Minimum length of cycles through specified vertices in graphs with wide-diameter at most d

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ABSTRACT. Let k be a positive integer and let G be a graph. For two distinct vertices $x, y \in V(G)$, the k-wide-distance $d_k(x, y)$ between x and y is the minimum integer l such that there exist k vertex-disjoint (x, y)-paths whose lengths are at most l. We define $d_k(x, x) = 0$. The k-wide-diameter $d_k(G)$ of G is the maximum value of the k-wide-distance between two vertices of G. In this paper we show that if G is a graph with $d_k(G) \geq 2$ ($k \geq 3$), then there exists a cycle which contains specified k vertices and has length at most $2(k-3)(d_k(G)-1) + \max\left\{3d_k(G), \left\lfloor \frac{18d_k(G)-16}{5} \right\rfloor\right\}$.

1 Introduction

We consider finite undirected graphs possibly with multiple edges but without loops. For a graph G, let $V(G)$ and $E(G)$ be the vertex set of G and the edge set of G, respectively. Dirac [1] proved that a k-connected graph has a cycle through specified k vertices ($k \geq 2$). In [2] Dirac also showed that a 2-connected graph with minimum degree at least d has a cycle of length at least $\min\{2d, |V(G)|\}$. Locke [10] combined these two theorems of Dirac for 2-connected graphs and proved the following theorem: Every two vertices in a 2-connected graph G with minimum degree d lie on a cycle of length at least $\min\{2d, |V(G)|\}$. Egawa, Glas, and Locke [3] generalized this Locke's result for general k-connected graphs. Saito [13] showed some results for the length of the longest cycle through specified vertices.

On the other hand, long paths and cycles may not be useful in some situations, so it is natural to consider only paths and cycles of bounded length. However, few results for the length of the shortest cycle through specified vertices are known. In this paper we study the minimum length of cycles through specified vertices.

Let x, y be vertices of G and let P be an (x, y)-path in G. We denote the length of P by $l(P)$. Write $\mathrm{int}(P) = V(P) - \{x, y\}$. Two paths P and Q are said to be *internally vertex-disjoint* (or simply *vertex-disjoint*) if $\mathrm{int}(P) \cap V(Q) = \emptyset$ and $\mathrm{int}(Q) \cap V(P) = \emptyset$. Let k be a positive integer and let x, y be two distinct vertices of G. A set of k vertex-disjoint (x, y)-paths is said to be a *Menger path system of width k between x and y*. Let $\mathcal{P}_k(x, y)$ be the collection of all Menger path systems of width k between x and y. We define the *k-wide-distance between two distinct vertices x and y* by

$$d_k(x, y) = \min\left\{ \max_{1 \le i \le k} l(P_i) \,\middle|\, \{P_1, P_2, \dots, P_k\} \in \mathcal{P}_k(x, y) \right\}$$

(if $\mathcal{P}_k(x, y) = \emptyset$, then we define $d_k(x, y) = \infty$) and define $d_k(x, x) = 0$. Moreover we define the *k-wide-diameter of G* by

$$d_k(G) = \max\{\, d_k(x, y) \mid x, y \in V(G)\}\,.$$

For example, $d_n(Q_n) = n + 1$ for $n \ge 2$, where Q_n is the n-cube [12]. Note that $d_1(x, y)$ is equal to the distance between x and y, and $d_1(G)$ is equal to the diameter of G. By Menger's theorem [11], G is a k-connected graph if and only if $0 < d_k(G) < \infty$.

The concepts of Menger path system, k-wide-distance, and k-wide-diameter arise quite naturally from the study of transmission delay, reliability, and fault tolerance in interconnection networks for parallel and distributed computer systems. Many sufficient conditions for a graph to have a given k-wide-diameter were investigated by Faudree, Jacobson, Ordman, Schelp, and Tuza [6]. These results were extended in [5] where various combinations of connectivity, minimum degree, sum of degree properties, and neighborhood union conditions implying $d_k(G) \le d$ were studied. Some of these results are summarized in the next theorem, where $\delta(G)$ represents the minimum degree in G, $DC(G) \ge t$ means that the sum of the degrees of each pair of nonadjacent vertices of G is at least t, and $NC(G) \ge t$ implies that the union of the neighborhoods of each pair of nonadjacent vertices of G is at least t.

Theorem A (Faudree, Gould, and Schelp [5]). *Let $d \ge 3$ and k be positive integers, and let G be an m-connected graph of order n. Then $d_k(G) \le d$ if any of the following conditions are satisfied.*

(i) minimum degree condition

$$\delta(G) \begin{cases} \geq \lfloor \frac{n+k}{2} \rfloor & \text{if } m < k \\ > \min \begin{cases} \frac{n-k+2}{\lfloor \frac{d+4}{3} \rfloor} + k - 2 \\ \frac{n-(m-k+1)(d-2)+k-2}{2} \end{cases} & \text{if } k \leq m \leq \frac{n-k}{d} + k - 1 \\ \geq m & \text{if } m > \frac{n-k}{d} + k - 1. \end{cases}$$

(ii) degree sum condition

$$DC(G) \begin{cases} \geq n + k - 2 & \text{if } m < k \\ > \min \begin{cases} 2\left(\frac{n-k+2}{\lfloor \frac{d+4}{3} \rfloor} + k - 2\right) \\ n - (d-3)(m-(k-1)(d-1)) + k \end{cases} & \text{if } k \leq m \leq \frac{n-k}{d} + k - 1 \\ \geq 2m & \text{if } m > \frac{n-k}{d} + k - 1. \end{cases}$$

