# Long Paths and Cycles Through Specified Vertices in $k$-Connected Graphs

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## ABSTRACT

Let $k$ and $d$ be integers with $d \geq k \geq 4$, let $G$ be a $k$-connected graph with $|V(G)| \geq 2d-1$, and let $x$ and $z$ be distinct vertices of $G$. We show that if for any nonadjacent distinct vertices $u$ and $v$ in $V(G) - \{x, z\}$, at least one of $u$ and $v$ has degree greater than or equal to $d$ in $G$, then for any subset $Y$ of $V(G) - \{x, z\}$ having cardinality at most $k-1$, $G$ contains a path which has $x$ and $z$ as its endvertices, passes through all vertices in $Y$, and has length at least $2d-2$.

We also show a similar result for cycles.

## 1. INTRODUCTION

All graphs considered in this paper are finite simple undirected graphs with no loops and no multiple edges. For a graph $G$ , we let $V(G)$ and $E(G)$ denote the set of vertices and edges of $G$, respectively. For a vertex $v$ of $G$, we let $d_G(v)$ denote the degree of $v$ in $G$. For a path $P = u_0u_1 \cdots u_m$, $u_0$

is called the initial vertex of $P$, and $u_m$ is called the terminal vertex of $P$. For two vertices $x$ and $z$, a path having $x$ as its initial vertex and $z$ as its terminal vertex is called an $(x,z)$-path, and an $(x,z)$-path of length at least $m$ is referred to as an $(x,z;m)$-path. Furthermore, for a vertex set $Y$ (we allow the possibility that $Y \cap \{x,z\} \neq \phi$), an $(x,z)$-path passing through all vertices in $Y$ is referred to as an $(x,Y,z)$-path, and an $(x,Y,z)$-path of length at least $m$ is referred to as an $(x,Y,z;m)$-path. If $Y$ consists of a single vertex, say $y$, then an $(x,Y,z)$-path and an $(x,Y,z;m)$-path are also referred to as an $(x,y,z)$-path and an $(x,y,z;m)$-path, respectively.

The following theorems appear in [1]:

**Theorem A.** Let $k$ and $d$ be integers with $d \geq k \geq 3$, and let $G$ be a $k$-connected graph with $|V(G)| \geq 2d-1$. Let $x$ and $z$ be distinct vertices of $G$, and let $Y$ be a subset of $V(G)-\{x,z\}$ with cardinality $k-1$. Suppose that the minimum degree of $G$ is at least $d$. Then $G$ contains an $(x,Y,z;2d-2)$-path.

**Theorem B.** Let $k$ and $d$ be integers with $d \geq k \geq 2$. Let $G$ be a $k$-connected graph with $|V(G)| \geq 2d$, and let $Y$ be a subset of $V(G)$ with cardinality $k$. Suppose that the minimum degree of $G$ is at least $d$. Then $G$ contains a cycle which passes through all vertices in $Y$ and has length at least $2d$.

In this paper, we give the following theorems.

**Theorem 1.** Let $k$ and $d$ be integers with $d \geq k \geq 4$, and let $G$ be a $k$-connected graph with $|V(G)| \geq 2d-1$. Let $x$ and $z$ be distinct vertices of $G$, and let $Y$ be a subset of $V(G)-\{x,z\}$ with cardinality at most $k-1$. Suppose that

$$\begin{array}{l}\max\{d_G(u), d_G(v)\} \geq d \text{ for any nonadjacent} \\ \text{distinct vertices } u \text{ and } v \text{ in } V(G)-\{x,z\}.\end{array} \tag{1.1}$$

Then $G$ contains an $(x,Y,z;2d-2)$-path.

**Theorem 2.** Let $k$ and $d$ be integers with $d \geq k \geq 3$. Let $G$ be a $k$-connected graph with $|V(G)| \geq 2d$, and let $Y$ be a subset of $V(G)$ with cardinality at most $k$. Suppose that

$$\max\{d_G(u), d_G(v)\} \geq d \text{ for any nonadjacent distinct vertices } u \text{ and } v \text{ of } G.$$

Then $G$ contains a cycle which passes through all vertices in $Y$ and has length at least $2d$.

