On Quadrilaterals and Cycle Covers in a Bipartite Graph

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ABSTRACT. In [13], we conjectured that if $G = (V_1, V_2; E)$ is a bipartite graph with $|V_1| = |V_2| = 2k$ and minimum degree at least $k+1$, then G contains k vertex-disjoint quadrilaterals. In this paper, we propose a more general conjecture: If $G = (V_1, V_2; E)$ is a bipartite graph such that $|V_1| = |V_2| = n \geq 2$ and $\delta(G) \geq \lceil n/2 \rceil + 1$, then for any bipartite graph $H = (U_1, U_2; F)$ with $|U_1| \leq n$, $|U_2| \leq n$ and $\Delta(H) \leq 2$, G contains a subgraph isomorphic to H. To support this conjecture, we prove that if $n = 2k + t$ with $k \geq 0$ and $t \geq 3$, then G contains $k+1$ vertex-disjoint cycles covering all the vertices of G such that k of them are quadrilaterals.

1 Introduction

We consider finite simple graphs only. A set of graphs is said to be independent if no two of them have a common vertex. Let $G = (V_1, V_2; E)$ be a bipartite graph with (V_1, V_2) as its bipartition and E as its edge set. A 2-factor of G is a 2-regular spanning subgraph of G. Clearly, each component of a 2-factor of G is a cycle. In [13], we conjectured that if $|V_1| = |V_2| = 2k$ and $\delta(G) \geq k+1$, then G has a 2-factor with exactly k components, i.e., G contains k independent quadrilaterals. This conjecture is still open. We propose the following more general conjecture:

Conjecture A *[14] Let $G = (V_1, V_2; E)$ be a bipartite graph such that $|V_1| = |V_2| = n \geq 2$ and $\delta(G) \geq \lceil n/2 \rceil + 1$. Then for any bipartite graph $H = (U_1, U_2; F)$ with $|U_1| \leq n$, $|U_2| \leq n$ and $\Delta(H) \leq 2$, G contains a subgraph isomorphic to H.*

The degree condition in the above conjecture is sharp in general. If n is even, G might not be hamiltonian if we have $\delta(G) = n/2$. For instance, the union of two independent copies of $K_{n/2,n/2}$ is not hamiltonian. For an odd n, let $n = 2k + 1$ with k even. Let G be a bipartite graph with a bipartition $(A \cup B \cup \{u\}, X \cup Y \cup \{v\})$ such that the two induced subgraphs $G[A \cup X]$ and $G[B \cup Y]$ are isomorphic to $K_{k,k}$, $N(u) = X \cup \{v\}$ and $N(v) = B \cup \{u\}$. Moreover, the vertices of A are matched with vertices of Y by k independent edges of G. Clearly, G is of order $2(2k+1)$ and has minimum degree $k+1 < k+2 = \lceil (2k+1)/2 \rceil + 1$. If a cycle C of G contains u, then either C has at least 8 vertices, or C contains uv and exactly two vertices from each of $A \cup X$ and $B \cup Y$. As $k-1$ is odd, we then see that if C is a cycle of G containing u, then $G - V(C)$ does not have a 2-factor with exactly $k-1$ components.

In [13], we proved the following

Theorem B. *Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = 2k$, where k is a positive integer. Suppose that the minimum degree of G is at least $k+1$. Then G contains $k-1$ independent quadrilaterals and a path of order 4 such that the path is independent of all the $k-1$ quadrilaterals.*

In [14], we proved the following:

Theorem C. *Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n \geq 2k+1$ where k is a positive integer. Suppose that the minimum degree of G is at least $\lceil n/2 \rceil + 1$. Then G contains a 2-factor with exactly k components.*

As for general graphs, El-Zahar [8] conjectured that if a graph G of order $n = n_1 + \cdots + n_k$ with $n_i \geq 3$ $(1 \leq i \leq k)$ has minimum degree at least $\lceil n_1/2 \rceil + \cdots + \lceil n_k/2 \rceil$, then G contains k independent cycles of lengths $n_1, \ldots, n_k$, respectively. He proved it for $k = 2$. Corrádi and Hajnal proved the following:

Theorem D. [7] *If a graph G of order $n \geq 3k$ has minimum degree at least $2k$, then G contains k independent cycles. In particular, when $n = 3k$, G contains k independent triangles.*

We proved the following:

Theorem E. [11] *If a graph G of order $n \geq 3(k+1)$ has minimum degree at least $\lceil (n+k)/2 \rceil$, then G contains k independent triangles and a cycle of order $n - 3k$ such that the cycle is independent of all the k triangles.*

In this paper, we prove the following.