(iii) neighborhood union condition

$$NC(G) \begin{cases} \geq n - 1 & \text{if } m < k \\ > \min \begin{cases} \frac{5n-k-d-4}{d+2} + 2k - 2 \\ n - 2 - (d-3)(m-(k-1)(d-1)) \end{cases} & \text{if } k \leq m \leq \frac{n-k}{d} + k - 1 \\ \geq m & \text{if } m > \frac{n-k}{d} + k - 1. \end{cases}$$

Let $k \geq 2$ be a positive integer and let $x_1, x_2, \ldots, x_k$ be k distinct vertices of G. Let $c(x_1, x_2, \ldots, x_k)$ denote the minimum integer l such that there exists a cycle which contains $x_1, x_2, \ldots, x_k$ and has length l, namely

$$c(x_1, x_2, \ldots, x_k) = \min\{l(C) | C \text{ is a cycle with } \{x_1, x_2, \ldots, x_k\} \subseteq V(C)\}.$$

(If there is not a cycle through $x_1, x_2, \ldots, x_k$, then we define $c(x_1, x_2, \ldots, x_k) = \infty$.) Furthermore we define

$$c_k(G) = \max\{c(x_1, x_2, \ldots, x_k) | x_1, x_2, \ldots, x_k \in V(G) \text{ (distinct)}\}.$$

Note that $c(x, y) \leq 2d_2(x, y)$ $(x \neq y)$ and $c_2(G) \leq 2d_2(G)$. If $d_k(G) = 1$, then $c_k(G) = k$. In this paper we prove the following theorem.

Theorem 1 *Let $d \geq 2$ and $k \geq 3$ be positive integers. Let G be a graph and let $x_1, x_2, \ldots, x_k$ be k distinct vertices of G. Suppose that $d_k(x_i, x_j) \leq d$ for $1 \leq i, j \leq k$. Then*

$$c(x_1, x_2, \ldots, x_k) \leq 2(k-3)(d-1) + \max\left\{3d, \left\lfloor \frac{18d-16}{5} \right\rfloor\right\}.$$

From Theorem 1, we obtain the following corollary.

Corollary 2 *Let G be a graph with $d_k(G) \geq 2$ $(k \geq 3)$. Then*

$$c_k(G) \leq 2(k-3)(d_k(G)-1) + \max\left\{3d_k(G), \left\lfloor \frac{18d_k(G)-16}{5} \right\rfloor\right\}.$$

From Theorem A and Corollary 2, we can obtain sufficient conditions for a graph to have short cycles through specified vertices.

Corollary 3 *Let $d, k \geq 3$ be positive integers, and let G be an m-connected graph of order n. If it satisfies one of the three conditions* (i), (ii), *and* (iii) *in Theorem A, then $c_k(G) \leq 2(k-3)(d-1) + \max\{3d, \lfloor \frac{18d-16}{5} \rfloor\}$.*

This paper consists of three sections. In Section 2, we show preliminary lemmas to prove Theorem 1. In Section 3, we prove Theorem 1. Note that if we want to prove only the weaker bound $c_k(x_1, x_2, \ldots, x_k) \leq 2(k-1)(d-1)+2$, the proof becomes fairly short; see the remark following the proof of Lemma 7 in Section 2.

2 Preliminaries

Let G be a graph and let $P = (v_0, e_1, v_1, e_2, v_2, \ldots, e_p, v_p)$ be a path of G, where edge e_i joins the vertices v_{i-1} and v_i for $i = 1, 2, \ldots, p$. For $v_i, v_j \in V(P)$, let $P[v_i, v_j]$ denote the subpath between v_i and v_j in P. P is said to be *chordless* if v_i and v_j are not adjacent for i and j differing by more than one. Let $G_1, G_2, \ldots, G_m$ be graphs. We denote the union graph of $G_1, G_2, \ldots, G_m$ by $G_1 \cup G_2 \cup \cdots \cup G_m$, i.e., $V(G_1 \cup G_2 \cup \cdots \cup G_m) = \bigcup_{i=1}^{m} V(G_i)$ and $E(G_1 \cup G_2 \cup \cdots \cup G_m) = \bigcup_{i=1}^{m} E(G_i)$. To prove Theorem 1, we need a few lemmas.

Lemma 4 *Let d be a positive integer. Let G be a graph and let x, y, z be three distinct vertices of G. Suppose that $d_3(x,y), d_3(x,z) \leq d$. Then there exist three vertex-disjoint paths P, Q, R of length at most d such that P is an (x,y)-path, Q is an (x,z)-path, and R is either an (x,y)-path or an (x,z)-path.*

Proof: By the assumption that $d_3(x,y), d_3(x,z) \leq d$, there exist three vertex-disjoint (x,y)-paths P_1, P_2, P_3 whose length are at most d and there exist three vertex-disjoint (x,z)-paths Q_1, Q_2, Q_3 of length at most d. We choose $u_i \in V(P_i) \cap \left\{\bigcup_{j=1}^{3} V(Q_j)\right\}$ so that $V(P_i[y, u_i]) \cap \left\{\bigcup_{j=1}^{3} V(Q_j)\right\} = \{u_i\}$ for $i = 1, 2, 3$. Furthermore, we choose $v_j \in V(Q_j) \cap \left\{\bigcup_{i=1}^{3} V(P_i)\right\}$ so that $V(Q_j[z, v_j]) \cap \left\{\bigcup_{i=1}^{3} V(P_i)\right\} = \{v_j\}$ for $j = 1, 2, 3$. Without loss of generality, we may assume that $l(P_1[y, u_1]) = \min_{i,j \in \{1,2,3\}} \{l(P_i[y, u_i]), l(Q_j[z, v_j])\}$ and $u_1 \in V(P_1) \cap V(Q_1)$. Then we obtain $l(P_1[y, u_1] \cup Q_1[u_1, x]) \leq$

