Spanning multipartite tournaments of semicomplete multipartite digraphs

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Abstract

A digraph obtained by replacing each edge of a complete n-partite graph by an arc or a pair of mutually opposite arcs is called a semicomplete n-partite digraph. An n-partite tournament is an orientation of a complete n-partite graph. In this paper we shall prove that a strongly connected semicomplete n-partite digraph with a longest directed cycle C, contains a spanning strongly connected n-partite tournament which also has the longest directed cycle C with exception of a well determined family of semicomplete bipartite digraphs. This theorem shows that many well-known results on strongly connected n-partite tournaments are also valid for strongly connected semicomplete n-partite digraphs.

Key words: Multipartite tournaments, cycles

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1. Terminology and Introduction

All mixed graphs mentioned here are finite without loops. A *mixed graph* D consists of a vertex set $V(D)$ and a set of edges and arcs $E(D)$. A mixed graph without edges is a *digraph*. We denote by $e = uv$ an edge joining

the vertices u and v and by $e = (u, v)$ an arc from u to v. Two vertices are *adjacent* if they are connected by an edge or an arc. An alternating sequence $u_0e_1u_1e_2 \dots u_{k-1}e_ku_k$ of vertices u_i and arcs or edges e_i such that $e_i = u_{i-1}u_i$ or $e_i = (u_{i-1}, u_i)$ for $i = 1, 2, \dots, k$ is called a *u_0-u_k path of length k* in a mixed graph D, if the vertices $u_0, u_1, \dots, u_k$ are pairwise distinct. If $u_0 = u_k$ and the vertices $u_1, u_2, \dots, u_k$ are pairwise distinct, then we speak of a *cycle of length k* or shortly of a *k-cycle*. A mixed graph D is said to be *strongly connected* or just *strong*, if for every pair u, v of vertices, there is a u-v path in D. A mixed graph D is *connected*, if the underlying graph of D is connected. If $D_1, D_2, \dots, D_k$ are the strong components of a mixed grap D, then D_i is called a *source component* (a *sink component*) when there is no arc from $V(D) - D_i$ to D_i (from D_i to $V(D) - D_i$, respectively). The *condensation* D^* of a mixed graph D has the strong components $D_1, D_2, \dots, D_k$ of D as its vertices, with an arc from D_i to D_j if and only if there is at least one arc in D from a vertex of D_i to a vertex of D_j for $i \neq j$. Obviously, the condensation D^* is a digraph without cycles. A digraph D is *k-connected* if for any set A of at most $k - 1$ vertices, the subdigraph $D - A$ is strong. If a digraph contains neither parallel edges nor 2-cycles, it is also called an *oriented graph*. If (u, v) is an arc, then we say that u *dominates* v. If A and B are two disjoint subsets of a digraph D such that every vertex of A dominates every vertex of B, then we say that A *dominates* B, denoted by $A \to B$. If v is a vertex of a digraph D, then we denote by $d^+(v, D)$ and $d^-(v, D)$ the *outdegree* and the *indegree* of v, respectively.
A digraph obtained by replacing each edge of a complete n-partite graph by an arc or a pair of mutually opposite arcs is called *semicomplete n-partite digraph* or *semicomplete multipartite digraph*. An *n-partite* or *multipartite tournament* is an orientation of a complete n-partite graph. If $n = 2$, then we also speak of a *semicomplete bipartite digraph* or *bipartite tournament*, respectively.

Throughout this paper a *spanning subdigraph* D' of a digraph D is a subdigraph with $V(D') = V(D)$ such that two vertices in D' are adjacent if and only if they are adjacent in D.

In the next section we shall prove that a strongly connected semicomplete multipartite digraph with a longest cycle C, contains a spanning strongly connected multipartite tournament which also has the longest cycle C with exception of a well determined family of semicomplete bipartite digraphs. This theorem shows that various results in [1], [4], [5], [6], [7], [8], [9], [10], [11], [12], [16], [17], and [18] on strongly connected multipartite tournaments are also valid for the more general class of strongly connected semicomplete multipartite digraphs.

2. Main Results

First we will present a characterization of strong digraphs having a strong spanning oriented subgraph. This is a simple consequence of a result of Boesch and Tindell [3], which I learned from G. Gutin [13]. For the reason of completeness we will give a new and more structural proof of the theorem of Boesch and Tindell.

