K-dominating sets of $P_{2k+2} \times P_n$ and $P_m \times P_n$

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February 10, 1999

Abstract. In this paper we determine the k-domination number γ_k of $P_{2k+2} \times P_n$ and $\lim_{m,n \longrightarrow \infty} \frac{\gamma_k(P_m \times P_n)}{mn}$.

1 Introduction and terminology.

For any graph G we denote the vertex-set and edge-set of G by $V(G)$ and $E(G)$, respectively. A subset $D \subset V(G)$ is called a *k-dominating set*, $k \geq 2$, if for every vertex y not in D, there exists at least one vertex $x \in D$, such that $d(x, y) \leq k$. For convenience we also say that D *k-dominates* G. The *k-domination number* $\gamma_k(G)$ is the cardinality of a smallest k-dominating set.

The cardinal product of two graphs G and H is the graph $G \times H$ with $V(G \times H) = V(G) \times V(H)$ and $((g_1, h_1), (g_2, h_2)) \in E(G \times H)$, if and only if $(g_1, g_2) \in E(G)$ and $(h_1, h_2) \in E(H)$.

The problem of determining the domination numbers of graphs first occurs in a paper of de Jaenisch [1]. He wanted to find the minimal number of queens on a chessboard, such that every square is either occupied by a queen or can be reached by a queen with a single move.

The study of domination numbers of products of graphs was initiated by Vizing [15]. He conjectured that the domination number of the cartesian product of two graphs is always greater than or equal to the product of the domination numbers of the two factors; a conjecture which is still unproven. For cardinal products of graphs this conjecture does not hold as was shown in [11].

Starting in the eighties the domination numbers of cartesian products were intensively investigated (see e. g. [2], [3], [4], [6], [7], [10]). Recently papers on domination numbers of cardinal products of graphs were published. We refer the interested reader to [5], [8], [9], [11], [12], [13], [14].

In [14] we have shown the following results:

Theorem 1 *If P_n is the path of order n, $n \geq 3, k \geq 1$, then*

$$\gamma_k(P_2 \times P_n) = \cdots = \gamma_k(P_{2k} \times P_n) = 2 \cdot \lceil \frac{n}{2k+1} \rceil.$$

$$\gamma_k(P_{2k+1} \times P_n) = \begin{cases} 2 \cdot \lfloor \frac{n}{2k} \rfloor + 1 & n \equiv 1 (mod 2k) \\ 2 \cdot \lceil \frac{n}{2k} \rceil & otherwise \end{cases}$$

Continuing the investigations of Theorem 1 we determine $\gamma_k(P_{2k+2} \times P_n)$ in this paper. For the general case we show that $\lim_{m,n \longrightarrow \infty} \frac{\gamma_k(P_m \times P_n)}{mn} = \frac{1}{\lceil \frac{4k^2+4k+1}{2} \rceil}$ holds (see section 3 of this paper).

Definition 1 *Let $1, ..., m$ and $1, ..., n$ be the vertices of P_m and P_n, respectively. Then the vertices of $P_m \times P_n$ are denoted by (i, j) where $i = 1, ..., m$ and $j = 1, ..., n$.*

Observation 1 *The cardinal product $P_m \times P_n$, $m, n \geq 3$, consists of two components. If both, m and n are odd, these components are not isomorphic. If at least one of these two numbers is even, the components are isomorphic.*

Definition 2 *By C_1 we denote the component which contains the vertex (1,1), by C_2 the other component.*

Definition 3 *For a fixed m, $1 \leq m \leq n$, the set $(P_k)_m = \{(i, m)|i = 1, ..., k\}$ is called a column of $P_k \times P_n$. The set $(P_n)_m = \{(m, j)|j = 1, ..., n\}$ is called a row of $P_k \times P_n$.*

A set $B = \{(P_k)_m, (P_k)_{m+1}, ..., (P_k)_{m+l}, |l \geq 0, m \geq 1, m + l \leq n\}$, of columns is called a block of size $k \times (l + 1)$ of $P_k \times P_n$.

If another block B_1 contains the column $(P_k)_{m-1}$ or the column $(P_k)_{m+l+1}$, then we say that B_1 is adjacent to B. A block B is called internal, if it is adjacent to two other blocks, it is called external if it is only adjacent to one block.