Theorem F. *Let $G = (V_1, V_2; E)$ be a bipartite graph. Suppose that $|V_1| = |V_2| = n \geq 2$ and $\delta(G) \geq \lceil n/2 \rceil + 1$. If $k \geq 0$ and $t \geq 3$ are two integers such that $n = 2k + t$, then G contains k independent quadrilaterals and a cycle of order $2t$ such that the cycle is independent of all the k quadrilaterals.*

Let G be a graph. Let u be a vertex of G and let H be either a subgraph of G or a subset of $V(G)$. We define $N(u,H)$ to be the set of neighbors of u contained in H and let $e(u,H) = |N(u,H)|$. Thus $e(u,G)(= e(u,V(G)))$ is the degree of u in G. If X is a subset of $V(G)$, we define $e(X,H) = \sum_{u\in X} e(u,H)$. If X is a subgraph of G, we let $e(X,H) = e(V(X),H)$. Thus if X is a subset of $V(G)$ or a subgraph of G and Y is a subset of $V(G)$ or a subgraph of G such that X and Y don't have a common vertex, then $e(X,Y)$ is the number of edges of G between X and Y. For a subset U of $V(G)$, $G[U]$ denotes the subgraph of G induced by U. If G is a cycle or a path, we use $l(G)$ to denote the length of G. A quadrilateral is a cycle of length 4.

2 Lemmas

In the following, $G = (V_1, V_2; E)$ is a bipartite graph with $|V_1| = |V_2| = n \geq 2$.

Lemma 2.1. *Let $P = x_1x_2\ldots x_p$ be a path in G and let u and v be two vertices in $G - V(P)$ with $u \in V_1$ and $v \in V_2$. The following two statements hold.*

(a) *If $uv \in E$ and $e(uv,P) \geq p/2$, then $G[V(P)\cup\{u,v\}]$ has a hamiltonian path with x_1 as one of its two endvertices.*

(b) *If $uv \notin E$ and $e(\{u,v\},P) > p/2+1$, then $G[V(P)\cup\{u,v\}]$ has a hamiltonian path with x_1 as one of its two endvertices.*

Proof: Say $\{x_1,u\} \subseteq V_1$. Let us prove the statement (a) first. If p is even, then the condition implies that $e(uv, x_ix_{i+1}) = 2$ for some $i \in \{1,3,\ldots,p-1\}$. Thus $x_1x_2\ldots x_ivux_{i+1}x_{i+2}\ldots xp$ is a required path from x_1 to x_p. If p is odd, we may assume that $x_pv \notin E$ for otherwise we are done. Hence $e(uv, P - x_p) > (p-1)/2$. Thus $G[V(P-x_p)\cup\{u,v\}]$ has a hamiltonian path P' from x_1 to x_{p-1}. Then $P' + x_{p-1x_p}$ satisfies the requirement.

Next, we prove the statement (b). If p is even or p is odd with $x_pv \notin E$, then the condition implies that $e(\{u,v\},x_ix_{i+1}) = 2$ and $e(\{u,v\},x_jx_{j+1}) = 2$ for some $i,j \in \{1,3,\ldots,p-r\}$ with $i<j$ where $r=1$ when p is even and otherwise $r=2$. Thus

$$x_1x_2\ldots x_ivx_jx_{j-1}x_{j-2}\ldots x_{i+1}ux_{j+1}x_{j+2}\ldots x_p$$

is a required path from x_1 to x_p. If p is odd and $x_pv \in E$, then the condition implies that $e(\{u,v\},x_ix_{i+1}) = 2$ for some $i \in \{1,3,\ldots,p-2\}$. Then $x_1x_2\ldots x_ivx_px_{p-1}\ldots x_{i+1}u$ satisfies the requirement. □

Lemma 2.2. *Let $P = x_1x_2\ldots x_p$ be a path in G and let u be a vertex in $G - V(P)$ such that $\{u,x_p\} \not\subseteq V_i$ for each $i \in \{1,2\}$. If $e(\{u,x_p\},P) > \lceil p/2\rceil$, then $G[V(P)\cup\{u\}]$ has a path from x_1 to u.*

Proof: If p is even, the condition implies that $e(\{u, x_p\}, x_i x_{i+1}) = 2$ for some $i \in \{1, 3, \ldots, p-1\}$. Thus $x_1 x_2 \ldots x_i x_p x_{p-1} \ldots x_{i+1} u$ is a required path from x_1 to u. If p is odd, then $e(\{u, x_p\}, P - x_1) > (p-1)/2$, and so $G[V(P - xl) \cup \{u\}]$ has a hamiltonian path P' from x_2 to u. Then $P' + x_1 x_2$ is a required path. □

Lemma 2.3. [13] *Let C be a quadrilateral of G. Let $x \in V_1$ and $y \in V_2$ be two vertices not on C. Suppose $e(\{x, y\}, C) \geq 3$. Then there exists $z \in V(C)$ such that either $C - z + x$ is a quadrilateral and $yz \in E$, or $C - z + y$ is a quadrilateral and $xz \in E$.*

Lemma 2.4. [13] *Let C be a quadrilateral of G. Let uv and xy be two independent edges of G such that they are independent of C. Suppose $e(\{u, v, x, y\}, C) \geq 5$. Then $G[V(C) \cup \{u, v, x, y\}]$ contains a quadrilateral C' and a path P' of order 4 such that P' is independent of C'.*

