Multipartite Ramsey Numbers

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ABSTRACT. For a graph G, a partiteness $k \geq 2$ and a number of colours c, we define the multipartite Ramsey number $r_k^c(G)$ as the minimum value m such that, given any colouring using c colours of the edges of the complete balanced k-partite graph with m vertices in each partite set, there must exist a monochromatic copy of G. We show that the question of the existence of $r_k^c(G)$ is tied up with what monochromatic subgraphs are forced in a c-colouring of the complete graph K_k. We then calculate the values for some small G including $r_3^2(C_4) = 3$, $r_4^2(C_4) = 2$, $r_3^3(C_4) = 7$ and $r_3^2(C_6) = 3$.

1 Introduction

For ordinary Ramsey numbers, one colours the edges of a complete graph using two or more colours and asks about the monochromatic subgraphs that are forced. There are many generalisations. Several authors have considered bipartite Ramsey numbers; see for example [1, 2, 6]. Here the edges of a complete bipartite graph are coloured and one asks about the monochromatic subgraphs that are forced.

In this paper we extend the concept of bipartite Ramsey numbers. We consider the problem of colouring the edges of a complete k-partite graph for general k. In particular, we fix the partiteness and look at how large the complete (balanced) k-partite graph must be before a particular monochromatic subgraph is forced.

Let $\mathcal{G} = \{G_1, G_2, \ldots, G_c\}$ be a set of c graphs. We define $r_k(\mathcal{G})$ as the minimum value m such that, given any colouring using c colours of the edges of the complete balanced k-partite graph with m vertices in each partite set, there must exist an i such that there is a monochromatic copy of G_i in colour i. We will only investigate the case where all the graphs in $\mathcal{G}$ are the same graph G. So we will write $r_k(G)$ for the case of two colours, and $r_k^c(G)$ for the general case of c colours.

All our graphs are simple and undirected without loops or multiple edges. For a graph G the vertex set is denoted $V(G)$ and the edge set $E(G)$. The complete graph on n vertices is denoted by K_n and the complete k-partite graph with $a_1, a_2, \ldots, a_k$ vertices in the partite sets is denoted $K(a_1, a_2, \ldots, a_k)$.

2 Existence

A basic question is: For which graphs G does $r_k(G)$ exist and, more generally, for which graphs does $r_k^c(G)$ exist.

Consider first the case $k = 3$. An immediate observation is that any subgraph of a complete 3-partite graph must be 3-colourable. But it is immediate that one can avoid a triangle by colouring the edges incident with one partite set one colour, and the remaining edges another colour. In fact, such a colouring avoids any nonbipartite graph. So $r_3(G)$ can exist only for bipartite G. But, by the work on bipartite Ramsey numbers, $r_3(G)$ exists for all bipartite G (see [6]).

The answer as to when $r_k^c(G)$ exists is tied up with the question of which graphs are forced in a c-colouring of K_k. We say that a graph G is **homomorphic** to H if there exists a mapping f from $V(G)$ to $V(H)$ such that if $uv \in E(G)$ then $f(u)f(v) \in E(H)$. (For example, a graph is k-(vertex-)colourable iff it is homomorphic to K_k.) We say that a colouring of the edges of a complete k-partite graph is **expansive** if for every pair of partite sets the edges between the two sets have the same colour. This yields in a natural way a colouring of the edges of the complete graph K_k.

Indeed, we say that an expansive colouring is **induced** by the corresponding colouring of K_k.

Theorem 1 *Let c and k be given. Then for a graph G the following are equivalent:*
(1) a monochromatic copy of G is forced in a c-colouring of sufficiently large balanced complete k-partite graph (that is, $r_k^c(G)$ exists);
(2) in every c-colouring of the complete graph K_k there exists a monochromatic subgraph H such that G is homomorphic to H.

Proof: Assume (1) and consider any colouring of K_k: we need to show that it contains a monochromatic subgraph H to which G is homomorphic. But simply consider the expansive colouring of the complete k-partite graph induced by the given colouring of K_k. This contains a monochromatic copy of G. By identifying vertices of G in the same partite set we obtain a monochromatic subgraph of K_k to which G is homomorphic.

Now, assume (2) and let C denote the complete k-partite graph with M vertices in each partite set where M is sufficiently large. Order the vertices in each partite set with $1, \ldots, M$. That is, the vertices of C are labelled v_i^l for $1 \leq l \leq k$ and $1 \leq i \leq M$. Let layer i be the set $\{ v_i^l : 1 \leq l \leq k \}$.