2 The k-domination number of $P_{2k+2} \times P_n$.

Theorem 2 *For $k \geq 1$ and $n \geq 3$,*

$$\gamma_k(P_{2k+2} \times P_n) = \begin{cases} 4 \cdot \lfloor \frac{n}{2k+2} \rfloor + 2 & n \equiv 1(mod(2k+2)) \\ 4 \cdot \lceil \frac{n}{2k+2} \rceil & otherwise \end{cases}$$

Proof. We denote a k-dominating set of C_1 by S_1 and k-dominating set of C_2 by S_2. Let $S = S_1 \cup S_2$ where

$S_1 = \{(k+1, k+1+(2k+2)m), (k+2, k+2+(2k+2)m); m = 0, 1, ..., \lceil \frac{n}{2k+2} \rceil - 1\}$ and $S_2 = \{(k+1, k+2+(2k+2)m), (k+2, k+1+(2k+2)m); m = 0, 1, ..., \lceil \frac{n}{2k+2} \rceil - 1\}$. It is easy to see that $|S| = 4\lceil \frac{n}{2k+2} \rceil$, and that for $n \equiv 0 mod(2k+2)$ $n \equiv (2k+1) mod(2k+2)$, $\cdots$, $n \equiv (k+2) mod(2k+2)$ S is a k-dominating set.

For $n \equiv (k+1)(mod(2k+2))$,...,$n \equiv 2(mod(2k+2))$ we distinguish between even and odd k. If k is odd

$S_1 = \{(k+1, k+1+(2k+2)m), (k+2, k+2+(2k+2)m); m = 0, 1, ..., \lfloor \frac{n}{2k+2} \rfloor - 1\} \cup \{(k+1, n), (k+2, n-1)\}$ and $S_2 = \{(k+1, k+2+(2k+2)m), (k+2, k+1+(2k+2)m); m = 0, 1, ..., \lfloor \frac{n}{2k+2} \rfloor - 1\} \cup \{(k+1, n-1), (k+2, n)\}$

are k-dominating sets of C_1 and C_2 respectively.

For k even the k-dominating sets are almost the same. In S_1 we take the vertices (k+1,n-1),(k+2,n), instead of the vertices (k+1,n),(k+2,n-1) and in S_2 we take the vertices (k+1,n),(k+2,n-1) instead of the vertices (k+1,n-1), (k+2,n).

The set S is k-dominating, and $|S| = 4\lceil \frac{n}{2k+2} \rceil$.

Let $n \equiv 1(mod(2k+2))$. Then $S_1 = \{(k+1, k+1+(2k+2)m), (k+2, k+2+(2k+2)m); m = 0, 1, ..., \lfloor \frac{n}{2k+2} \rfloor - 1\} \cup \{(k+1, n-k)\}$ and $S_2 = \{(k+1, k+2+(2k+2)m), (k+2, k+1+(2k+2)m); m = 0, 1, ..., \lfloor \frac{n}{2k+2} \rfloor - 1\} \cup \{(k+2, n-k)\}$.

Such a set $S = S_1 \cup S_2$ is k-dominating, and $|S| = 4\lfloor \frac{n}{2k+2} \rfloor + 2$.

In the sequel we prove that $\gamma_k(P_{2k+2} \times P_n) \geq |S|$.

Since $P_{2k+2} \times P_n$ consists of two isomorphic components, all considerations are done for only one component, namely C_2. We partition the graph $P_{2k+2} \times P_n$ into $(2k+2) \times (2k+2)$ blocks. If a block is external, we denote it by R, if it is internal we denote it by M.

Lemma 1 *$|D \cap R| \geq 2$, for every dominating set D.*

Proof. Without loss of generality let $R = \{(P_{2k+2})_1, ..., (P_{2k+2})_{2k+2}\}$. K-dominating vertices from the adjacent blocks can at most k-dominate $(P_{2k+2})_{k+3}$, ...,$(P_{2k+2})_{2k+2}$ in R. To k-dominate the remaining vertices of the $(2k+2) \times (k+2)$ block $(P_{2k+2})_1, ..., (P_{2k+2})_{k+2}$, we need at least 2 vertices. ∎

Lemma 2 *$|D \cap M| \geq 2$, for every dominating set D.*

Proof. K-dominating vertices from the adjacent blocks can at most k-dominate the first k and the last k columns in M. To k-dominate the remaining vertices on the block $(2k+2) \times 2$, we need at least 2 vertices. ∎

Lemma 3 *A $(2k+2) \times (2k+2)$ block is k-dominated by a set S^* of 2 vertices if and only if S^* contains exactly the respective vertices of S_2.*

Proof. Without loss of generality we consider the block $R = \{(P_{2k+2})_1, ..., (P_{2k+2})_{2k+2}\}$.