Lemma 2.5. [13] *Let $P = x_1 x_2 x_3$ and $Q = y_1 y_2 y_3$ be two independent paths of G with $x_1 \in V_1$ and $y_1 \in V_2$. Let C be a quadrilateral of G such that C is independent of both P and Q. Suppose $e(\{x_1, x_3, y_1, y_3\}, C) \geq 5$. Then $G[V(C \cup P \cup Q)]$ contains a quadrilateral C' and a path P' of order 6 such that P' is independent of C'.*

Lemma 2.6. [13] *Let C be a quadrilateral and P a path of order $s \geq 6$ in G such that C is independent of P. If $e(P, C) \geq s + 1$, then $G[V(C \cup P)]$ contains two independent cycles.*

Lemma 2.7. [13] *Let s and t be two integers such that $t \geq s \geq 2$ and $t \geq 3$. Let C_1 and C_2 be two independent cycles of G with orders $2s$ and $2t$, respectively. Suppose that $e(C_1, C_2) \geq 2t+1$. Then $G[V(C_1 \cup C_2)]$ contains two independent cycles C' and C'' such that $l(C') + l(C'') < 2s + 2t$.*

Lemma 2.8. [4] *If $e(\{x, y\}, G) \geq n + 2$ for any two non-adjacent vertices x and y with $x \in V_1$ and $y \in V_2$, then G is hamiltonian connected.*

Lemma 2.9. [12] *Suppose that G has a hamiltonian path and for any two endvertices u and v of a hamiltonian path of G, $e(\{u, v\}, G) \geq k$ holds, where k is an integer greater than n Then for every $x \in V_1$ and every $y \in V_2$, $e(\{x, y\}, G) \geq k$.*

Lemma 2.10. *Let $P_1 = x_1 x_2 \ldots x_{2q}$, $P_2 = y_1 y_2 y_3 y_4$ and $Q = a_1 a_2 a_3 a_4 a_1$ be three independent subgraphs in G with $q \geq 2$ and $\{x_1, y_1, a_1\} \subseteq V_1$. If $e(\{x_1, x_2, x_{2q-1} x_{2q}\}, Q) + e(P_2, Q) \geq 9$, then $G[V(P_1 \cup P_2 \cup Q)]$ contains a quadrilateral Q' and a path P' such that $l(P') \geq l(P_1) + 2$ and $V(P') \cap V(Q') = \emptyset$.*

Proof: On the contrary, suppose that $G[V(P_1 \cup P_2 \cup Q)]$ does not contain two required subgraphs under the given condition. Set $A = \{x_1, x_2, x_{2q-1}, x_{2q}\}$. We divide the proof into the following three cases.

Case 1: $N(y_i, Q) \cap N(y_{i+2}, Q) \neq \emptyset$ for some $i \in \{1, 2\}$.

W.l.o.g., say $\{a_4y_1, a_4y_3\} \subseteq E$. Then $a_4y_1y_2t_3a_4$ is a quadrilateral. By our assumption on $G[V(P_1 \cup P_2 \cup Q)]$, we must have that $e(\{x_2, x_{2q}\}, Q) = 0$ and $x_1a_2 \notin E$, and therefore $e(A, Q) \leq 3$. Similarly, if $N(y_2, Q) \cap N(y_4, Q) \neq \emptyset$, then $e(\{x_1, x_{2q-1}\}, Q) = 0$, and consequently, $e(P_2, Q) \geq 9$, a contradiction. Hence $N(y_2, Q) \cap N(y_4, Q) = \emptyset$. This implies that $e(\{y_2, y_4\}, Q) \leq 2$ and so $e(P_2, Q) \leq 6$. Hence $e(A, Q) = 3$ and $e(P_2, Q) = 6$. It follows that $e(x_{2q-1}, Q) = 2$, $x_1a_4 \in E$ and $e(y_1, Q) = e(y_3, Q) = 2$. Consequently, $a_2y_1y_2y_3a_2$ and $P + x_1a_4 + a_4a_3$ satisfy the requirement, a contradiction.

Case 2: $e(y_1, Q) = 2$ or $e(y_4, Q) = 2$.

W.l.o.g., say $e(y_1, Q) = 2$. By Case 1, $e(y_3, Q) = 0$. Then $e(\{a_2, a_4\}, \{x_1, x_{2q-1}, y_1, y_3\}) \leq 6$, and so $e(\{a_1, a_3\}, \{x_2, x_{2q}, y_2, y_4\}) \geq 3$. W.l.o.g, say $e(a_1, \{x_2, x_{2q}, y_2, y_4\}) \geq 2$. As both $Q - a_1 + y_1$ and $Q - a_3 + y_1$ are quadrilaterals, we must have, by our assumption on $G[V(P_1 \cup P_2 \cup Q)]$, that $N(x_i, Q) \cap N(y_j, Q) = \emptyset$ for each $i \in \{2, 2q\}$ and $j \in \{2, 4\}$. Then we have that $\{a_1x_2, a_1x_{2q}\} \subseteq E$ since $N(y_2, Q) \cap N(y_4, Q) = \emptyset$ by Case 1. If $x_{2q}a_3 \in E$, then both $Q - a_2 + x_{2q}$ and $Q - a_4 + x_{2q}$ are quadrilaterals, and so we must have that $e(\{x_1, x_{2q-1}\}, Q) = 0$, and consequently, $e(A, Q) + e(P_2, Q) \leq 4 + 4 = 8$, a contradiction. Therefore $x_{2q}a_3 \notin E$. We conclude that $e(\{a_1, a_3\}, \{x_2, x_{2q}, y_2, y_4\}) = 3$. It follows that $e(\{a_2, a_4\}, \{x_1, x_{2q-1}, y_1\}) = 6$. Thus $x_1x_2a_1a_2x_1$ and $x_3x_4 \dots x_{2q-1}a_4y_1y_2y_3y_4$ are two required subgraphs.