$l(Q_1[z,v_1]) + l(Q_1[u_1,x]) \le l(Q_1) \le d$. Therefore $P_1[y,u_1] \cup Q_1[u_1,x], Q_2, Q_3$ are the desired paths.

Lemma 5 *Let G be a graph and let x_0, x_1, x_2, x_3 be vertices of G ($x_0 \ne x_i$ for $i = 1,2,3$). Suppose that there exist three vertex-disjoint paths P_1, P_2, P_3, where P_i is an (x_0, x_i)-path for $i = 1,2,3$. Furthermore assume that there exists an (x_0, x_1)-path P_0 with $x_2, x_3 \notin V(P_0)$ and $E(P_0) \cap \{\bigcup_{i=1}^{3} E(P_i)\} = \emptyset$. Let $H = P_0 \cup P_1 \cup P_2 \cup P_3$ and let e_1, e_2 be two distinct edges of H incident to x_0. Then H contains two edge-disjoint paths Q_1, Q_2 such that* (i) *Q_i is an (x_1, x_{i+1})-path with $x_0 \in V(Q_i)$ for $i = 1,2$ and* (ii) *either $e_1 \in E(Q_1), e_2 \in E(Q_2)$ or $e_1 \in E(Q_2), e_2 \in E(Q_1)$.*

Proof: Let f_i be the edge of P_i incident to x_0 for $i = 0,1,2,3$. Note that $e_1, e_2 \in \{f_0, f_1, f_2, f_3\}$. We choose $y \in V(P_0) \cap \left\{\bigcup_{i=1}^{3} V(P_i)\right\} - \{x_0\}$ so that $V(P_0[x_0,y]) \cap \left\{\bigcup_{i=1}^{3} V(P_i)\right\} - \{x_0\} = \{y\}$. Let $\alpha, \beta, \gamma \in \{1,2,3\}$ be distinct integers such that $y \in V(P_\alpha)$ and $\beta, \gamma \ne \alpha$. We prove this lemma by induction on $|E(H)|$. If $|E(H)| = 4$, then $y = x_1$, and the conclusion of this lemma is true. Therefore we may assume that $y \ne x_1$. We divide our proof into two cases.

Case 1. $V(P_\alpha[x_0,y]) \cap V(P_0) \ne \{x_0, y\}$. We choose $z \in V(P_\alpha[x_0,y]) \cap V(P_0) - \{x_0, y\}$ so that $V(P_\alpha[x_0,y]) \cap V(P_0[x_1,z]) - \{x_0,y\} = \{z\}$. Let $P'_0 = P_0[x_1,z] \cup P_\alpha[z,x_0]$, $P'_\alpha = P_\alpha[x_\alpha,y] \cup P_0[y,x_0]$, $\{P'_1, P'_2, P'_3\} - \{P'_\alpha\} = \{P_1,P_2,P_3\} - \{P_\alpha\}$, and $H' = P'_0 \cup P'_1 \cup P'_2 \cup P'_3$. Then P'_1, P'_2, P'_3 are mutually vertex-disjoint paths, $x_2, x_3 \notin V(P'_0)$, $E(P'_0) \cap \{\bigcup_{i=1}^{3} E(P'_i)\} = \emptyset$, $e_1, e_2 \in E(H')$, and $|E(H')| < |E(H)|$. Therefore by the induction hypothesis, the result holds in H', and so in H.

Case 2. $V(P_\alpha[x_0,y]) \cap V(P_0) = \{x_0, y\}$. Let H' be the graph obtained from H by contracting $P_\alpha[x_0,y]$ and $P_0[x_0,y]$ to one vertex x'_0. Note that by $y \ne x_1$ and $x_2, x_3 \notin V(P_0)$, $x'_0 \ne x_i$ for $i = 1,2,3$. By the induction hypothesis, H' contains two edge-disjoint paths Q'_1, Q'_2 such that (i') Q'_i is an (x_1, x_{i+1})-path with $x'_0 \in V(Q'_i)$ for $i = 1,2$ and (ii') either $f_\beta \in E(Q'_1), f_\gamma \in E(Q'_2)$ or $f_\beta \in E(Q'_2), f_\gamma \in E(Q'_1)$. Let R_i be the subgraph of H induced by $E(Q'_i)$ for $i = 1,2$. Then either $R_1 \cup P_\alpha[x_0,y], R_2 \cup P_0[x_0,y]$ or $R_1 \cup P_0[x_0,y], R_2 \cup P_\alpha[x_0,y]$ are the desired paths in H.

By using Lemma 5, we can obtain the following corollary.

