_2 of the same order and same size is defined as the minimum number of edge rotations required to transform G_1 into G_2. The *jump distance* $d_j(G_1, G_2)$ is defined analogously. Both distances are metrics on the space of all graphs of a fixed order (at least 5 for the jump distance) and fixed size. In [3], it is shown that each edge rotation can be achieved by two edge jumps and that each edge jump can be achieved by two edge rotations. Thus, for any two graphs G_1 and G_2 of the same order (≥ 5) and same size, the rotation distance is at most twice the jump distance and the jump distance is at most twice the rotation distance, that is,

$$d_r(G_1, G_2) \leq 2d_j(G_1, G_2) \text{ and } d_j(G_1, G_2) \leq 2d_r(G_1, G_2).$$

These inequalities can also be described as

$$\frac{1}{2}d_j(G_1, G_2) \leq d_r(G_1, G_2) \leq 2d_j(G_1, G_2),$$

or equivalently,

$$\frac{1}{2}d_r(G_1, G_2) \leq d_j(G_1, G_2) \leq 2d_r(G_1, G_2).$$

It is also shown in [3] that these bounds are best possible, that is, for every two positive integers a and b with $a/2 \leq b \leq 2a$, there exists graphs G_1 and G_2 of the same order and size such that $d_j(G_1, G_2) = a$ and $d_r(G_1, G_2) = b$.

Since a P_3-adjacency is either an edge rotation or an edge jump and since an edge jump requires two edge rotations, it follows that the rotation distance is at most twice the P_3-distance. This, together with Lemma 2.5, gives the following bounds for each metric in terms of the other.

2.7 Corollary. *For any two graphs G_1 and G_2 of the same order and same size having maximum degree at least 2,*

$$\frac{1}{2}d_r(G_1, G_2) \leq d_{P_3}(G_1, G_2) \leq 3d_r(G_1, G_2),$$

or, equivalently,

$$\frac{1}{3}d_{P_3}(G_1, G_2) \leq d_r(G_1, G_2) \leq 2d_{P_3}(G_1, G_2).$$

The bounds provided for each metric in terms of the other are the only restrictions, as the next result shows.

2.8 Theorem. *For every two positive integers a and b with $a/2 \leq b \leq 3a$, there exists graphs G_1 and G_2 of the same order and same size such that $d_r(G_1, G_2) = a$ and $d_{P_3}(G_1, G_2) = b$.*

Proof. First note that $d_{P_3}(2P_3, P_5 \cup K_1) = 1$, $d_{P_3}(2P_3, P_4 \cup K_2) = 2$, and $d_{P_3}(2P_3 \cup K_1, P_3 \cup 2K_2) = 3$, while $d_r(2P_3, P_5 \cup K_1) = d_r(2P_3, P_4 \cup K_2) = d_r(2P_3 \cup K_1, P_3 \cup 2K_2) = 1$. We now proceed by cases.

Case 1. Assume that $2a \leq b \leq 3a$. Choose

$$G_1 = (b - 2a)(2P_3 \cup K_1) \cup (3a - b)(2P_3) = 2aP_3 \cup (b - 2a)K_1$$

and

$$G_2 = (b-2a)(P_3 \cup 2K_2) \cup (3a-b)(P_4 \cup K_2) = (3a-b)P_4 \cup (b-2a)P_3 \cup (b-a)K_2.$$