Consider a colouring of the edges of C. For $i < j$ let H_{ij} be the subgraph induced by the $k(k-1)$ edges between layer i and layer j. Now, define an auxiliary graph A whose vertex set is the integers from 1 up to M, and with all possible edges. Then label each edge ij of A ($1 \leq i < j \leq M$) with a $k(k-1)$-tuple giving the colouring of the edges of H_{ij}. This is a colouring of the edges of A.

There are only finitely many possible colourings of the edges in H_{ij}. Hence, by the original graph Ramsey theorem, A has a large monochromatic clique. In particular, for any positive integer m, if M is sufficiently large there exists a subset J of $\{1, 2, \ldots, M\}$ of cardinality km such that for all $i < j \in J$ the edges $ij \in E(A)$ have the same $k(k-1)$-tuple. That is, the graphs H_{ij} are identically coloured with partite sets and layers preserved. Without loss of generality $J = \{1, 2, \ldots, km\}$.

Now let D be the subgraph of C induced by the km vertices $v_{(l-1)m+a}^l$ for $1 \leq l \leq k$ and $1 \leq a \leq m$ (the first m vertices from the first partite set, the second m vertices from the second partite set, etc.). Then D is a complete k-partite graph with m vertices in each partite set. Furthermore, its edges have an expansive colouring: for any pair of partite sets all the edges between them have the same colour.

Now, in the corresponding colouring of K_k there is by assumption a monochromatic subgraph H to which G is homomorphic. If m is large enough (for example at least the order of G), then it follows that G is a monochromatic subgraph of D and hence of C. □

For example, if $k = 4$ then one cannot force a nonbipartite graph in a colouring of K_4. It follows that $r_4(G)$ exists if and only if G is bipartite.

If $k = 5$ then in any 2-colouring of K_5 there is a always a nonbipartite subgraph. In particular either a monochromatic copy of K_3 or C_5. Since C_5 is homomorphic to K_3 (and homomorphism is transitive), it follows that $r_5(G)$ exists if and only if G is homomorphic to the 5-cycle.

An analogous result holds for general $\mathcal{G}$.

3 The 4-cycle

By Theorem 1 we know that in a colouring of a 3-partite complete graph we can never force a non-bipartite graph, though all bipartite graphs are eventually forced. So our investigation of exact results starts with simple bipartite graphs.

We consider the 4-cycle. The following counting argument will prove useful. The idea is a simple extension of ideas used before (see for example [6]) to exclude the 4-cycle.

Theorem 2 *Suppose there is a c-colouring of the complete k-partite graph each of size m without a monochromatic 4-cycle. Let* $q = \binom{k}{2}m^2$, $e = \lceil q/c \rceil$, $d = k(k-1)m$, $a = \lfloor 2e/d \rfloor$ *and* $b = 2e - ad$. *Then*

$$k \times \binom{m}{2} \geq b\binom{a+1}{2} + (d-b)\binom{a}{2}.$$

In particular, if m is a multiple of c, then $a = m/c$ *and* $b = 0$ *so that the above inequality simplifies to:*

$$k \times \binom{m}{2} \geq km(k-1)\binom{m/c}{2}.$$

Proof: Consider the colour with the maximum occurrence, call it red. It has at least e edges. Count the number Q of red P_3's with endpoints in the same partite set.

By the lack of red 4-cycles, a red P_3 is uniquely determined by its endpoints. These are determined by choosing the partite set and then the two endpoints. Therefore

$$Q \leq k \times \binom{m}{2}.$$

On the other hand, we can bound Q from below by counting each red P_3 by its centre. Let $A_1, A_2, \ldots, A_k$ denote the partite sets and let v_{ir} denote the rth vertex in A_i. Let a_{irj} denote the number of red edges joining v_{ir} to A_j. Then v_{ir} is the centre of $\sum_j \binom{a_{irj}}{2}$ red P_3's with endpoints in the same partite set. That is,

$$Q \geq \sum_{i,j,r} \binom{a_{irj}}{2}.$$

By the first theorem of graph theory, $\sum_{i,j,r} a_{irj} \geq 2e$. Also, there are $d = k(k-1)m$ values of a_{irj} which can be nonzero. It follows by calculus that the minimum value of $\sum_{i,j,r} \binom{a_{irj}}{2}$ is obtained when all the values a_{irj} are as equal as possible. If all the a_{irj} are equal, then they equal $2e/d$. Otherwise, $d - b$ of them have value $a = \lfloor 2e/d \rfloor$ and b of them have value $a + 1$, where b is such that $(d-b)a + b(a+1) = 2e$. Hence we are done. □

It is known that $r_2(C_4) = 5$ (see [1]).

Theorem 3 $r_3(C_4) = 3$.