Corollary 6 *Let G be a graph and let x_0, x_1, x_2, x_3 be vertices of G. Suppose that there exist three vertex-disjoint paths P_1, P_2, P_3, where P_i is an (x_0, x_i)-path for $i = 1,2,3$ and there exists an (x_0, x_1)-path P_0 with $x_2, x_3 \notin V(P_0)$. Then $P_0 \cup P_1 \cup P_2 \cup P_3$ contains an (x_1, x_2)-path Q_1 and an (x_1, x_3)-path Q_2 such that $x_0 \in V(Q_1) \cap V(Q_2)$ and $\sum_{i=1}^{2} l(Q_i) \le \sum_{j=0}^{3} l(P_j)$.*

Lemma 7 *Let d and $k \geq 2$ be positive integers. Let G be a graph and let $x_1, x_2, \ldots, x_k, x_{k+1}$ be $k+1$ distinct vertices of G. Suppose that there exists a cycle C which contains $x_1, x_2, \ldots, x_k$. Furthermore assume that $d_{k+1}(x_i, x_{k+1}) \leq d$ for $i = 1, 2, \ldots, k$. Then there exists a cycle which contains $x_1, x_2, \ldots, x_k, x_{k+1}$ and has length at most $l(C)+2d-\min\left\{2, \left\lceil \frac{l(C)}{k} \right\rceil\right\}$.*

Proof: We may assume that C contains $x_1, x_2, \ldots, x_k$ in this order and $C = P_1 \cup P_2 \cup \cdots \cup P_k$, where P_i is an (x_i, x_{i+1})-path for $i = 1, 2, \ldots, k-1$ and P_k is an (x_k, x_1)-path. Without loss of generality, we may assume that $l(P_1) = \max_{1 \leq i \leq k} l(P_i)$. Then we obtain $l(P_1) \geq \left\lceil \frac{l(C)}{k} \right\rceil$. If $x_{k+1} \in V(C)$, then C is the desired cycle. Therefore we may assume that $x_{k+1} \notin V(C)$. By the assumption that $d_{k+1}(x_1, x_{k+1}) \leq d$, there exist $k+1$ vertex-disjoint (x_1, x_{k+1})-paths $Q_1, Q_2, \ldots, Q_{k+1}$ whose length are at most d. We choose $u_i \in V(Q_i) \cap V(C)$ so that $V(Q_i[x_{k+1}, u_i]) \cap V(C) = \{u_i\}$ for $i = 1, 2, \ldots, k+1$.

Case 1. $u_1, u_2, \ldots, u_{k+1}$ are $k+1$ distinct vertices. In this case, there are integers $\alpha, \beta \in \{1, 2, \ldots, k+1\}$ $(\alpha \neq \beta)$ and $\gamma \in \{1, 2, \ldots, k\}$ such that $u_\alpha, u_\beta \in V(P_\gamma)$. Suppose that u_α is closer to x_γ than u_β on P_γ. Let $C' = P_1 \cup \cdots \cup P_{\gamma-1} \cup P_\gamma[x_\gamma, u_\alpha] \cup Q_\alpha[u_\alpha, x_{k+1}] \cup Q_\beta[x_{k+1}, u_\beta] \cup P_\gamma[u_\beta, x_{\gamma+1}] \cup P_{\gamma+1} \cup \cdots \cup P_k$. Then C' is a cycle which contains $x_1, x_2, \ldots, x_k, x_{k+1}$ and $l(C') \leq l(C) + l(Q_\alpha[u_\alpha, x_{k+1}]) + l(Q_\beta[u_\beta, x_{k+1}]) - l(P_\gamma[u_\alpha, u_\beta]) \leq l(C) + 2d - 2$.

Case 2. Otherwise. In this case, there are two vertex-disjoint (x_1, x_{k+1})-paths R_1, R_2 such that $V(R_i) \cap V(C) = \{x_1\}$ and $l(R_i) \leq d$ for $i = 1, 2$. Moreover by the assumption that $d_{k+1}(x_2, x_{k+1}) \leq d$ and Case 1 of this lemma, we may assume that there are two vertex-disjoint (x_2, x_{k+1})-paths S_1, S_2 such that $V(S_i) \cap V(C) = \{x_2\}$ and $l(S_i) \leq d$ for $i = 1, 2$. Similar arguments in the proof of Lemma 4 shows that $R_1 \cup R_2 \cup S_1 \cup S_2$ contains two vertex-disjoint paths R, S of length at most d such that R is an (x_1, x_{k+1})-path and S is an (x_2, x_{k+1})-path. Let $C' = R \cup S \cup P_2 \cup \cdots \cup P_k$. Then C' is a cycle through $x_1, x_2, \ldots, x_k, x_{k+1}$ and $l(C') \leq l(C) + l(R) + l(S) - l(P_1) \leq l(C) + 2d - \left\lceil \frac{l(C)}{k} \right\rceil$.

Remark. We observe that $d_n(x, y) \leq d_{n+1}(x, y)$ and $c(x, y) \leq 2d_2(x, y)$ $(x \neq y)$. Therefore by Lemma 7, we easily show that if $d_k(x_i, x_j) \leq d$ for $1 \leq i, j \leq k$, where $d, k \geq 2$ and $x_1, x_2, \ldots, x_k$ are k distinct vertices of G, then $c(x_1, x_2, \ldots, x_k) \leq 2(k-1)(d-1) + 2$.