Clearly $d_{P_3}(G_1, G_2) \le 3(b-2a) + 2(3a-b) = b$. Note that no matter how the components of G_1 are P_3-transformed, we require at least two P_3-adjacencies to obtain a component isomorphic to K_2 or P_4 and that after this transformation, every other component is isomorphic to P_3 or K_1. Thus to obtain the $3a-b$ components of G_2 isomorphic to P_4, we require at least $2(3a-b)$ P_3-adjacencies. If only these P_3-adjacencies were applied, there would be $2b-4a$ components of G_2 isomorphic to K_2 yet to obtain from the remaining components isomorphic to P_3 or P_4. Obtaining a K_2 from a P_4 requires one P_3-adjacency, but another P_4 must then be obtained from the remaining components, all of which are isomorphic to P_3. This requires yet another two P_3-adjacencies and gives another K_2 for a total of three P_3-adjacencies. On the other hand, obtaining a K_2 from the remaining components isomorphic to P_3 requires at least two P_3-adjacencies giving either $P_4 \cup K_2$ or $K_{1,3} \cup K_2$. Now $K_{1,3} \cup K_2$ will not give another K_2 with one P_3-adjacency, yet when $P_4 \cup K_2$ is obtained, one P_3-adjacency gives $2K_2 \cup P_3$, for a total of three P_3-adjacencies. Thus, to obtain the remaining $2b-4a$ components isomorphic to K_2, we need at least $3(b-2a)$ P_3-adjacencies. Thus, $d_{P_3}(G_1, G_2) \ge 3(b-2a) + 2(3a-b) = b$, and therefore $d_{P_3}(G_1, G_2) = b$. Notice also that G_1 and G_2 differ in exactly a edges and that G_2 can be obtained from G_1 by a edge-rotations. Therefore, $d_r(G_1, G_2) = a$.

CASE 2. ASSUME THAT $a \le b < 2a$. Choose

$$G_1 = (b-a)(2P_3) \cup (2a-b)(2P_3) = 2aP_3$$

and

$$G_2 = (b-a)(P_4 \cup K_2) \cup (2a-b)(P_5 \cup K_1).$$

Clearly, $d_{P_3}(G_1, G_2) \le 2(b-a) + 2a - b = b$. To obtain one copy of K_2 from the components of G_1, we require at least two P_3-adjacencies giving a copy of $K_2 \cup P_4$. We still have $2a-b$ copies of P_5 to obtain and thus $d_{P_3}(G_1, G_2) \ge 2(b-a) + 2a - b = b$. As before G_1 and G_2 differ in exactly a edges and G_2 can be obtained from G_1 by a edge rotations. Therefore, $d_r(G_1, G_2) = a$.

CASE 3. ASSUME THAT $a/2 \le b < a$. Consider the graphs F_1 and F_2 shown in Figure 2 below. It is easy to verify that $d_{P_3}(F_1, F_2) = 1$ while $d_r(F_1, F_2) = 2$. Choose $G_1 = (a-b)F_1 \cup (2b-a)(2P_3)$ and $G_2 = (a-b)F_2 \cup (2b-a)(P_5 \cup K_1)$. Then $d_{P_3}(G_1, G_2) = b$ while $d_r(G_1, G_2) = a$.

Figure 2

■

Let G_1 and G_2 be two graphs of the same order (at least 5) and same size with maximum degree at least 2. Suppose that G_2 can be obtained from G_1 by an edge jump. It is not hard to verify that G_2 can be obtained from G_1 by at most four P_3-adjacencies. Similarly, if G_2 can be obtained from G_1 by a P_3-adjacency, then G_2 can be obtained from G_1 by at most two edge jumps. Hence, we have the following bounds for each of these metrics in terms of the other:

$$\frac{1}{2}d_j(G_1, G_2) \leq d_{P_3}(G_1, G_2) \leq 4d_j(G_1, G_2),$$

or, equivalently,

$$\frac{1}{4}d_{P_3}(G_1, G_2) \leq d_j(G_1, G_2) \leq 2d_{P_3}(G_1, G_2).$$

It is not known whether these bounds are the only restrictions on the jump and P_3-distances.

The next theorem describes conditions under which two graphs of the same order and same size are P_4-connected. The proof of next theorem is long and tedious and can be found in [7].