Case 3: $e(y_1, Q) = 1$ or $e(y_4, Q) = 1$.

W.l.o.g., say $e(y_1, Q) = 1$ and $y_1a_4 \in E$. We divide this case into the following two situations.

Case 3.1: $e(y_2, Q) > 0$.

W.l.o.g., say $y_2a_3 \in E$, and so $y_1y_2a_3a_4y_1$ is a quadrilateral in G. Thus $e(\{x_1, x_{2q}\}, a_1a_2) = 0$ and so $e(A, Q) \leq 6$. Moreover, if $e(a_1a_2, y_3y_4) > 0$, then $e(\{x_2, x_{2q-1}\}, a_1a_2) = 0$. Hence if $e(a_1a_2, y_3y_4) > 0$, then $e(A, Q) \leq 4$ and so $e(P_2, Q) \geq 5$. Consequently, $N(y_i, Q) \cap N(y_{i+2}, Q) \neq \emptyset$ for some $i \in \{1, 2\}$, a contradiction by Case 1. Therefore, we must have that $e(a_1a_2, y_3y_4) = 0$. By Case 1, we see that $e(y_3y_4, Q) = 0$. As $e(A, Q) \leq 6$, we have that $e(y_1y_2, Q) \geq 3$. Thus $y_2a_1 \in E$. Then $y_2a_1a_2a_3y_2$ and $a_1y_2y_1a_4a_1$ are two quadrilaterals in G, and so $x_1a_4 \notin E$ and $x_{2q}a_3 \notin E$. Consequently, $e(A, Q) \leq 4$ and $e(A, Q) + e(P_2, Q) \leq 4 + 3 = 7$, a contradiction.

Case 3.2: $e(y_2, Q) = 0$.

By Case 1, $y_3a_4 \notin E$ and so $e(y_3, Q) \leq 1$. By Case 2, $e(y_4, Q) \leq 1$. By Case 3.1, we obtain that $e(y_3y_4, Q) \leq 1$. Therefore $e(A, Q) \geq 7$. If $e(x_{2q}, Q) = 2$, then $x_{2q}a_1a_2a_3x_{2q}$ is a quadrilateral in G, and so $x_1a_4 \notin E$ and $x_{2q-1}a_4 \notin E$. Consequently, $e(A, Q) \leq 6$, a contradiction.

Hence $e(x_{2q}, Q) = 1$ and $e(x_1, Q) = e(x_2, Q) = e(x_{2q-1}, Q) = 2$. Then $x_1x_2a_1a_2x_1$ and $x_3x_4 \dots x_{2q-1}a_4y_1y_2y_3y_4$ are two required subgraphs, a contradiction. □

3 Proof of Theorem F

Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V - 1| = |V_2| = n \geq 2$ and $\delta(G) \geq \lceil n/2 \rceil + 1$. Suppose, for a contradiction, that for some two integers $k \geq 0$ and $t \geq 3$ with $n = 2k+t$, G does not contain k independent quadrilaterals and a cycle of order $2t$ such that the cycle is independent of all the k quadrilaterals. We shall prove the following two claims.

Claim 1. *G contains $\lceil n/2 \rceil - 1$ independent quadrilaterals.*

Proof of Claim 1.: If n is even, the claim is true by Theorem B. So we assume that n is odd and the claim fails. Set $n = 2k+1$. Then $\delta(G) \geq k+2$. Choose two vertices $w_1 \in V_1$ and $w_2 \in V_2$ such that $w_1w_2 \in E$. Let $G' = G - w_1 - w_2$. Then $\delta(G') \geq k+1$. By Theorem B, G' contains $k-1$ independent quadrilaterals $Q_1, \dots, Q_{k-1}$ and a path P of order 4 such that $V(P) \cap V(\cup_{i=1}^{k-1} Q_i) = \emptyset$. Say $P = x_1x_2x_3x_4$ with $x_1 \in V_1$. Set $H = \cup^{k-1} i = 1\} Q_i$ and $D = G - V(H)$.