To k-dominate the vertex (1,2k+2) we have many possibilities. The best solution is to take the vertex (k+1,k+2). This vertex k-dominates $\lceil\frac{(2k+1)\times(2k+1)}{2}\rceil$ vertices of R. (One vertex can at most k-dominate $\lceil\frac{(2k+1)\times(2k+1)}{2}\rceil$ vertices.)

Only the last row and the first column of the $(2k+2)\times(2k+2)$ block are not k-dominated.

Only the vertex (k+2,k+1) k-dominates all these vertices.(see Figure 1. for k=3)■

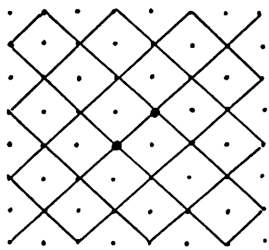

Figure 1.

Case 1. $n \equiv 0 mod(2k+2)$

If $n \equiv 0 mod(2k+2)$ it follows from Lemma 1 and Lemma 2 that on one component $|D| \geq 2\cdot\frac{n}{2k+2}$ holds for any k-dominating set D. Then on both components

$$|D| \geq 4\frac{n}{2k+2} = |S|.$$

Case 2. $n \equiv 1\ mod(2k+2)$

We partition the graph $P_{2k+2}\times P_n$ into $(2k+2)\times(2k+2)$ blocks $B_1,\ldots,$ $B_{\lfloor\frac{n}{2k+2}\rfloor}$, where $B_{\lfloor\frac{n}{2k+2}\rfloor} = \{(P_{2k+2})_{n-2k-2},\cdots,(P_{2k+2})_{n-1}\}$ and one block $R' = \{(P_{2k+2})_n\}$.

Lemma 4 *If* $|D\cap R'| = 0$, *then there exists at least one block* $B \in \{B_1,\cdots,$ $B_{\lfloor\frac{n}{2k+2}\rfloor-1}\}$ *such that* $|D\cap B| \geq 3$.

Proof. a) $n = 2k+3$ (we have only one block B of size $(2k+2)\times(2k+2)$).

From Lemma 3 it follows that the optimal solution is if vertices (k+1,k+2) and (k+2,k+1) are in D.

But then $(P_{2k+2})_n$, is not k-dominated. To k-dominate these vertices we need at least one additional vertex. So

$$|D\cap B| \geq 3.$$

b) $n > 2k+3$

Then we have two or more $(2k+2)\times(2k+2)$ blocks.

$B_{\lfloor\frac{n}{2k+2}\rfloor}\cup R'$ is a $(2k+2)\times(2k+3)$ block. If $|D\cap B_{\lfloor\frac{n}{2k+2}\rfloor}| = 2$ holds, (and $|D\cap R'| = 0$) it follows (cf. Lemma 3), that at least the first column on $B_{\lfloor\frac{n}{2k+2}\rfloor}$ must be k-dominated by vertices of $B_{\lfloor\frac{n}{2k+2}\rfloor-1}$.

If $|D \cap B_{\lfloor \frac{n}{2k+2} \rfloor -1}| = 2$ holds, then we have the same situation as before. Then at least the first column on $B_{\lfloor \frac{n}{2k+2} \rfloor -1}$ must be k-dominated by vertices of $B_{\lfloor \frac{n}{2k+2} \rfloor -2}$.

If for all blocks B_i, $i = 1, ..., \lfloor \frac{n}{2k+2} \rfloor$, $|D \cap B_i| = 2$ holds, it follows that at least the first column on B_1 is not k-dominated. To k-dominate these vertices we need at least one additional vertex.

Then $|D \cap B_1| \geq 3$ holds.■

From Lemmas 1, 2 it follows that on one component on $\lfloor \frac{n}{2k+2} \rfloor$ blocks of size $(2k+2) \times (2k+2)$ we have at least $2\lfloor \frac{n}{2k+2} \rfloor$ k-dominating vertices.