2.9 Theorem. *Let G_1 and G_2 be two graphs of the same order and the same size. Then G_1 is P_4-connected to G_2 if and only if each of G_1 and G_2 contains a subgraph isomorphic to P_4.*

We have seen that if H is P_3 or P_4, then every two graphs of the same order and same size containing H as a subgraph are H-connected. Although we cannot answer the question in general for $H = P_5$, we can show that every two trees of diameter at least 4 are H-connected by showing that every tree of diameter at least 4 is P_5-connected to a path.

2.10 Theorem. *If T is a tree of order n and diameter d, then T is P_k-connected to P_n for each positive integer k with $3 \leq k \leq d$.*

PROOF. Let k be an integer with $3 \leq k \leq d$. It suffices to show that if $d < n - 1$, then there exists a tree T' of diameter at least $d + 1$ such that T is P_k-connected to T'. Suppose that $d < n - 1$ and let $P : v_0, v_1, \ldots, v_d$ be a longest path in T. Since $d < n - 1$, there is a vertex v_ℓ on P with degree at least 3 where $0 < \ell < d$. Let w be a vertex of T such that w is adjacent to v_ℓ but not on P. We consider two cases.

CASE 1. SUPPOSE THAT $\ell < k - 1$. Then create T' by replacing the path $v_0, v_1, \ldots, v_{k-1}$ with $w, v_{\ell-1}, v_{\ell-2}, \ldots, v_0, v_{\ell+1}, v_{\ell+2}, \ldots, v_{k-1}$, that is, let $T' = T - \{v_\ell v_{\ell-1}, v_\ell v_{\ell+1}\} + \{wv_{\ell-1}, v_0 v_{\ell+1}\}$. Then the path from v_ℓ to v_d in T' has length $d + 1$ and hence T' has diameter at least $d + 1$.

CASE 2. SUPPOSE THAT $\ell \geq k - 1$. Then create T' by replacing the path $v_{\ell-k+1}, v_{\ell-k+2}, \ldots, v_\ell$ by the path $v_{\ell-k+1}, v_{\ell-k+2}, \ldots, v_{\ell-1}, w$, that is, $T' = T - \{v_\ell v_{\ell-1}\} + \{wv_{\ell-1}\}$. Then the path from v_0 to v_d in T' has length $d + 1$ and so T' has diameter at least $d + 1$. ■

Thus if T_1 and T_2 be trees of order p with $d = \min\{\text{diam } T_1, \text{diam } T_2\}$, then T_1 is P_k-connected to T_2 for each positive integer k with $3 \le k \le d$.

3 Other H-Connected Graphs

We have seen that if $H = P_3$ or $H = P_4$, then every pair of graphs of the same order and same size containing H as a subgraph are H-connected. It turns out, however, that if $H = K_3$, then not every pair of graphs of the same order and same size containing H as a subgraph are H-connected.

3.1 Theorem. *Let H be a connected graph such that every vertex has even degree and let G_1 and G_2 be two graphs. If G_1 is H-connected to G_2, then G_1 and G_2 have the same number of odd vertices.*

PROOF. Suppose first that G_1 has k odd vertices and that G_1 and G_2 are H-adjacent. Thus there exist subgraphs H_1 and H_2 of G_1 and G_2, respectively, such that $H_1 \cong H_2 \cong H$ and $G_2 = G_1 - E(H_1) + E(H_2)$. Since every vertex of H_1 has even degree, the graph $G_1 - E(H_1)$ has k odd vertices. Similarly, since every vertex of H_2 has even degree, $G_2 = G_1 - E(H_1) + E(H_2)$ has k odd vertices. Hence, if G_1 and G_2 are H-adjacent, then G_1 and G_2 have the same number of odd vertices. Consequently, if G_1 and G_2 are H-connected, then G_1 and G_2 have the same number of odd vertices. ■

It is not known whether the converse of the implication in the previous theorem is true as well.