First, suppose that D has a path of order 6. Let $L = y_1y_2y_3y_4y_5y_6$ be a path of D such that if D has a cycle of order 6 then $y_1y_6 \in E$. As G does not contain k independent quadrilaterals, we see that $e(y_i, D) \leq 2$ for each $y_i \in V(D)$. Thus $e(D, H) \geq 6(k+2) - 12 = 6(k-1) + 6$. This implies that there exists Q_i in H, say $Q_i = Q_1$, such that $e(D, Q_1) \geq 7$. By Lemma 2.6, it follows that $G[V(Q_1 \cup D)]$ contains two independent cycles of orders 4 and 6, respectively. So in the first place, we may assume $y_1y_6 \in E$. Then by Lemma 2.7, $G[V(Q_1 \cup D)]$ contains two independent quadrilaterals, a contradiction.

From the above argument, we may assume that for any $k-1$ independent quadrilaterals $Q'_1, \dots, Q'_{k-1}$ in G, $G - V(\cup_{i=1}^{k-1} Q'_i)$ does not contain a path of order 6. Clearly, $\{x_1x_4, x_1w_2, x_4w_1\} \cap E = \emptyset$ and $e(w_1w_2, P) \leq 1$. Thus $e(\{x_1, w_2\}, D) \leq 3$ and $e(\{x_4, w_1\}, D) \leq 3$. Since $e(\{x_1, w_2\}, H) \geq 2(k+2) - 3 = 2(k-1) + 3$, there exists Q_i in H, say $Q_i = Q_1$, such that $e(\{x_1, w_2\}, Q_1) \geq 3$. By Lemma 2.3, $G[V(Q_1) \cup \{x_1, w_2\}]$ contains a quadrilateral Q' and a path P' of order 2 such that $V(P') \cap V(Q') = \emptyset$. Moreover, exactly one of x_1 and w_2 is an endvertex of P'. First, assume that w_2 is an endvertex of P'. Let $z_0 \in V(Q_1)$ be such that $Q' = Q_1 - z_0 + x_1$ and $z_0w_2 \in E$. Let $D' = D - x_1 + z_0$ and $W = \{x_2, x_4, w_1, z_0\}$. Then $x_2x_3x_4$ and $w_1w_2z_0$ are two independent paths in D'. Furthermore, $e(W, D') = 4$ since D' does not contain a path of order 6. It follows that $e(W, (H - V(Q_1)) \cup Q') \geq 4(k+2) - 4 = 4(k-1) + 8$. Thus there exists a quadrilateral Q' in $(H - V(Q_1)) \cup Q'$ such that $e(W, Q') \geq 5$. We then obtain a contradiction by Lemma 2.5. Hence we must have that $e(w_2, Q_1) = 2$

and $e(x_1, Q_1) = 1$. Similarly, we must have that $e(\{x_4, w_1\}, Q_1) \le 3$. Thus $e(\{x_4, w_1\}, D \cup Q_1) \le 3 + 3 = 6$. Then we have that $e(\{w_1, x_4\}, H - V(Q_1)) \ge 2(k+2) - 6 = 2(k-2) + 2$. This implies that there exists Q_i in $H - V(Q_1)$, say $Q_i = Q_2$, such that $e(\{w_1, x_4\}, Q_2) \ge 3$. Again, we must have that $e(x_4, Q_2) = 1$ and $e(w_1, Q_2) = 2$. Let $z_1 \in V(Q_1)$ and $z_2 \in V(Q_2)$ be such that $z_1x_1 \in E$ and $z_2x_4 \in E$. Then $z_1x_1x_2x_3x_4z_2$ is a path of order 6 in G that is independent of both $Q_1 - z_1 + w_2$ and $Q_2 - z_2 + w_1$, a contradiction. So the claim holds. □

Claim 2. *For any two integers $k \ge 0$ and $t \ge 1$ with $n = 2k+t$, G contains k independent quadrilaterals and a path of order $2t$ such that the path is independent of all the k quadrilaterals.*

Proof of Claim 2: By Claim 1, we choose k independent quadrilaterals $Q_1, \ldots, Q_k$ such that

$$\text{The length of a longest path of } G - V(\cup_{i=1}^{k} Q_i) \text{ is maximum.} \qquad (1)$$

Let P be a longest path of $G - V(\cup_{i=1}^{k} Q_i)$. Subject to (1), we choose $Q_1, \ldots, Q_k$ and P such that

$$\text{The edge independence number of } G - V(\cup_{i=1}^{k} Q_i) \cup V(P) \text{ is maximum.} \qquad (2)$$

Let $H = \cup_{i=1}^{k} Q_i$, $D = G - V(H)$ and $P = x_1x_2 \ldots x_s$. Let $M = \{y_1z_1, \ldots, y_rz_r\}$ be a maximum matching of $D - V(P)$. W.l.o.g., say $\{x_1, y_1, \ldots, y_r\} \subseteq V_1$.

