On McLeish's construction for Latin squares without intercalates

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A Latin square is N_2 if it has no intercalates (Latin subsquares of order 2). We correct results published in an earlier paper by McLeish, dealing with a construction for N_2 Latin squares.

§1. Definitions

A *Latin square* is a matrix of order n in which each row and column is a permutation of some (fixed) symbol set of size n. A *subsquare* of a Latin square is a submatrix (not necessarily consisting of adjacent entries) which is itself a Latin square. A subsquare of order 2 is an *intercalate*. A Latin square without intercalates is said to be N_2.

We use the closed interval notation $[a, b]$ in a slightly non-standard way. Since all our variables will be integers we prefer to interpret this as a discrete set. That is, $[a, b] = \{a, a+1, \ldots, b-1, b\}$. If a Latin square uses $[1, n]$ as its symbol set then it naturally defines a binary operation $\otimes$ on that set in which $a \otimes b$ is the entry in row a, column b. The resulting algebraic structure is a quasigroup. Reversing the process, every quasigroup defines a Latin square via its Cayley table. See [1] for details. We will not draw any distinction between a quasigroup and its associated Latin square.

The purpose of this note is to correct errors published with regard to the construction of N_2 squares in [3]. A study of when this construction yields a square with a unique proper subsquare (of order ≥ 5) can be found in [4].

§2. McLeish's construction

In [3] McLeish gives a construction for a quasigroup $M_{n,s}$ on the set $[1, n]$. Her binary operation $\otimes$ is defined in four "regions" A, B, C and D as follows.

A: For $a \in [n-s+1, n]$ and $b \in [1, n-s]$ select $a \otimes b \in [1, n-s]$ satisfying $a \otimes b \equiv 2b + a - 2 - \frac{s-1}{2} \pmod{n-s}$.

B: For $a \in [1, n-s]$ and $b \in [n-s+1, n]$ select $a \otimes b \in [1, n-s]$ satisfying $a \otimes b \equiv 2a + 3b - 4 - 3\frac{s-1}{2} \pmod{n-s}$.

C: For $a, b \in [1, n-s]$ there are three cases

i) if $a \equiv b+j \pmod{n-s}$ for $j \in [0, \frac{s-1}{2}]$ then $a \otimes b = n-s+2j+1$,

ii) if $a \equiv b+j \pmod{n-s}$ for $j \in [-\frac{s-1}{2}, -1]$ then $a \otimes b = n-s-2j$,

iii) Otherwise $a \otimes b \equiv 3b-a-1 \pmod{n-s}$ and $a \otimes b \in [1, n-s]$.

D: For $a, b \in [n-s+1, n]$ select $a \otimes b \in [n-s+1, n]$ which satisfies $a \otimes b \equiv a+b-n-3 \pmod{s}$.

McLeish argues that $M_{n,s}$ is a quasigroup whenever

$$n \equiv 0 \pmod{2}, \tag{1}$$
$$n > 2s, \tag{2}$$
$$s \equiv 1 \pmod{2}, \tag{3}$$
$$n \not\equiv s \pmod{3}. \tag{4}$$

Her Theorem 5.1 then states that $M_{n,s}$ is N_2 precisely when the following additional conditions hold:

$$n \not\equiv s \pmod{5}, \tag{5}$$
$$3n-s \not\equiv 3 \pmod{4} \text{ or } n > 3s-11, \tag{6}$$
$$3n-s \not\equiv 1 \pmod{4} \text{ or } n > 3s-5, \tag{7}$$
$$s \not\equiv 3 \pmod{4}, \tag{8}$$
$$s \not\equiv 1 \pmod{4} \text{ or } 2n > 5s-5. \tag{9}$$

We will show that two other conditions must be added to this list, namely

$$s > 1 \text{ and} \tag{10}$$
$$n \not\equiv 2 \pmod{4} \text{ or } n > 4s-6. \tag{11}$$

The corrected theorem can then be stated as follows.

Theorem 1. *$M_{n,s}$ is N_2 if and only if*
(a) either (i) $n \equiv 0 \pmod{4}$, $n > 3s-11$ and $2n > 5s-5$ or
(ii) $n \equiv 2 \pmod{4}$ and $n > 4s-6$,
(b) $n-s$ is not divisible by 3 or 5,
(c) $s \equiv 1 \pmod{4}$ and $s > 1$.