We now consider H-adjacency when $H = K_{1,3}$. Consider the graphs G_1 and G_2 of Figure 3. Now $K_{1,4} \cong G_1 - \{v_1v_0, v_1v_2, v_1v_4\} + \{v_2v_0, v_2v_1, v_2v_4\}$ so that G_1 is H-adjacent to $K_{1,4}$. Also, $K_{1,5} \cong G_2 - \{v_1v_0, v_1v_2, v_1v_5\} + \{v_2v_0, v_2v_1, v_2v_5\}$ so that G_2 is H-adjacent to $K_{1,5}$. In fact, we show that every tree of order p is $K_{1,3}$-connected to the star $K_{1,p-1}$.

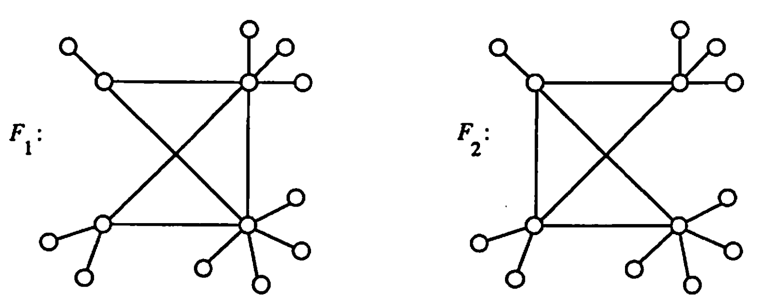

Figure 3

We begin by showing that every tree with maximum degree at least n is $K_{1,n}$-connected to a star.

3.2 Theorem. *Let n be a positive integer such that $n \geq 2$. If T is a tree of order p with $\Delta(T) \geq n$, then T is $K_{1,n}$-connected to $K_{1,p-1}$.*

PROOF. Let $H = K_{1,n}$. If $n = 2$, then the result follows by Theorem 2.6. Thus, assume that $n \geq 3$. If $T \cong K_{1,p-1}$, we are done and thus we assume $T \ncong K_{1,p-1}$. Let x be a vertex of maximum degree in T. Since $T \ncong K_{1,p-1}$, it follows that x has a neighbor t of degree at least 2. Let y and z be two other neighbors of x, s be a neighbor of t, and N be a set of $n-3$ neighbors of x different from t, y, and z. Then we increase the degree of x using the following $K_{1,n}$-adjacencies:

(1) Replace the edge xz with the edge xs.

(2) Replace the edges from x to $\{s, t, y\} \cup N$ with the edges from s to $\{x, y, z\} \cup N$.

(3) Replace the edges from s to $\{t, y, z\} \cup N$ with the edges from y to $\{s, x, z\} \cup N$.

(4) Replace the edges from y to $\{s, x, z\} \cup N$ with edges from x to $\{t, y, z\} \cup N$.

Let T' denote the resulting graph. Since each step above is an H-adjacency, we have that T is H-connected to T' and $\Delta(T') > \Delta(T)$. We continue in this manner, that is let $T = T'$, until $T = K_{1,p-1}$. ■

As a consequence of Theorem 3.2, we have that two trees T_1 and T_2 of order p are $K_{1,n}$-connected if and only if T_1 and T_2 both have max degree at least n. We now turn from trees to hamiltonian graphs.

3.3 Theorem. *If G_1 and G_2 are two nonisomorphic hamiltonian graphs of the same order and same size, then G_1 is $K_{1,3}$-connected to G_2.*

PROOF. Without loss of generality, let $C : v_1, v_2, \ldots, v_p, v_1$ be a hamiltonian cycle in both G_1 and G_2. Since G_1 and G_2 are not isomorphic, we may assume that $p \geq 5$ and that there exist chords v_iv_j and v_kv_ℓ such that $v_iv_j \in E(G_1) - E(G_2)$ and $v_kv_\ell \in E(G_2) - E(G_1)$. Without loss of generality, we may assume $i < j, k < \ell$, and $i < k$. Assume first that $G_2 = G_1 - v_iv_j + v_kv_\ell$.