Clearly, $r + \lceil s/2 \rceil \le t$. We show that $r + \lceil s/2 \rceil = t$. On the contrary, suppose $r < t - \lceil s/2 \rceil$. Let w_1 and w_2 be two vertices in $D - V(P) \cup \{y_i, z_i | 1 \le i \le r\}$ such that $w_1 \in V_1$ and $w_2 \in V_2$. Then $e(\{w_1, w_2\}, y_iz_i) \le 1$ for all $i \in \{1, \ldots, r\}$ by the maximality of M. By Lemma 2.1 (b), we readily see that $e(\{w_1, w_2\}, P) \le \lceil (s+1)/2 \rceil$. Thus $e(\{w_1, w_2\}, D) \le r + \lceil (s+1)/2 \rceil \le t$. It follows that $e(\{w_1, w_2\}, H) \ge 2(k + \lceil t/2 \rceil + 1) - t \ge 2k+2$. This implies that there exists Q_i in H such that $e(\{w_1, w_2\}, Q_i) \ge 3$. By Lemma 2.3, this is a contradiction with the maximality of M. This shows that $r + \lceil s/2 \rceil = t$.

We now divide the proof of Claim 2 into the following two cases: s is even or odd.

Case 1: $s = 2q$.

Let $R = \{x_1, x_{2q}, y_1, z_1\}$. By Lemma 2.1, $e(\{x_1, z_1\}, P) \le q$ and $e(\{x_{2q}, y_1\}, P) \le q$. Thus $e(R, D) \le 2t$, and so $e(R, H) \ge 4(k + \lceil t/2 \rceil + 1) - 2t \ge 4k + 4$. This implies that there exists Q_i in H such that $e(R, Q_i) \ge 5$. Say $Q_i = u_1u_2u_3u_4u_1$ with $u_1 \in V_1$. Clearly, $e(\{x_1, z_1\}, Q_i) \ge 3$ or $e(\{x_{2q}, y_1\}, Q_i) \ge 3$. W.l.o.g., say $e(\{x_1, z_1\}, Q_i) \ge 3$. By Lemma 2.3 and the maximality of P, we see that $e(x_1, Q_i) = 2$ and $e(z_1, Q_i) = 1$ Say

$z_1u_1 \in E$. If $e(y_1, Q_i) > 0$, say $y_1u_2 \in E$, then $y_1z_1u_1u_2y_1$ is a quadrilateral in G and $P + x_1u_4$ is longer than P in G, contradicting the maximality of P. Hence $e(y_1, Q_i) = 0$. Then we have that $e(x_{2q}, Q_i) = 2$, and we readily see a contradiction with the maximality of P.

Case 2: $s = 2q + 1$.

Let z_0 be the only vertex of D which is neither on P nor covered by M. Clearly, $z_0 \in V_2$. By (1) and Lemma 2.2, $e(\{z_0, x_{2q+1}\}, P) \leq q + 1$ and so $e(\{z_0, x_{2q+1}\}, D) \leq t$. Then $e(\{z_0, x_{2q+1}\}, H) \geq 2(k + \lceil t/2 \rceil + 1) - t \geq 2k + 2$. This implies that there exists Q_i in H, say $Q_i = Q_1$, such that $e(\{z_0, x_{2q+1}\}, Q_1) \geq 3$. By (1) and Lemma 2.3, we must have that $e(x_{2q+1}, Q_1) = 2$ and $e(z_0, Q_1) = 1$. Let $Q_1 = y_0a_2a_3a_4y_0$ be such that $z_0y_0 \in E$. Set $P' = P - x_{2q+1}$, $Q' = Q_1 - y_0 + x_{2q+1}$, $D' = D - x_{2q+1} + y_0$, and $F = D - V(P) + y_0$. Then $M' = \{y_ix_i | 0 \leq i \leq r\}$ is a perfect matching of F.

First, let us assume that F has a path of order 4. We may assume w.l.o.g. that $z_0y_1 \in E$. Let $R = \{x_1, x_2, x_{2q-1}, x_{2q}, y_0, y_1, z_0, z_1\}$. By (1) and Lemma 2.2, we readily see

$$e(\{x_1, z_0\}, P') \leq q, \text{ and } e(\{x_1, z_0\}, y_iz_i) \leq 1 \text{ for each } i \in \{0, 1, \ldots, r\};$$
$$e(\{x_{2q}, y_1\}, P') \leq q, \text{ and } e(\{x_{2q}, y_1\}, y_iz_i) \leq 1 \text{ for each } i \in \{0, 1, \ldots, r\};$$
$$e(\{x_2, y_0\}, P') \leq q, \text{ and } e(\{x_2, y_0\}, y_iz_i) \leq 1 \text{ for each } i \in \{0, 1, \ldots, r\};$$
$$e(\{x_{2q-1}, z_1\}, P') \leq q, \text{ and } e(\{x_{2q-1}, z_1\}, y_iz_i) \leq 1 \text{ for each } i \in \{0, 1, \ldots, r\}.$$

These inequalities imply that $e(R, D') \leq 4t$. Therefore $e(R, (H - V(Q_1)) \cup Q') \geq 8(k + \lceil t/2 \rceil + 1) - 4t \geq 8k + 8$. This implies that there exists a quadrilateral Q in $(H - V(Q_1)) \cup Q'$ such that $e(R, Q) \geq 9$. By Lemma 2.10, $G[V(P' \cup Q) \cup \{y_0, y_1, z_0, z_1\}]$ contains a quadrilateral Q'' and a path P'' such that $V(Q'') \cap V(P'') = \emptyset$ and $l(P'') \geq l(P'') + 2$. Clearly, $l(P'') \geq l(P) + 1$, contradicting (1) again.