3 Proof of Theorem 1

By Lemma 7 together with the fact $d_n(x, y) \leq d_{n+1}(x, y)$, in order to prove Theorem 1, it suffices to show that the conclusion of Theorem 1 is true for $k = 3$. We verify that $\left\lfloor \frac{7d-5}{2} \right\rfloor \leq \max\left\{3d, \left\lfloor \frac{18d-16}{5} \right\rfloor\right\}$. So, in this section, we prove that if $d_3(x,y), d_3(y,z), d_3(z,x) \le d$, where x,y,z are distinct vertices of G, then the inequality

$$c(x,y,z) \le \max\left\{3d, \left\lfloor \frac{7d-5}{2} \right\rfloor, \left\lfloor \frac{18d-16}{5} \right\rfloor\right\} \tag{1}$$

holds. To prove inequality (1) holds, we show that an average of lengths of some cycles which contain x,y,z is less than or equal to $\max\left\{3d, \lfloor\frac{7d-5}{2}\rfloor, \lfloor\frac{18d-16}{5}\rfloor\right\}$.

Suppose that $d_3(x,y), d_3(y,z), d_3(z,x) \le d$. By Lemma 4, without loss of generality, there exist a chordless (x,y)-path P and two chordless (x,z)-paths Q_1, Q_2 such that they are mutually vertex-disjoint and $l(P), l(Q_1)$, $l(Q_2) \le d$ Among such paths, we choose three vertex-disjoint chordless paths P, Q_1, and Q_2 so that $l(P)$ is minimal possible (if necessary, change labels in x,y, and z). Moreover by the assumption that $d_3(y,z) \le d$, there exist three vertex-disjoint chordless (y,z)-paths R_1, R_2, R_3 with $l(R_i) \le d$ for $i = 1,2,3$. If $x \in V(R_i)$, then $R_i \cup R_j$ is the desired cycle $(i,j \in \{1,2,3\}, i \ne j)$. Therefore we may assume that $x \notin V(R_i)$ for $i = 1,2,3$.

We choose $u \in V(P) \cap \left\{\bigcup_{j=1}^{3} V(R_j)\right\}$ so that $V(P[x,u]) \cap \left\{\bigcup_{j=1}^{3} V(R_j)\right\} = \{u\}$. We choose $v_i \in V(Q_i) \cap \left\{\bigcup_{j=1}^{3} V(R_j)\right\}$ so that $V(Q_i[x,v_i]) \cap \left\{\bigcup_{j=1}^{3} V(R_j)\right\} = \{v_i\}$ for $i = 1,2$. Furthermore, we choose $w_j \in V(R_j) \cap \left\{\bigcup_{i=1}^{2} V(Q_i)\right\}$ so that $V(R_j[y,w_j]) \cap \left\{\bigcup_{i=1}^{2} V(Q_i)\right\} = \{w_j\}$ for $j = 1,2,3$. Let $h_j \in \{1,2\}$ be an integer so that $w_j \in V(Q_{h_j})$ for $j = 1,2,3$, and let $k_i, s_i, t_i \in \{1,2,3\}$ be distinct integers so that $v_i \in V(R_{k_i})$ and $s_i, t_i \ne k_i$ for $i = 1,2$. Then $Q_i[x,v_i] \cup R_{k_i}[v_i,z], R_{s_i}, R_{t_i}$ are mutually vertex-disjoint for $i = 1,2$. Therefore by minimality of $l(P)$,

$$l(P) \le l(Q_i[x,v_i]) + l(R_{k_i}[v_i,z]) \quad (i = 1,2). \tag{2}$$

Claim 8 *If P, R_i, R_j are mutually vertex-disjoint $(i,j \in \{1,2,3\})$, then $c(x,y,z) \le 3d$.*

Proof: Suppose that P, R_i, R_j are mutually vertex-disjoint. Similar arguments in the proof of Lemma 4 shows that $Q_1 \cup Q_2 \cup R_i \cup R_j$ contains two vertex-disjoint paths Q, R of length at most d such that Q is an (x,z)-path and R is a (y,z)-path. Then $P \cup R \cup Q$ is a cycle through x,y,z and $c(x,y,z) \le l(P \cup R \cup Q) \le 3d$.

Claim 9 $c(x,y,z) \le 4d - l(P[u,y]) - l(Q_i[v_i,z])$ *for* $i = 1,2$.

Proof: We may assume that $u \in V(R_1)$. By Claim 8, we may assume that $u \ne y$.

Case 1. Either $k_i = 1$ or $v_i = z$. Suppose that u is closer to y than v_i on R_1. Let $C = P[x,u] \cup R_1[u,y] \cup R_2 \cup R_1[z,v_i] \cup Q_i[v_i,x]$. Then C is a cycle with $\{x,y,z\} \subseteq V(C)$ and

$$\begin{aligned} l(C) &\leq l(P[x,u]) + (l(R_1) - 1) + l(R_2) + l(Q_i[v_i,x]) \\ &\leq (d - l(P[u,y])) + 2d - 1 + (d - l(Q_i[v_i,z])) \\ &= 4d - 1 - l(P[u,y]) - l(Q_i[v_i,z]). \end{aligned}$$

If v_i is closer to y than u on R_1, a similar argument leads us to the conclusion.

Case 2. Otherwise. By symmetry, we may assume that $k_i = 2$ and $v_i \neq z$. Let $C_1 = P[x,u] \cup R_1[u,y] \cup R_3 \cup R_2[z,v_i] \cup Q_i[v_i,x]$ and $C_2 = P[x,u] \cup R_1[u,z] \cup R_3 \cup R_2[y,v_i] \cup Q_i[v_i,x]$. Then C_1 and C_2 are cycles through x, y, z and

$$\begin{aligned} l(C_1) + l(C_2) &= 2\{l(P[x,u]) + l(Q_i[v_i,x]) + l(R_3)\} + l(R_1) + l(R_2) \\ &\leq 8d - 2\{l(P[u,y]) + l(Q_i[v_i,z])\}. \end{aligned}$$

Therefore, in this case, $c(x,y,z) \leq \min\{l(C_1), l(C_2)\} \leq 4d - l(P[u,y]) - l(Q_i[v_i,z])$.