Lemma 4 implies that if $|D \cap R'| = 0$, then at least 1 block B exists, such that $|D \cap B| \geq 3$.

Then on one component $|D| \geq 2\lfloor \frac{n}{2k+2} \rfloor + 1 = |S_2|$ holds. Therefore

$$\gamma(P_{2k+2} \times P_n) = 4 \cdot \lfloor \frac{n}{2k+2} \rfloor + 2 \quad if \quad n \equiv 1(mod(2k+2))$$

Case 3. $n \equiv 2 mod(2k+2)$

We first assume that $n > 2k+4$. We partition the graph $P_{2k+2} \times P_n$ into $\lfloor \frac{n}{2k+2} \rfloor$ $(2k+2)\times(2k+2)$ blocks $B_1,...,B_{\lfloor \frac{n}{2k+2} \rfloor}$, where $B_{\lfloor \frac{n}{2k+2} \rfloor} = \{(P_{2k+2})_{n-2k-3}, \cdots, (P_{2k+2})_{n-2}\}$. There remains one $(2k+2)\times 2$ block $R' = \{(P_{2k+2})_{n-1}, (P_{2k+2})_n\}$.

Lemma 5 *For every k-dominating set D,* $|D \cap (B_{\lfloor \frac{n}{2k+2} \rfloor} \cup R')| \geq 2$ *holds.*

Proof. Vertices from $B_{\lfloor \frac{n}{2k+2} \rfloor -1}$ can at most k-dominate vertices of the columns $(P_{2k+2})_{n-2k-3}, ..., (P_{2k+2})_{n-k-4}$. To k-dominate the vertices of the remaining k+4 columns we need at least 2 vertices which are contained in $B_{\lfloor \frac{n}{2k+2} \rfloor} \cup R'$.■

Lemma 6 *If* $|D \cap (B_{\lfloor \frac{n}{2k+2} \rfloor} \cup R')| = 2$, *then there exists at least one block* $B \in \{B_1, \cdots, B_{\lfloor \frac{n}{2k+2} \rfloor -1}\}$ *such that* $|D \cap B| \geq 4$, *or at least two blocks* B_i, $B_j \in \{B_1, \cdots, B_{\lfloor \frac{n}{2k+2} \rfloor -1}\}$ *such that* $|D \cap B_i| \geq 3$ *and* $|D \cap B_j| \geq 3$.

Proof. Let $|D \cap (B_{\lfloor \frac{n}{2k+2} \rfloor} \cup R')| = 2$. Vertices from $B_{\lfloor \frac{n}{2k+2} \rfloor -1}$ can at most k-dominate the first k columns on $B_{\lfloor \frac{n}{2k+2} \rfloor}$. It follows that these 2 vertices must k-dominate all vertices on the k+4 columns $(P_{2k+2})_{n-k-3}, ..., (P_{2k+2})_n$.

The best solution is if $(k+1, n-k) \in D$, because this vertex k-dominates all vertices on a $(2k+1)\times(2k+1)$ block. Then the last row and the first three column on $B_{\lfloor \frac{n}{2k+2} \rfloor} \cup R'$ are not k-dominated. Only if the vertex $(k+2, n-k-1) \in D$, all vertices on $(P_{2k+2})_{n-2k-1}, ..., (P_{2k+2})_n$ are k-dominated (cf. Lemma 3)

For this case only the first two columns on $B_{\lfloor \frac{n}{2k+2} \rfloor}$ must be k-dominated by vertices from $B_{\lfloor \frac{n}{2k+2} \rfloor -1}$.

If $|D \cap B_{\lfloor \frac{n}{2k+2} \rfloor -1}| = 2$, then we have the same situation as in previous case. Then at least the first two columns on $B_{\lfloor \frac{n}{2k+2} \rfloor -1}$ must be k-dominated by vertices from $B_{\lfloor \frac{n}{2k+2} \rfloor -2}$.

If for all B_i, $1 \le i < \lfloor \frac{n}{2k+2} \rfloor$, $|D \cap B_i| = 2$ holds, then it follows that at least the first two columns on B_1 are not k-dominated.

To k-dominate these columns we need at least two more vertices. Then $|D \cap B_1| \ge 4$ holds.