Next, we assume that F does not have a path of order 4. Then M' has all the edges of F. By (1) and Lemma 2.1 (a), $e(y_iz_i, P') \leq q$ for all $i \in \{0, 1, \ldots, r\}$. We then see that $e(\{y_0, y_1, z_0, z_1\}, D') \leq 2t$. Consequently, $e(\{y_0, y_1, z_0, z_1\}, (H - V(Q_1)) \cup Q')) \geq 4(k + \lceil t/2 \rceil + 1) - 2t \geq 4k + 4$. This implies that there exists a quadrilateral Q''' in $(H - V(Q_1)) \cup Q'$ such that $e(\{y_0, y_1, z_0, z_1\}, Q''') \geq 5$. By Lemma 2 4, $G[V(Q''') \cup \{y_0, y_1, z_0, z_1\}]$ contains a quadrilateral $Q^{(4)}$ and a path P''' of order 4 such that $V(Q^{(4)}) \cap V(P''') = \emptyset$. Then we replace Q''' by $Q^{(4)}$ and F by $(F - \{y_0, y_1, z_0, z_1\}) \cup P'''$ and repeat the previous argument to obtain a contradiction with (1). This proves Claim 2. □

We are now in the position to complete the proof of the theorem. Since G is hamiltonian, the theorem is obviously true if $k = 0$. So we have $k \geq 1$. By Claim 1 and Claim 2, we choose $k + 1$ quadrilaterals $Q_1, \ldots, Q_{k+1}$ and a path P of order $2t - 4$ to cover all the vertices of G. Set $H = \cup_{i=1}^{k+1} Q_i$,

$D = G[V(P)]$ and $s = t - 2$. Then $|V(D)| = 2s$. Let u and v be any two endvertices of a hamiltonian path of D. If $e(u, Q_i) > 0$ and $e(v, Q_i) > 0$ for some $i \in \{1, 2, \ldots, k+1\}$, then $G[V(Q_i \cup P)]$ is hamiltonian and we are done. Therefore we must have that either $e(u, Q_i) = 0$ or $e(v, Q_i) = 0$ for all $i \in \{1, 2, \ldots, k+1\}$. Consequently, $e(\{u, v\}, H) \leq 2(k+1)$ and so $e(\{u, v\}, D) \geq 2(k + 2 + \lceil s/2 \rceil) - 2(k+1) \geq s + 2$. This implies that $s \geq 2$. By Lemma 2.8 and Lemma 2.9, for any two vertices x and y of D with $x \in V_1$ and $y \in V_2$, we have that $e(\{x, y\}, D) \geq s + 2$. In particular, D is hamiltonian connected. Let $C = x_1 x_2 \ldots x_{2s} x_1$ with $x_1 \in V_1$ be a hamiltonian cycle of D. We now divide the proof into the following two cases: $s \geq 3$ or $s = 2$.

Case 1: $s \geq 3$.

As G is connected, we may assume that $e(x_1, Q_1) > 0$. We show that D contains a quadrilateral Q_0 such that $D - V(Q_0)$ has a hamiltonian path with x_1 as one of its two endvertices. Set $x_{2s+1} = x_1$. As $\sum_{i=1}^{s} e(x_{2s-4}x_{2s-3}, x_{2i}x_{2i+l}) \geq s + 2$, there exists $i \in \{1, 2, \ldots, s\}$ with $i \neq s - 2$ such that $e(x_{2s-4}x_{2s-3}, x_{2i}x_{2i+1}) = 2$. If $i \neq s$, then $x_{2s-4}x_{2s-3}x_{2i}x_{2i+1}x_{2s-4}$ is a quadrilateral not containing x_1 in D. If $i = s$, then $x_{2s-3}x_{2s-2}x_{2s-1}x_{2s}x_{2s-3}$ is a quadrilateral not containing x_1 in D. In either case, D still has an edge incident with x_1 after removing the quadrilateral from D. We choose a quadrilateral Q_0 from D such that $D - V(Q_0)$ contains a longest path starting at x_1. Let $P_1 = x_1 u_2 \ldots u_p$ be a longest path starting at x_1 in D. If $p < 2s - 4$, let w be a vertex in $D - V(P_1 \cup Q_0)$ such that $\{w, u_p\} \in V_i$ for each $i \in \{1, 2\}$. By Lemma 2.2, $e(\{w, u_p\}, P_1) \leq \lceil p/2 \rceil$. Therefore $e(\{w, u_p\}, D - V(Q_0)) \leq s - 2$. It follows that $e(\{w, u_p\}, Q_0) \geq s + 2 - (s - 2) = 4$. Let $u_{p+1} \in V(Q_0)$ be such that $u_{p+1}u_p \in E$. Then $Q_0 = Q_0 - u_{p+1} + w$ is a quadrilateral in D but $P_1 + u_p u_{p+1}$ is a path starting at x_1 in $D - V(Q_0)$, a contradiction. So $p = 2s - 4$. Say $Q_1 = a_1 a_2 a_3 a_4 a_1$ with $x_1 a_2 \in E$. As $G[V(P_1 \cup Q_1)]$ has a hamiltonian path, we replace Q_1 by Q_0 and D by $G[V(P_1 \cup Q_1)]$ and repeat the argument in the paragraph preceding Case 1. Then we see that $G[V(P_1 \cup Q_1)]$ must be hamiltonian connected. In particular, $G[V(P_1 \cup Q_i)]$ has a hamiltonian path from a_1 to a_2. This implies that either $e(a_1, P_1) > 0$ or $e(a_3, P_1) > 0$. As D is hamiltonian connected, we see that $G[V(D \cup Q_1)]$ is hamiltonian since we have that $e(a_2, D) > 0$, a contradiction.