Lemma 2.1 Let D be a connected but not strongly connected mixed graph with the strong components $D_1, D_2, \ldots, D_k$. If D contains exactly one source component, say D_1, and exactly one sink component, say D_k, then for every $w \in V(D)$, $u \in V(D_1)$, and $v \in V(D_k)$, there exists a u-w path and w-v path in D, respectively.

Proof. Let D^* be the condensation of D. By the hypothesis, we have $d^-(D_1, D^*) = 0$ and $d^+(D_1, D^*) \geq 1$, $d^+(D_k, D^*) = 0$ and $d^-(D_k, D^*) \geq 1$, and $d^+(x, D^*), d^-(x, D^*) \geq 1$ for $x \neq D_1, D_k$. Using the fact that D^* has no cycle, we conclude that for every $y \in V(D^*)$, the longest y-z path and z-y path in D^* is a y-D_k path and D_1-y path, respectively. This completes the proof. □

Theorem 2.2 (Boesch, Tindell [3] 1980) Let e be an edge of a strongly connected mixed graph D. Then e may be oriented to produce another strongly connected mixed graph if and only if $D - e$ is connected.

Proof. Clearly, if $e = uv$ may be oriented to produce another strong mixed graph, then $D - e$ is connected.
Conversely, let $D - e$ be connected. If $D - e$ is strong, then e may be oriented arbitrarily. Otherwise, let $D_1, D_2, \ldots, D_k$ be the strong components of $D - e$. Since D is strong, there is exactly one source component, say D_1, and exactly one sink component, say D_k. Obviously, $u \in V(D_1)$ and $v \in V(D_k)$ (or vice versa). In view of Lemma 2.1, we see that there is a w-v path and u-w path in $D - e$ for every $w \in V(D)$. If we orient now e to (v, u), then the resulting mixed graph is strong. □

Corollary 2.3 Let D be a strong digraph and uvu a 2-cycle in D. Then at least one of the digraphs $D - (u, v)$ and $D - (v, u)$ is strong if and only if $D - \{(u, v), (v, u)\}$ is connected.

Proof. If we identify the 2-cycle uvu with an edge $e = uv$ in D, then we obtain a strong mixed graph G, and by the hypothesis, $G - e$ is connected. In view of Theorem 2.2, the desired result follows. □

From Corollary 2.3 we can immediately deduce the characterization of strong digraphs having a strong spanning oriented subgraph.

Corollary 2.4 A strongly connected digraph D has a strongly connected spanning oriented subgraph if and only if for every pair of adjacent vertices u and v in D, the deletion of all arcs between u and v leads to a connected subdigraph.

Theorem 2.5 Let D be a strongly connected digraph such that for every pair of adjacent vertices u and v, the deletion of all arcs between u and v leads to a connected subdigraph. If $C_1, C_2, \ldots, C_k$ is collection of pairwise vertex disjoint cycles in D such that $|V(C_i)| \geq 3$ for $i = 1, 2, \ldots, k$, then D contains a strongly connected spanning oriented subgraph with the vertex disjoint cycles $C_1, C_2, \ldots, C_k$.

Proof. If we delete all arcs opposite to the arcs of the cycles $C_1, C_2, \ldots, C_k$ in D, then the hypothesis of the theorem remains valid for the resulting strongly connected spanning subdigraph F. Obviously, each spanning subdigraph of F contains the cycles $C_1, C_2, \ldots, C_k$. Applying Corollary 2.4 on the subdigraph F, we obtain the desired strongly connected spanning oriented subgraph. □

Theorem 2.6 Let D be a strongly connected semicomplete n-partite digraph with a longest cycle C. Then D contains a strongly connected spanning n-partite tournament which also has the longest cycle C if and only if D is not an element of $\mathcal{B}$, where $\mathcal{B}$ denotes the family of semicomplete bipartite digraphs with the partite sets X and Y such that $|X| = 1$ and $X \rightarrow Y \rightarrow X$.

Proof. Certainly, if D is a member of the family $\mathcal{B}$, then D contains no strongly connected spanning bipartite tournament.
Conversely, let $D \notin \mathcal{B}$ be a strongly connected semicomplete n-partite digraph with the longest cycle C. It follows immediately that for every pair of adjacent vertices u and v in D, the deletion of all arcs between u and v leads to a connected subdigraph. Hence, according to Corollary 2.4, there exists a strongly connected spanning n-partite tournament D' of D. Since D' has no 2-cycles, the longest cycle of D' has length at least three, and thus we conclude that $|V(C)| \geq 3$. Applying now Theorem 2.5 on D with $k = 1$ and $C_1 = C$, we find the desired spanning strong n-partite tournament of D which also has the longest cycle C. □

Corollary 2.7 Let D be a strongly connected semicomplete multipartite digraph. The longest cycle of D has length two if and only if $D \in \mathcal{B}$.