Claim 10 *If $V(R_i[y,w_i]) \cap V(P) = \{y\}$, then $c(x,y,z) \leq \max\left\{3d, \left\lfloor \frac{7d-5}{2} \right\rfloor\right\}$ for $i = 1,2,3$.*

Proof: By symmetry, we may assume that $V(R_1[y,w_1]) \cap V(P) = \{y\}$ and $h_1 = 1$. Let $C_1 = P \cup R_1[y,w_1] \cup Q_1[w_1,z] \cup Q_2$. Then C_1 is a cycle with $\{x,y,z\} \subseteq V(C_1)$.

Case 1. Either $k_1 = 1$ or $v_1 = z$. By inequality (2), we get $l(P) \leq l(Q_1[x,v_1]) + l(R_1[v_1,z]) \leq l(Q_1[x,w_1]) + l(R_1[w_1,z])$. Therefore, in this case, $c(x,y,z) \leq l(C_1) \leq l(R_1) + l(Q_1) + l(Q_2) \leq 3d$.

Case 2. Otherwise. If $l(R_1[y,w_1] \cup Q_1[w_1,z]) \leq d$, then $l(C_1) \leq 3d$. Therefore we may assume that $l(R_1[y,w_1] \cup Q_1[w_1,z]) > d$. Since Q_1 is a chordless path, $l(R_1[y,w_1]) \leq d-2$. Since, in this case, $w_1 \neq v_1$, we have $l(Q_1[w_1,z]) \leq l(Q_1[v_1,z]) - 1$. Therefore we obtain $l(C_1) \leq d + (d-2) + (l(Q_1[v_1,z]) - 1) + d = 3d - 3 + l(Q_1[v_1,z])$. By Claim 8, we may assume that $l(P[u,y]) \geq 2$. Furthermore by Claim 9, there is a cycle C_2 through x, y, z with $l(C_2) \leq 4d - 2 - l(Q_1[v_1,z])$, and we get $l(C_1) + l(C_2) \leq 7d - 5$. Therefore, in this case, $c(x,y,z) \leq \max\{3d, \min\{l(C_1), l(C_2)\}\} \leq \max\left\{3d, \left\lfloor \frac{7d-5}{2} \right\rfloor\right\}$.

Let $A_i = V(P) \cap V(R_i[y,w_i])$ for $i = 1,2,3$. We choose $a \in \bigcup_{i=1}^{3} A_i$ so that $V(P[x,a]) \cap \left\{\bigcup_{i=1}^{3} A_i\right\} = \{a\}$. Without loss of generality, we may assume that $a \in V(R_1)$. By Corollary 6, $P[y,a] \cup R_1[y,a] \cup R_2[y,w_2] \cup$

$R_3[y,w_3]$ contains an (a,w_2)-path S_1 and an (a,w_3)-path S_2 such that $y \in V(S_1) \cap V(S_2)$ and $l(S_1)+l(S_2) \le l(P[y,a])+l(R_1[y,a])+l(R_2[y,w_2])+l(R_3[y,w_3])$. We denote $\widehat{Q_1} = Q_2$ and $\widehat{Q_2} = Q_1$. Let $C_i = P[x,a] \cup S_i \cup Q_{h_{i+1}}[w_{i+1},z] \cup \widehat{Q_{h_{i+1}}}$ for $i = 1,2$. Then C_1 and C_2 are cycles through x,y,z and

$$\begin{aligned} l(C_1)+l(C_2) &\le 2l(P) - l(P[y,a]) + l(R_1[y,a]) \\ &\quad + \sum_{i=2}^{3} \left\{ l(R_i[y,w_i]) + l(Q_{h_i}[w_i,z]) + l(\widehat{Q_{h_i}}) \right\}. \quad (3) \end{aligned}$$

Claim 11 *If $l(R_1[a,z]) \le 1$, then $l(P) \le 2$.*

Proof: Suppose that $l(R_1[a,z]) \le 1$. Then $R_1[z,a] \cup P[a,y], Q_1, Q_2$ are mutually vertex-disjoint. Hence by minimality of $l(P)$, we obtain $l(P) \le l(R_1[z,a]) + l(P[a,y])$ and $l(P[x,a]) \le l(R_1[z,a]) \le 1$. So $P[x,a] \cup R_1[a,z]$, R_2, R_3 are mutually vertex-disjoint. Therefore by minimality of $l(P)$, $l(P) \le l(P[x,a]) + l(R_1[a,z]) \le 2$.

From Claim 10 and 11, we may assume that

$$l(P[y,a]) \ge 3, \quad (4)$$
$$l(R_1[a,z]) \ge 2. \quad (5)$$

Claim 12 *If $l(R_2[w_2,z]), l(R_3[w_3,z]) \le 1$, then $c(x,y,z) \le \max\left\{3d, \lfloor \frac{7d-5}{2} \rfloor\right\}$.*

Proof: Suppose that $l(R_i[w_i,z]) \le 1$ for $i = 2,3$. Since Q_{h_i} is a chordless path, we obtain $l(R_i[y,w_i]) + l(Q_{h_i}[w_i,z]) \le d$ for $i = 2,3$. Therefore by inequalities (3),(4), and (5), $l(C_1) + l(C_2) \le 7d - 5$, and the conclusion of this claim is true.