Let $|D \cap (B_{\lfloor \frac{n}{2k+2} \rfloor} \cup R')| = 2$ and (w.l.o.g.) $|D \cap B_{\lfloor \frac{n}{2k+2} \rfloor - 1}| = 3$. It follows from the previous discussion that vertices from $B_{\lfloor \frac{n}{2k+2} \rfloor - 1}$ must at least k-dominate the first two columns of $B_{\lfloor \frac{n}{2k+2} \rfloor}$.

$B_{\lfloor \frac{n}{2k+2} \rfloor - 1} \cup \{(P_{2k+2})_{n-2k-3}, (P_{2k+2})_{n-2k-2}\}$ is a $(2k+2) \times (2k+4)$ block. Three vertices cannot k-dominate all vertices on this block. This holds because two vertices can at most k-dominate a $(2k+2) \times (2k+2)$ block. To k-dominate the remaining $(2k+2) \times 2$ block we need at least two more vertices.

Hence at least one vertex on $B_{\lfloor \frac{n}{2k+2} \rfloor - 1}$ is k-dominated by vertices of $B_{\lfloor \frac{n}{2k+2} \rfloor - 2}$.

If for all B_i, $1 \le i < \lfloor \frac{n}{2k+2} \rfloor - 1$, $|D \cap B_i| = 2$, holds, then at least one vertex of B_1 is not k-dominated. To k-dominate this vertex we need at least one additional vertex. Then $|D \cap B_1| \ge 3$ holds. ■

Lemma 7 *If* $|D \cap (B_{\lfloor \frac{n}{2k+2} \rfloor} \cup R')| = 3$, *then there exists at least one block* B *such that* $|D \cap B| \ge 3$.

Proof. $B_{\lfloor \frac{n}{2k+2} \rfloor} \cup R'$ is a $(2k+2) \times (2k+4)$ block. Three vertices cannot k-dominate all vertices on this block (By the same argument as in the proof of Lemma 6.) It follows that at least one verex of $B_{\lfloor \frac{n}{2k+2} \rfloor}$ must be k-dominated by vertices of $B_{\lfloor \frac{n}{2k+2} \rfloor - 1}$. If $|D \cap B_{\lfloor \frac{n}{2k+2} \rfloor - 1}| = 2$ holds, then it follows from Lemma 3 that at least one vertex of $B_{\lfloor \frac{n}{2k+2} \rfloor - 1}$ must be k-dominated by vertices of $B_{\lfloor \frac{n}{2k+2} \rfloor - 2}$. If for all B_i, $1 \le i < \lfloor \frac{n}{2k+2} \rfloor - 1$, $|D \cap B_i| = 2$ holds, then at least one vertex of B_1 is not k-dominated. To k-dominate this vertex we need at least one additional vertex. Then $|D \cap B_1| \ge 3$ holds. ■

From Lemma 1 and Lemma 2 it follows that on the first $\lfloor \frac{n}{2k+2} \rfloor - 1$ blocks there are at least $2(\lfloor \frac{n}{2k+2} \rfloor - 1)$ k-dominating vertices. From this together with Lemmas 5, 6 and 7 it follows, that for $n \equiv 2 mod(2k+2)$

$$|D| \ge 2(\lfloor \frac{n}{2k+2} \rfloor - 1) + 4 = 2\lceil \frac{n}{2k+2} \rceil = |S_2|.$$

If $|D \cap (B_{\lfloor \frac{n}{2k+2} \rfloor} \cup R')| \ge 4$, then also

$$|D| \ge 2(\lfloor \frac{n}{2k+2} \rfloor - 1) + 4 = 2\lceil \frac{n}{2k+2} \rceil = |S_2|.$$

Let $n = 2k+4$. Similarily to the proof of Lemma 6 it follows that $|D| \ge 4 = |S_2|$.

For $n \equiv 3 mod(2k+2)$,...,$n \equiv (2k+1) mod(2k+2)$ our graph always contains a subgraph H isomorphic to $P_{2k+2} \times P_n$ for some $n \equiv 2 mod(2k+2)$. In fact only the external block R' contains more than two columns. From the above considerations it is obvious that a minimal k-dominating set contains at least the same number of vertices as a minimal k-dominating set of H, which completes the proof. ■

Remark. It can be easily deduced from Lemma 3 that there exists no independent k-domination set of $P_{2k+2} \times P_n$ which has the same cardinality as S.