Suppose that v_iv_j and v_kv_ℓ are adjacent, that is, $j = k$. Since $i < j < \ell$ and v_iv_j and v_jv_ℓ are chords, it follows that $i \neq j-1$ and $l \neq j+1$. Therefore, $G_2 = G_1 - \{v_jv_i, v_jv_{j-1}, v_jv_{j+1}\} + \{v_jv_\ell, v_jv_{j-1}, v_jv_{j+1}\}$; so G_1 and G_2 are $K_{1,3}$-adjacent. Thus we may assume that i, j, k, and ℓ are distinct. If any one of the edges $v_iv_k, v_jv_k, v_iv_\ell$, and v_jv_ℓ is not present in G_1, say v_iv_k, then let

$$F_1 = G_1 - \{v_iv_{i-1}, v_iv_{i+1}, v_iv_j\} + \{v_iv_{i-1}, v_iv_{i+1}, v_iv_k\}.$$

Since $G_2 = F_1 - \{v_kv_{k-1}, v_kv_{k+1}, v_kv_i\} + \{v_kv_{i-1}, v_kv_{k+1}, v_kv_\ell\}$, we have that G_1 and G_2 are $K_{1,3}$-connected. Therefore, $v_iv_k, v_jv_k, v_iv_\ell, v_jv_\ell \in E(G_1)$. Suppose first that $j \neq \ell - 1$ and that $i \not\equiv (\ell + 1)(\mathrm{mod}\, p)$. Let $F_1 = G_1 - \{v_\ell v_i, v_\ell v_j, v_\ell v_{\ell-1}\} + \{v_\ell v_i, v_\ell v_{\ell-1}, v_\ell v_k\}$; so F_1 and G_1 are $K_{1,3}$-adjacent. Then either $v_k \neq v_{j-1}$ or $v_k \neq v_{j+1}$, say $v_k \neq v$, where $v \in N(v_j)$. Hence $G_2 = F_1 - \{v_iv_j, v_jv_k, v_jv\} + \{v_jv_k, v_jv_\ell, v_jv\}$; so F_1 and G_2 are $K_{1,3}$-adjacent, and G_1 and G_2 are $K_{1,3}$- connected. Thus we assume that $j = l - 1$ and that $i \equiv \ell + 1(\mathrm{mod}\, p)$. Hence $i = 1$, $\ell = p$, $j = p - 1$, $v_iv_j = v_1v_{p-1}$, and $v_kv_\ell = v_kv_p$. Since $p \geq 5$, there exists a vertex $v \in N(v_k)$ such that v lies between v_1 and v_k on C, or v lies between v_k and v_{p-1} on C. Let $F_1 = G_1 - \{v_kv_1, v_kv_{p-1}, v_kv\} + \{v_kv, v_kv_p, v_kv_1\}$; so F_1 and G_1 are $K_{1,3}$-adjacent. Then let $G_2 = F_1 - \{v_{p-1}v_p, v_{p-1}v_1, v_{p-1}v_{p-2}\} + \{v_{p-1}v_p, v_{p-1}v_k, v_{p-1}v_{p-2}\}$; so F_1 is $K_{1,3}$-adjacent to G_2. Hence G_1 and G_2 are $K_{1,3}$-connected.

So if G_1 and G_2 are two nonisomorphic hamiltonian graphs of the same order and same size such that $G_2 = G_1 - e + f$ for two edges e and f, then G_1 and G_2 are $K_{1,3}$-connected. Thus if G_1 and G_2 are two nonisomorphic hamiltonian graphs of the same order and the same size, G_1 and G_2 are $K_{1,3}$-connected. ■

Thus every two trees of the same order and with maximum degree at least 3 are $K_{1,3}$-connected as are every two hamiltonian graphs of the same order, same size, and with maximum degree at least 3. For graphs that are not trees and not hamiltonian, it remains to be determined which pairs of these graphs are $K_{1,3}$-connected.

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