From Claim 12, we may assume that

$$l(R_2[w_2,z]) \ge 2 \quad \text{or} \quad l(R_3[w_3,z]) \ge 2. \quad (6)$$

Claim 13 *If $k_{h_2} \ne k_{h_3}$, then $c(x,y,z) \le \max\left\{3d, \lfloor \frac{7d-5}{2} \rfloor\right\}$.*

Proof: By symmetry, we may assume that $h_2 = 1$ and $h_3 = 2$. Let $m = l(R_1[y,a]) + \sum_{i=1}^{2} \{l(R_{i+1}[y,w_{i+1}]) - l(Q_i[v_i,w_{i+1}])\}$. By inequalities (2),(3), and (4),

$$\begin{aligned} l(C_1)+l(C_2) &\le \sum_{i=1}^{2} \{l(Q_i[x,v_i]) + l(R_{k_i}[v_i,z])\} - 3 + m \\ &\quad + \sum_{i=1}^{2} \{l(Q_i[v_i,z]) + l(Q_i)\} \\ &= 2\{l(Q_1)+l(Q_2)\} - 3 + m + l(R_{k_1}[v_1,z]) + l(R_{k_2}[v_2,z]). \end{aligned}$$

Let n stand for $m + \sum_{i=1}^{2} l(R_{k_i}[v_i, z])$. If $n \leq 3d - 2$, then $l(C_1) + l(C_2) \leq 7d - 5$ and the conclusion of this claim is true. So we show that $n \leq 3d - 2$.

Case 1. Either $k_1 = 1, v_1 \neq z$ *or* $k_2 = 1, v_2 \neq z$. By symmetry, we may assume that $k_1 = 1$ and $v_1 \neq z$. Then $l(Q_1[v_1, w_2]) \geq 1$ and $l(R_1[a, v_1]) \geq 1$. Hence by the assumption that $k_1 \neq k_2$, $n \leq (l(R_1) - 1) + l(R_2) + l(R_3) - 1 \leq 3d - 2$.

Case 2. Otherwise. By inequality (5) and $k_1 \neq k_2$, $n \leq l(R_1[y, a]) + l(R_2) + l(R_3) \leq 3d - 2$.

Claim 14 *If* $h_2 \neq h_3$, *then the inequality* (1) *holds.*

Proof: By symmetry and by Claim 13, we may assume that $h_2 = 1$, $h_3 = 2$, $k_1 = k_2$, and $v_1, v_2 \neq z$. Similar arguments in the proof of Case 1 of Claim 9 shows that there is a cycle C_3 through x, y, z with $l(C_3) \leq 4d - 1 - l(Q_1[v_1, z]) - l(Q_2[v_2, z])$. Let $m = l(R_1[y, a]) + \sum_{i=1}^{2} \{l(R_{i+1}[y, w_{i+1}]) - l(Q_i[v_i, w_{i+1}])$. Then by inequalities (3) and (4), we obtain $l(C_1) + l(C_2) + l(C_3) \leq 8d - 4 + m$. Arguing as in the proof of Claim 13, we get $l(C_1) + l(C_2) \leq 4d - 3 + m + l(R_{k_1}[v_1, z]) + l(R_{k_1}[v_2, z])$. Therefore,

$$2\{l(C_1) + l(C_2)\} + l(C_3) \leq 12d - 7 + 2m + l(R_{k_1}[v_1, z]) + l(R_{k_1}[v_2, z]).$$

Let n stand for $2m + \sum_{i=1}^{2} l(R_{k_1}[v_i, z])$. If $n \leq 6d - 9$, then $2\{l(C_1) + l(C_2)\} + l(C_3) \leq 18d - 16$ and the conclusion of this claim is true. So we show that $n \leq 6d - 9$. We divide our proof into two cases depending on the value of k_1.

Case 1. $k_1 = 1$. In this case, $l(Q_1[v_1, w_2]), l(Q_2[v_2, w_3]) \geq 1$ and a, v_1, v_2 are distinct vertices. Then by inequality (6), $n \leq (2l(R_1) - 3) + 2\{l(R_2[y, w_2]) + l(R_3[y, w_3]) - 2\} \leq 2d - 3 + 2(2d - 4) = 6d - 11$.

Case 2. $k_1 = 2$ *or* 3. By symmetry, we may assume that $k_1 = 2$. By inequality (2), we obtain $l(P) \leq l(Q_1[x, v_1]) + l(R_2[v_1, z]) \leq l(Q_1[x, w_2]) + l(R_2[w_2, z])$. So if $l(R_3[y, w_3]) + l(Q_2[w_3, z]) \leq d$, then by inequalities (3),(4), and (5), $l(C_1) + l(C_2) \leq 7d - 5$. Therefore we may assume that $l(R_3[y, w_3]) + l(Q_2[w_3, z]) > d$. Since Q_2 is a chordless path, we get $l(R_3[y, w_3]) \leq d - 2$. Furthermore, in this case, $v_2 \neq w_2, w_3$. Hence by inequality (5), $n \leq (2l(R_2) - 1) + 2\{l(R_1[y, a]) + l(R_3[y, w_3]) - 1\} \leq 2d - 1 + 2(2d - 5) = 6d - 11$.