Proof: $K(2,2,2)$ is isomorphic to K_6 minus a perfect matching, and can be 2-coloured such that one colour is a 6-cycle and the other is $2K_3$. Hence $r_3(C_4) \geq 3$.

By the above theorem, any 2-colouring of $K(3,3,3)$ has a monochromatic C_4. (Note $k = 3$, $m = 3$, $c = 2$, $q = 27$, $e = 14$, $d = 18$, $a = 1$, $b = 10$, $LHS = 9$, $RHS = 10$.) Hence $r_3(C_4) \leq 3$. □

Theorem 4 $r_4(C_4) = 2$.

Proof: It is easy to colour K_4 with two colours without a monochromatic C_4. Hence $r_4(C_4) \geq 2$.

$K(2,2,2,2)$ has 8 vertices and 24 edges. A graph with 8 vertices and 12 edges is guaranteed to have a 4-cycle. (See [4].) Thus $r_4(C_4) \leq 2$. □

Since the ordinary Ramsey number of C_4 is 6 (see [3]), it follows from the above theorem that $r_5(C_4) = 2$ and $1 = r_6(C_4) = r_7(C_4) = \ldots$.

We now turn our attention to three or more colours. It is shown in [5] that $r_2^3(C_4) = 11$. We consider $r_k^3(C_4)$ for $k \geq 3$.

Theorem 5 $r_3^3(C_4) = 7$.

Proof: We first show that $r_3^3(C_4) \geq 7$.

Consider the complete bipartite graph $K(6,6)$. This has a natural decomposition into three 12-cyles defined as follows. Let the partite sets be U and V, and let the vertices in U be u_i ($0 \leq i \leq 5$ and all arithmetic modulo 6) and in V be v_i. Colour 1: u_i to v_i and v_{i+1}. Colour 2: u_i to v_{i+2} and v_{i+3}. Colour 3: u_i to v_{i+4} and v_{i+5}.

Which vertices in U have common neighbours joined by edges of the same colour? Only u_i and u_{i+1}. This is true for all three colours. Similarly the pairs of vertices in V with common neighbours are $\{v_i, v_{i+1}\}$.

Now we will consider the edges of $K(6,6,6)$ as composed of three $K(6,6)$. We will give each $K(6,6)$ the above colouring. Suppose the partite set are A, B and C with vertices a_i, b_i and c_i. Consider $[A,B]$. Map U to A and V to B any way.

Consider $[B,C]$. Map U to B and V to C such that pairs in B with common neighbours do not again get common neighbours. In particular, if the first mapping was v_i to b_i, then the second is u_0 to b_0, u_2 to b_1, u_5 to b_2, u_3 to b_3, u_1 to b_4, u_4 to b_5.

Similarly, consider $[C,A]$. Map U to C and V to A such that pairs of vertices with common neighbours do not again get common neighbours. Thus $r_3^3(C_4) \geq 7$.

Theorem 2 shows that a 3-colouring of $K(7,7,7)$ guarantees a monochromatic 4-cycle. (Note $k=3$, $m=7$, $c=3$, $q=147$, $e=49$, $d=42$, $a=2$, $b=14$, $LHS=63$, $RHS=70$.) Thus $r_3^3(C_4) \leq 7$. □

For 3-coloring in 4-partite graphs we will need the following result from [5]. Let graph G_{20} be constructed as follows. Take the cartesian product $K_4 \times K_2$ and in each copy of K_4 subdivide each edge with a new vertex. Finally, join each new vertex in the first copy of K_4 to the new vertex in the second copy which is at distance 5 from it. The graph G_{20} has 20 vertices and 34 edges.

Theorem 6 *[5] The maximum number of edges in a C_4-free bipartite graph on* 20 *vertices is* 34. *Furthermore, G_{20} is the unique such graph of size* 34.

Theorem 7 $r_4^3(C_4) \leq 5$.

Proof: Consider $K(5,5,5,5)$ with partite sets A, B, C, D. Suppose there is a 3-colouring of it without a 4-cycle.

This is not precluded by Theorem 2, but there is equality in the bound. (Note $k=4$, $m=5$, $c=3$, $q=150$, $e=50$, $d=60$, $a=1$, $b=40$, $LHS=RHS=40$.) This means that the lower bound is reached. In particular, in the proof of Theorem 2 it holds that a_{irj} is 1 or 2 for all i, r, j and all colours. That is, given any vertex, any other partite set and any colour, the vertex has either one or two edges of that colour to that partite set.