Case 2: $s = 2$.

As in Case 1, let $Q_1 = a_1 a_2 a_3 a_4 a_1$ with $a_1 \in V_1$. For convenience, denote D by Q_0. Then $G[V(Q_i \cup Q_j)]$ is not hamiltonian for all $\{i, j\} \subseteq \{0, 1, \ldots, k+1\}$ with $i \neq j$. In particular, since $\delta(G) \geq k + 3$, we see that $k+1$ is even and there exist $(k+1)/2$ distinct quadrilaterals in H, say $Q_1, \ldots, Q_{(k+1)/2}$, such that $e(x_1, Q_i) = 2$, $e(x_2, Q_i) = 0$, $e(x_1, Q_{(k+1)/2+i}) = 0$ and $e(x_2, Q_{(k+1)/2+i}) = 2$ for all $i \in \{1, 2, \ldots, (k+1)/2\}$. Then we readily see that $e(x_3, Q_i) = 2$, $e(x_4, Q_i) = 0$, $e(x_3, Q_{(k+1)/2+i}) = 0$ and

$e(x_4, Q_{(k+1)/2+i}) = 2$ for all $i \in \{1, 2, \ldots, (k+1)/2\}$. As $\delta(G) \geq k + 3$, $e(a_1, Q_i) > 0$ for some $i > (k+1)/2$. Say $e(a_1, Q_{k+1}) > 0$. Let $Q_{k+1} = b_1 b_2 b_3 b_4 b_1$ with $b_1 \subseteq V_1$. Then similarly, we must have that $e(\{a_1, a_3\}, Q_{k+1}) = 4$ and $e(\{a_2, a_4\}, Q_{k+1}) = 0$. Then $a_4 x_1 x_2 x_3 a_4$ and $x_4 a_1 a_2 a_3 b_2 b_1 b_4 b_3 x_4$ are two independent cycles of orders 4 and 8, respectively in $G[V(Q_0 \cup Q_1 \cup Q_{k+1})]$, and so the theorem holds, a contradiction. This completes the proof of the theorem.

References

[1] M. Aigner and S. Brandt, Embedding arbitrary graphs of maximum degree two, *J. London Math. Soc., (2)* **48** (1993), 39–51.

[2] D. Amar, Partition of a bipartite Hamiltonian graph into two cycles, *Discrete Mathematics* **58** (1986), 1–10.

[3] D. Amar and A. Raspaud, Covering the vertices of a digraph by cycles of prescribed length, *Discrete Mathematics* **87** (1991), 111–118.

[4] J.A. Bondy and V. Chvátal, A method in graph theory, *Discrete Mathematics* **15** (1976), 111–135.

[5] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, The Macmillan Press, London, 1976.

[6] S. Brandt, G. Chen, R. Faudree, R. Gould and L. Lesniak, Degree conditions for 2-factors, *Journal of graph Theory* **Vol. 24, No. 2** (1997), 165–173.

[7] K. Corrádi and A. Hajnal, On the maximal number of independent circuits in a graph, *Acta Math. Acad. Sci. Hungar.* **14** (1963), 423–439.

[8] M. El-Zahar, On circuits in graphs, *Discrete Mathematics* **50** (1984), 227–230.

[9] C. Little and H. Wang, Vertex-disjoint cycles in a directed graph, *The Australasian Journal of Combinatorics* **12** (1995), 113–119 .

[10] H. Wang, Partition of a bipartite graph into cycles, *Discrete Mathematics* **117** (1993), 287–291.

[11] H. Wang, Covering a graph with cycles, *Journal of Graph Theory* **Vol. 20, No. 2** (1995), 203–211.

[12] H. Wang, Charles Little and Kee Teo, Partition of a Directed Bipartite Graph into Two Directed Cycles, *Discrete Mathematics* **160** (1996), 283–289.

[13] H. Wang, On the maximum number of independent cycles in a bipartite graph, *Journal of Combinatorial Theory, Series B* **67** (1996), 152–164.

[14] H. Wang, On 2-factors of a bipartite graph, to appear.