By Claim 14 and inequality (6), we may assume that $h_2 = h_3 = 1$ and $v_1 \neq z$. We choose $b_i \in V(Q_i) \cap \{V(R_{s_1}) \cup V(R_{t_1})\}$ so that $V(Q_i[x, b_i]) \cap \{V(R_{s_1}) \cup V(R_{t_1})\} = \{b_i\}$ for $i = 1, 2$. Moreover we choose $b \in \left\{\bigcup_{i=1}^{2} V(Q_i[x, b_i])\right\} \cap V(R_{k_1})$ so that $\left\{\bigcup_{i=1}^{2} V(Q_i[x, b_i])\right\} \cap V(R_{k_1}[y, b]) = \{b\}$. There are two cases $b \in V(Q_1)$ and $b \in V(Q_2)$. Proofs of these two cases are almost similar, and we show only that if $b \in V(Q_1)$, then the inequality (1) holds. Suppose that $b \in V(Q_1)$. Let $\alpha, \beta \in \{1,2,3\}$ be integers so that $b_2 \in V(R_\alpha)$ and $\beta \neq k_1, \alpha$. Let $C_3 = Q_1[x,b] \cup R_{k_1}[b,y] \cup R_\beta \cup R_\alpha[z,b_2] \cup Q_2[b_2,x]$. Then C_3 is a cycle which contains x, y, z. Let $m = l(R_1[y,a]) + l(R_2[y,w_2]) + l(R_3[y,w_3]) - l(Q_1[v_1,w_2]) - l(Q_1[b,w_3])$. By inequalities (2),(3), and (4),

$$\begin{aligned} & l(C_1) + l(C_2) \\ & \leq 2\{l(Q_1[x,v_1]) + l(R_{k_1}[v_1,z])\} - l(P[y,a]) + m \\ & \quad + l(Q_1[v_1,z]) + l(Q_1[b,z]) + 2l(Q_2) \\ & \leq 2\{l(Q_1) + l(Q_2)\} - l(P[y,a]) + m + 2l(R_{k_1}[v_1,z]) \\ & \leq 4d - 3 + m + 2l(R_{k_1}[v_1,z]) \end{aligned}$$

and

$$\begin{aligned} & l(C_1) + l(C_2) + l(C_3) \\ & \leq l(P) + \{l(Q_1[x,v_1]) + l(R_{k_1}[v_1,z])\} - l(P[y,a]) + m \\ & \quad + l(Q_1[v_1,z]) + l(Q_1[b,z]) + 2l(Q_2) + l(C_3) \\ & \leq l(P[x,a]) + 2\{l(Q_1) + l(Q_2)\} + l(R_{k_1}) + l(R_\beta) + m \\ & \quad + l(R_\alpha[z,b_2]) + l(Q_2[b_2,x]) \\ & \leq 7d - 3 + m + l(R_\alpha[z,b_2]) + l(Q_2[b_2,x]). \end{aligned}$$

Therefore,

$$\begin{aligned} 2\{l(C_1) + l(C_2)\} \quad + l(C_3) \leq & \; 11d - 6 + 2m + 2l(R_{k_1}[v_1,z]) \\ & + l(R_\alpha[z,b_2]) + l(Q_2[b_2,x]). \end{aligned}$$

Let n stand for $2m + 2l(R_{k_1}[v_1,z]) + l(R_\alpha[z,b_2]) + l(Q_2[b_2,x])$. In order to prove the inequality (1) holds, it suffices to show that $n \leq 7d - 10$. We divide our proof into two cases depending on the value of k_1.

Case 1. $k_1 = 1$. By symmetry, we may assume that $\alpha = 2$. In this case, $l(Q_1[v_1,w_2]) + l(Q_1[b,w_3]) \geq 3$. Then by inequality (6), $n \leq 2(l(R_1) - 1) + l(R_2) + \{l(R_2[y,w_2]) + 2l(R_3[y,w_3])\} - 6 + l(Q_2[b_2,x]) \leq 2(d-1) + d + (3d-2) - 6 + d = 7d - 10$.

Case 2. $k_1 = 2$ *or* 3. By symmetry, we may assume that $k_1 = 2$. By similar arguments in the proof of Case 2 of Claim 14, we may assume that $l(R_3[y,w_3]) \leq d - 2$. In this case, $l(Q_1[b,w_3]) \geq 1$. Suppose that $\alpha = 1$. If $b_2 \neq z$, then $l(R_1[y,a]) + l(R_1[z,b_2]) + l(Q_2[b_2,x]) \leq (l(R_1) - 1) + (l(Q_2) - 1) \leq 2d-2$; otherwise, by inequality (5), we obtain $l(R_1[y,a]) + l(Q_2) \leq 2d-2$. Therefore we get $l(R_1[y,a]) + l(R_1[z,b_2]) + l(Q_2[b_2,x]) \leq 2d - 2$. Hence by inequality (5), $n \leq 2l(R_2) + \{l(R_1[y,a]) + l(R_1[z,b_2]) + l(Q_2[b_2,x])\} + \{l(R_1[y,a]) + 2l(R_3[y,w_3])\} - 2 \leq 2d + (2d-2) + 3(d-2) - 2 = 7d - 10$. Similarly, we can show that if $\alpha = 3$, then $n \leq 7d - 10$.

We have completed the proof of Theorem 1.

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