To see that conditions (1) to (11) are equivalent to parts (a) to (c) of the theorem we make the following observations. Firstly, (4) and (5) are obviously equivalent to (b). Secondly, (3) and (8) together are equivalent to $s \equiv 1 \pmod{4}$ so (9) simplifies to $2n > 5s-5$ which is now stronger than (2), assuming (10). Also by (1) we see that $3n-s \equiv n-1 \pmod{4}$ so that (6) and (7) become respectively $n \equiv 2 \pmod 4$ or $n > 3s - 11$, and, $n \equiv 0 \pmod 4$ or $n > 3s - 5$. In the case $n \equiv 2 \pmod 4$ the new condition $n > 4s - 6$ from (11) is stronger than both $n > 3s - 5$ and $2n > 5s - 5$, by (c). Note however that we need to retain the condition $2n > 5s - 5$ when $n \equiv 0 \pmod 4$ to exclude the case $n = 20$, $s = 9$.

We now look at how conditions (10) and (11) arise, in turn.

§3. $s > 1$

McLeish may have intended to omit $s = 1$ as a trivial case, but does not do so explicitly. Although this case gives rise to no N_2 squares the subsquare structures it produces are not without interest. We first locate the point in [**3**] at which $s = 1$ should have been excluded.

In Case 1 of the proof of Theorem 4.2, bounds are found for the variable k. An inequality (implicitly) derived at this point is that $(k-2)s \leq 1$, from which the conclusion is drawn that $k \leq 2$. However, when $s = 1$ there is the additional possibility that $k = 3$. It is then a simple matter to verify that the complete solution to equations (1) to (5) of [**3**] is $j = 0$, $k' = -2$, $a_1 = b_1 = n$, $a_2 = b_2$; where $b_2 \in [1, n - s]$ is arbitrary. Given that the remainder of McLeish's proof is valid and shows that there are no other intercalates, it is then a short step to this result:

Lemma 1. *Suppose n is a positive integer satisfying $n \equiv 0 \pmod 2$, $n \not\equiv 1 \pmod 3$ and $n \not\equiv 1 \pmod 5$. There are exactly $n - 1$ intercalates in $M_{n,1}$. For each $i \in [1, n - 1]$ there is an intercalate at the intersection of rows i and n with columns i and n.*

Lemma 1 is interesting in that $M_{n,1}$ would be N_2, but for a single entry (the symbol n in row n, column n). This entry is involved in the maximum possible number of intercalates, since it forms one with every other copy of the symbol n. In some sense then, $M_{n,1}$ is the antithesis of the homogeneous squares introduced in [**2**]. A square is defined to be homogeneous if every entry is involved in the same number of intercalates.

§4. $n \not\equiv 2 \pmod 4$ or $n > 4s - 6$

We turn to the more serious error in [**3**]. McLeish skips the details of case 2 of her Theorem 4.2, with the claim that it works the same way as case 1 (which is true) and gives rise to no further restrictions (this is false). In order to demonstrate this we now work through case 2, in the manner of case 1 in [**3**].

The aim is to find necessary and sufficient conditions on n and s under which there is an intercalate with one entry in each of the regions A, B, C and D. Hence we are looking for $a_1, b_1 \in [n-s+1, n]$ and $a_2, b_2 \in [1, n-s]$ such that $a_1 \otimes b_1 = a_2 \otimes b_2$ and $a_2 \otimes b_1 = a_1 \otimes b_2$. In particular, case 2 applies when the entry in region C is determined by part (ii) of the rules for that region, so $a_2 \equiv b_2 + j \pmod{n-s}$ for some $j \in [-\frac{s-1}{2}, -1]$. From the definition of $\otimes$ we have

$$a_1 + b_1 - n - 3 + ks = n - s - 2j, \tag{12}$$

$$2b_2 + a_1 - 2 - \frac{s-1}{2} = 2(b_2 + j) + 3b_1 - 4 - 3\frac{s-1}{2} + k'(n-s), \tag{13}$$

where k and k' are both integers. Simple algebra then gives

$$a_1 - 3b_1 = 2j - 1 + k'n - (k'+1)s, \tag{14}$$

$$4a_1 = -4j + (6+k')n + 8 - (3k + k' + 4)s. \tag{15}$$

Also $a_1, b_1 \in [n-s+1, n]$ means $n-s+1-3n \leq a_1 - 3b_1 \leq n - 3(n-s+1)$ which coupled with (14) yields,

$$k's + 2 \leq (k'+2)n + 2j \leq (k'+4)s - 2. \tag{16}$$

At this point we split into subcases based on the values of k and k'. Note that $0 \leq k \leq 2$ for the same reason as in [3] (but this requires $s > 1$, refer §3). Also by considering equation (15) modulo 4 we know that $k' \equiv k \pmod 2$. Indeed $k' \equiv k \pmod 4$ if $k' \equiv 2 \pmod 4$ or $n \equiv 0 \pmod 4$. Hence it is sufficient to treat the subcases:

(a) $k' \geq 0$
Since $n > s$ and $2j \geq -s+1$ the right hand inequality in (16) means $2n \leq 5s - 3$. Since $s \equiv 1 \pmod 4$ and $2n \equiv 0 \pmod 4$ this is equivalent to $2n \leq 5s - 5$, which is a case already excluded by Theorem 1(a).