Now, consider the subgraph $K(10,10)$ induced by $[A \cup B, C \cup D]$. Consider a 3-colouring of the edges of this subgraph. At least one colour, say red, must colour at least 34 edges. Let G_R be the subgraph induced by the red edges. Since G_R has no monochromatic 4-cycle, Theorem 6 implies that $G_R \cong G_{20}$.

Say the partite sets of G_{20} are M and N. We next try to split M into A and B and N into C and D knowing that each vertex of M has at most two red neighbours in C and at most two red neighbours in D, and similarly with vertices of N. But a straight-forward calculation which we omit shows that this splitting is impossible. □

We believe that $r_4^3(C_4) = 5$.

4 Other Small Values

In this section we consider other small multipartite Ramsey numbers. The first theorem considers two vertex-disjoint copies of C_4. In [7] it was shown that $r_2(2C_4) = 7$.

Theorem 8 $4 \leq r_3(2C_4) \leq 6$.

Proof: Colour $K(3,3,3)$ so that the edges between two partite sets receive one colour and the remaining edges the other colour. This does not have two disjoint copies of C_4 with the same colour. Thus $r_3(2C_4) \geq 4$.

Consider a (red,blue)-colouring of $K(6,6,6)$ with partite sets A, B, C. We show first that there is monochromatic C_4 that has vertices from all three partite sets.

Suppose otherwise. For a vertex v, colour D and partite set X, let $d_D(v, X)$ be the number of edges of colour D from v to X. Let Δ be the maximum of $d_D(v, X)$ over all colours, all vertices and all partite sets. Suppose the maximum is achieved by vertex $a \in A$ for colour blue and to partite set B. Let B' denote a's blue neighbours in B; $|B'| = \Delta$.

By our supposition, two vertices in B' cannot have a common blue neighbour in C. So there are at most 6 blue edges from B' to C and therefore at least $6(\Delta - 1)$ red edges from C to B'. Hence there is a vertex of B' with at least $6(\Delta - 1)/\Delta$ edges to C. Thus

$$\Delta \geq \frac{6(\Delta - 1)}{\Delta}.$$

it follows that $\Delta \geq 5$. Also, there is a vertex b of B with $d_{red}(b, C) \geq 5$, and the majority of edges between B and C are red.

By the same argument (using b, C and red rather than a, B and blue), there is a vertex c of C with $d_{blue}(c, A) \geq 5$ and the majority of edges between C and A are blue. Then by the same argument the majority of edges between A and B are red; then the majority of edges between B and C are blue. This yields a contradiction.

Hence there exists a monochromatic 4-cycle, call it F, that has vertices from all three sets. Say F_1 has two vertices from A, one from B and one from C. Since a 2-colouring of $K(5,5)$ guarantees a monochromatic C_4 (see [6]), there is a monochromatic 4-cycle F_2 with two vertices from B and two from C that is disjoint from F_1. Now, if we ignore the vertices of F_1 and F_2 then we still have $K(4,3,3)$. By Theorem 3 this has a monochromatic C_4 as well. Since we have three disjoint monochromatic 4-cycles, two must have the same colour and thus $r_3(2C_4) \leq 6$. □

The value for the 6-cycle is determined next:

Theorem 9 $r_3(C_6) = 3$.

Proof: Clearly $K(2,2,2)$ can be 2-coloured without a monochromatic C_6 (e.g., the normal Ramsey number of C_6 is 8).

The proof that any 2-colouring of $K(3,3,3)$ has a monochromatic C_6 is by exhaustive computer search. We omit the details.

(To make the search feasible we do the following: There are nine edges between every pair of partite sets. There is one colour, say red, that is in the majority at least twice. So we have at least 10 red edges incident with some partite set. Consider all 2-colourings of $K(3,6)$ without monochromatic 6-cycle and with at least 10 red edges. Then for each such colouring, split the 6 vertices every possible way and colour the edges between them in every possible way.) □

Ramsey numbers of stars are really only about maximum degrees and the following result is typical.

Theorem 10 *For a star $K(1,a)$ with a edges, $r_3(K(1,a)) = a$ if a is odd and $a-1$ if a is even.*

Proof: Suppose a is odd. Note first that one can 2-colour $K(2,2,2)$ such that every vertex has degree 2 in each colour. (One colour is a C_6, the other $2K_3$.) Since $a-1$ is even, this can be expanded to a colouring of $K(a-1, a-1, a-1)$ such that every vertex has degree $a-1$ in each colour. Obviously in $K(a,a,a)$ each vertex is the centre of a monochromatic $K(1,a)$.

This leaves the question of $K(a-1, a-1, a-1)$ when a is even. But at least one monochromatic subgraph has average degree at least $a-1$: however, $3(a-1)$ and $a-1$ are odd so a suitable regular graph does not exist. □

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