3 Determining $\lim_{m,n \longrightarrow \infty} \frac{\gamma_k(P_m \times P_n)}{mn}$

Theorem 3 *For any two paths P_m, P_n,*

$$\lim_{m,n \longrightarrow \infty} \frac{\gamma_k(P_m \times P_n)}{mn} = \frac{1}{\lceil \frac{4k^2+4k+1}{2} \rceil}.$$

Proof. We follow the ideas used in [7] for an cartesian products of paths and k=1.

One vertex can at most k-dominate all vertices of a $(2k+1) \times (2k+1)$ block. On one component a $(2k+1) \times (2k+1)$ block contains at most $\lceil \frac{4k^2+4k+1}{2} \rceil$ vertices.

We consider the set $H = \{(i,j) | \ j \equiv (2k+1)i \ \ mod \lceil \frac{4k^2+4k+1}{2} \rceil \}$. H contains $\lceil \frac{nm}{\lceil \frac{4k^2+4k+1}{2} \rceil} \rceil$ vertices.

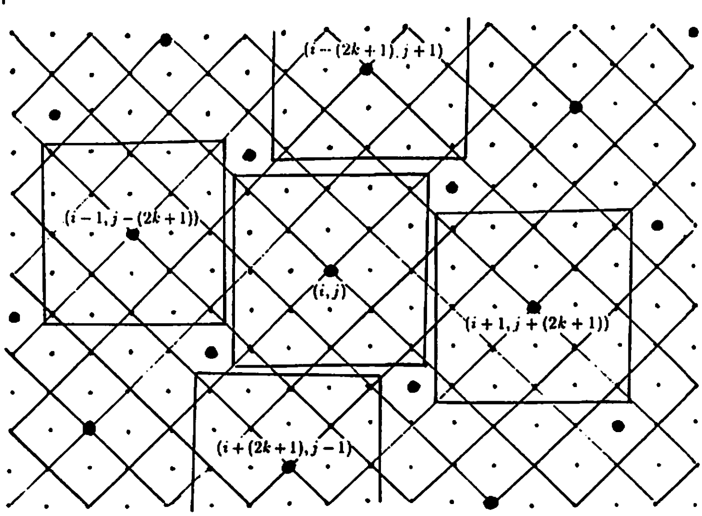

Figure 2. (k=2)

We make all considerations for C_1. A vertex (i, j) is in C_1 if i and j are both even or both odd.

We take $(i, j) \in H$. ($j \equiv (2k+1)i \ mod\lceil \frac{4k^2+4k+1}{2} \rceil$) This vertex can k-dominates all vertices at distance $\leq k$.

We show that all vertices at distance k+1 from the vertex $(i, j) \in H$ are k-dominated by vertices of H or vertices of the kind $(1, r), (m, r), (s, 1), (s, n), 1 \leq r \leq n$, $1 \leq s \leq m$.

The vertices at distance k+1 from (i, j) are
$\{(i-k-1, j-k+1), \cdots, (i-k-1, j+k+1),$
$(i-k-1, j-k-1), \cdots, (i+k-1, j-k-1),$
$(i+k+1, j-k-1), \cdots, (i+k+1, j+k-1),$
$(i-k+1, j+k+1), \cdots, (i+k+1, j+k+1).\}$

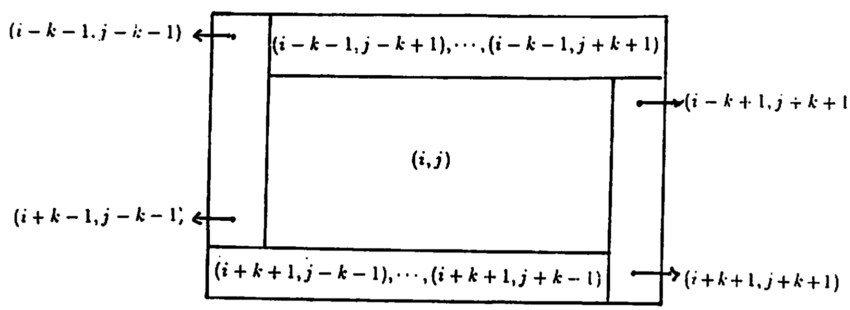

Figure 3.