Theorem 2.6 furthermore shows that the next conjecture of Bang-Jensen and Gutin [2] (cf. Conjecture 14.2) is valid for $k = 1$.

Conjecture 2.8 (Bang-Jensen, Gutin [2] 1998) Every $2k$-connected semicomplete multipartite digraph contains a spanning k-connected multipartite tournament.

Corollary 2.9 Let D be a strongly connected semicomplete n-partite digraph with a collection of pairwise vertex disjoint cycles $C_1, C_2, \ldots, C_k$ such that $|V(C_i)| \geq 3$ for $i = 1, 2, \ldots, k$. Then D contains a strongly connected spanning n-partite tournament with the vertex disjoint cycles $C_1, C_2, \ldots, C_k$.

Next we give a selection of known results on multipartite tournaments, which by Theorem 2.6 or Corollary 2.9 are also valid for semicomplete multipartite digraphs.

Theorem 2.10 (Bondy [4] 1976) Each strongly connected n-partite tournament with $n \geq 3$ contains an m-cycle for each $m \in \{3, 4, \ldots, n\}$.

Theorem 2.11 (Balakrishnan, Paulraja [1] 1984) Let D be a strongly connected n-partite tournament. If D has an n-cycle which visits at most $n-1$ partite sets, then D also contains an $(n+1)$-cycle.

Theorem 2.12 (Gutin [10] 1984) Let D be a strongly connected n-partite tournament with the partite sets $V_1, V_2, \ldots, V_n$ such that $n \geq 5$. If $|V_i| \geq 2$ for all $i = 1, 2, \ldots, n$, then D has an $(n+1)$-cycle or an $(n+2)$-cycle.

Theorem 2.13 (Goddard and Oellermann [6] 1991) Let D be a strongly connected n-partite tournament such that $n \geq 3$. Then, every vertex of D belongs to a cycle that contains vertices from exactly q partite sets for each $q \in \{3, 4, \ldots, n\}$.

Theorem 2.14 (Goddard and Oellermann [6] 1991) Every strongly connected n-partite tournament contains at least $n-2$ cycles of length 3.

Theorem 2.15 (Gutin [11] 1993) Let D be a strongly connected n-partite tournament with $n \geq 3$ such that one partite set consists of a single vertex v. Then, for each $m \in \{3, 4, \ldots, n\}$, there exists an m-cycle of D containing the vertex v.

Theorem 2.16 (Guo, Pinkernell, Volkmann [7] 1997) Every vertex of a strongly connected n-partite tournament is contained in a longest cycle.

Theorem 2.17 (Yeo [18]) Let D be a strongly connected n-partite tournament with $n \geq 3$. Then, D contains a pancyclic subdigraph of order k for all $k = 3, 4, \dots, n$.

3. Open Problems

Clearly, every strong semicomplete multipartite digraph D contains a spanning multipartite tournament T such that a longest path in T has the same length as the longest path in D. But does there exist such a spanning strong multipartite tournament?

Problem 3.1 Let D be a strong semicomplete multipartite digraph with a longest path of length p. Does there exist a spanning strong multipartite tournament with a longest path of length p?

Definition 3.2 Let P be a property of a strongly connected semicomplete multipartite digraph D. Then P is said to be *hereditary* if there exists a spanning strongly connected multipartite tournament of D with property P.

Problem 3.3 Are there further hereditary properties of strongly connected semicomplete multipartite digraphs?

Conjecture 3.4 Let D be a strong semicomplete multipartite digraph with a longest cycle of length k and a longest path of length p. Then $p \leq 2k - 2$.

The next example of Bondy [4] will show that the estimation of Conjecture 3.4 would be best possible.

Let $A_1, A_2, \dots, A_n$ be the partite sets of a strong n-partite tournament H with $A_1 = \{a_1\}$ and $|A_i| \geq 2$ for $2 \leq i \leq n$. If $A_2 \rightarrow a_1$, $a_1 \rightarrow A_i$ for $3 \leq i \leq n$, and $A_j \rightarrow A_i$ for $2 \leq i < j \leq n$, then H is strong with a longest cycle of length n and with a longest path of length $2n - 2$.

In the meantime, Gutin and Yeo [14] have confirmed Conjecture 3.4 in affirmative.

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