(b) $k' = -1,\ k = 1,\ n \equiv 2 \pmod 4$
The right hand inequality in (16) gives $n \leq 4s - 3$, since $2j \geq -s + 1$. Then since $n \equiv 2 \pmod 4$ we in fact have $n \leq 4s - 6$ is necessary for there to be intercalates. To show sufficiency, suppose $n = 4s - 6 - 4m$ for an integer $m \in [0, s-2]$. Then observe that $a_1 = n - m \in [n-s+2, n]$, $b_1 = n - s + 2 + m \in [n-s+2, n]$ and $j = -\frac{s-1}{2}$, is a valid solution of (12) and (13). Moreover a_2 can always be chosen to satisfy $a_2 \equiv b_2 + j \pmod{n-s}$ for given $b_2 \in [1, n-s]$. We conclude that $M_{n,s}$ will contain at least $n - s$ intercalates.

(c) $k' = -2,\ k = 2$
Here (15) implies $a_1 \leq n + \frac{3}{2} - \frac{3}{2}s$ since $-4j \leq 2s - 2$. But now $a_1 \geq n - s + 1$ implies $s \leq 1$. See §3.

(d) $k' = -3,\ k = 1$
Similarly, (15) implies $a_1 \le \frac{3}{4}n + \frac{3}{2} - \frac{1}{2}s$ and hence $n \le 2s+2$. This is excluded by $2n > 5s - 5$ except in one feasible case: $n = 12$, $s = 5$. For that case note that (16) insists on $n \le 3s - 4$.

(e) $k' \le -4$
Since $j < 0$ the left hand inequality in (16) gives $k's < (k'+2)n$, which is impossible for $n > 2s$ and $k' \le -4$.

Taken together with [3], this completes the proof of Theorem 1.

§5. Spectrum of McLeish's construction

McLeish was aiming for a direct N_2 construction for as many orders as possible with her method. She argued that for even n sufficiently large, $M_{n,s}$ will be an N_2 square if s is chosen according to the rule:

$$\begin{aligned} &s = 5 \text{ if } n \not\equiv 2 \pmod 3 \text{ and } n \not\equiv 0 \pmod 5,\\ &s = 9 \text{ if } \begin{cases} n \not\equiv 0 \pmod 3 \text{ and } n \equiv 0 \pmod 5, \text{ or} \\ n \equiv 2 \pmod 3 \text{ and } n \equiv 1, 2 \text{ or } 3 \pmod 5, \end{cases}\\ &s = 13 \text{ if } \begin{cases} n \equiv 0 \pmod 3 \text{ and } n \equiv 0 \pmod 5, \text{ or} \\ n \equiv 2 \pmod 3 \text{ and } n \equiv 4 \pmod 5. \end{cases} \end{aligned} \tag{17}$$

This conclusion remains valid. All that has changed is the criteria for n being sufficiently large. The new condition that $n > 4s-6$ for $n \equiv 2 \pmod 4$ rules out only $(n, s) = (22, 9)$ and $(n, s) = (26, 9)$ of the pairs used in the proof of Theorems 5.2 and 5.3 in [3]. (In fact the first of these pairs should never have been listed since it breaches (7). It is not needed, as $M_{22,5}$ is N_2.) The updated result is this:

Theorem 2. *If s is chosen according to (17) then $M_{n,s}$ is N_2 for all $n \ge 12$ except $n \in \{14, 20, 26, 30\}$. The spectrum of constructions cannot be increased by choosing other s.*

§6. References

[1] J. Dénes and A. D. Keedwell, *Latin squares and their applications*, Akadémiai Kiadó, Budapest, 1974.

[2] A. Hobbs, A. Kotzig and J. Zaks, Latin squares with high homogeneity, *Congr. Numer.* **36** (1982) 333–345.

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[4] I. M. Wanless, Latin squares with one subsquare, *J Combin. Des.*, to appear.