If $2k+2 \leq i \leq m-(2k+2)$ and $2k+2 \leq j \leq n-(2k+2)$ holds, then the vertices $(i-(2k+1), j+1)$, $(i-1, j-(2k+1))$, $(i+(2k+1), j-1)$ and $(i+1, j+(2k+1))$ are in H too.

If $(i, j) \in H$ then also $(i-1, j-(2k+1)) \in H$ since $j \equiv (2k+1)i mod\lceil \frac{4k^2+4k+1}{2} \rceil$ implies that $j-(2k+1) \equiv (2k+1)(i-1) mod\lceil \frac{4k^2+4k+1}{2} \rceil$. Similarly we get that also the other three vertices $(i-(2k+1), j+1)$, $(i+(2k+1), j-1)$ and $(i+1, j+(2k+1))$ are in H.

Then the vertex $(i-(2k+1), j+1)$ k-dominates all vertices $\{(i-k-1, j-k+1), \cdots, (i-k-1, j+k+1)\}$.

The vertex $(i-1, j-(2k+1))$ k-dominates all vertices $\{(i-k-1, j-k-1), \cdots, (i+k-1, j-k-1)\}$.

The vertex $(i+(2k+1), j-1)$ k-dominates all vertices $\{(i+k+1, j-k-1), \cdots, (i+k+1, j+k-1)\}$ and the vertex $(i+1, j+(2k+1))$ k-dominates all vertices $\{(i-k+1, j+k+1), \cdots, (i+k+1, j+k+1).\}$

If $k+1 \leq i \leq 2k+1$ or $m-(2k+1) \leq i \leq m-(k+1)$ holds, then the vertices $(i-k-1, j-k+1)$, $\cdots$, $(i-k-1, j+k+1)$ or

$(i+k+1, j-k-1), \cdots, (i+k+1, j+k-1)$ are dominated by vertices (1,s) or (m,s) $1 \le s \le n$ respectively.

If $k+1 \le j \le 2k+1$ or $n-(2k+1) \le i \le n-(k+1)$ holds, then the vertices $(i-k-1, j-k-1), \cdots, (i+k-1, j-k-1)$ or $(i-k+1, j+k+1), \cdots, (i+k+1, j+k+1)$ are dominated by vertices (r,1) or (r,n) $1 \le r \le m$ respectively.

It follows that every vertex at distance (k+1) from(i,j) is either k-dominated by some other vertex of H, or by a vertex $(1,s), (m,s)$,$(r,1), (r,n) 1 \le s \le n$, $1 \le r \le m$.

Then $D = H \cup \{(1,s), (m,s), (r,1), (r,n) | 1 \le s \le n, 1 \le r \le m\}$ is a k-dominating set and

$$|D| = \lceil \frac{nm}{\lceil \frac{4k^2+4k+1}{2} \rceil} \rceil + 2m + 2n.$$

From the fact that one vertex can at most k-dominate $\lceil \frac{4k^2+4k+1}{2} \rceil$ vertices it follows that D must contain at least $\frac{mn}{\lceil \frac{4k^2+4k+1}{2} \rceil}$ vertices. Then

$$\frac{mn}{\lceil \frac{4k^2+4k+1}{2} \rceil} \le \gamma_k(P_m \times P_n) \le (\frac{mn}{\lceil \frac{4k^2+4k+1}{2} \rceil} + 2m + 2n)$$

$$\frac{1}{\lceil \frac{4k^2+4k+1}{2} \rceil} \le \frac{\gamma_k(P_m \times P_n)}{mn} \le \frac{1}{mn} \cdot (\frac{mn}{\lceil \frac{4k^2+4k+1}{2} \rceil} + 2m + 2n)$$

For $m, n \longrightarrow \infty$ the right hand side of this inequality tends to $\frac{1}{\lceil \frac{4k^2+4k+1}{2} \rceil}$. Therefore

$$\Longrightarrow \lim_{m,n \longrightarrow \infty} \frac{\gamma_k(P_m \times P_n)}{mn} = \frac{1}{\lceil \frac{4k^2+4k+1}{2} \rceil} . \blacksquare$$

Remark. The reader can easily convince himself/herself that Theorem 3 also holds for the k-domination number of cartesian products of paths.

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