**Remark.** Theorem 1 does not hold for $k = 3$. To see this, let $d$, $l$ be integers with $d \geq 4$ and $l \geq 2$, and define a graph $G$ of order $l(d-2)+4$ by

$$\begin{aligned} V(G) &= \{ a_{hi} \mid 1 \leq h \leq l,\ 1 \leq i \leq d-2\} \cup \{ x_j \mid 1 \leq j \leq 3\} \cup \{y\}, \\ E(G) &= \{ a_{hi}a_{hj} \mid 1 \leq h \leq l,\ 1 \leq i < j \leq d-2\} \\ &\quad \cup \{ x_ix_j \mid 1 \leq i < j \leq 3\} \\ &\quad \cup \{ a_{hi}x_j \mid 1 \leq h \leq l,\ 1 \leq i \leq d-2,\ 1 \leq j \leq 3\} \\ &\quad \cup \{ yx_j \mid 1 \leq j \leq 3\} \end{aligned}$$

(then $G = (lK_{d-2} \cup K_1) + K_3$). Then $G$ is 3-connected and satisfies (1.1) with $x_1 = x$ and $x_2 = z$, but the maximum length of an $(x_1, y, x_2)$-path is $d+1$. We similarly see that Theorem 2 does not hold for $k = 2$.

Since the proofs of Theorems 1 and 2 follow essentially the same line of argument, we prove only Theorem 1 in this paper.

Our notation is standard with the possible exception of the following: Let $G$ be a graph. For a subset $U$ of $V(G)$, we let $\langle U \rangle$ denote the subgraph of $G$ induced by $U$. For a vertex $v$ of $G$, we denote by $N_G(v)$ the set of neighbours of $v$. Thus $d_G(v) = |N_G(v)|$. Let $A$ be a subset of $V(G)$. For a subset $U$ of $V(G)$, we let

$$\begin{aligned} N_U(A) &= \bigcup_{v \in A} (N_G(v) \cap U), \\ n_U(A) &= |N_U(A)|. \end{aligned}$$

For a subgraph $H$ of $G$, we write $N_H(A)$ and $n_H(A)$ for $N_{V(H)}(A)$ and $n_{V(H)}(A)$, respectively.

For two subsets $A$ and $B$ of $V(G)$, a path is called an $(A, B)$-path if its initial vertex and terminal vertex belong to $A$ and $B$, respectively, and no other vertex on it belongs to either $A$ or $B$ (if $A$ consists of a single vertex, say $a$, and $a \in B$, then the path $a$ of length 0 is the only $(A, B)$-path). A vertex $a$ is often identified with the set $\{a\}$. For example, we write $N_U(a)$ and $n_U(a)$ for $N_U(\{a\})$ and $n_U(\{a\})$, respectively, and an $(\{a\}, B)$-path is called an $(a, B)$-path. Let $P = u_0u_1\cdots u_m$ be a path. We denote the length $m$ of $P$ by $l(P)$. For two vertices $u_i, u_j$ on $P$ with $i \le j$, we let $P[u_i, u_j]$, $P[u_i, u_j)$, $P(u_i, u_j]$, and $P(u_i, u_j)$ denote the "subpaths" $u_iu_{i+1}u_{i+2}\cdots u_{j-1}u_j$, $u_iu_{i+1}u_{i+2}\cdots u_{j-1}$, $u_{i+1}u_{i+2}\cdots u_{j-1}u_j$, and $u_{i+1}u_{i+2}\cdots u_{j-1}$ of $P$, respectively (if $j = i+1$, $P(u_i, u_j)$ denotes an empty path; if $j = i$, $P[u_i, u_j)$ and $P(u_i, u_j]$ as well as $P(u_i, u_j)$ denote an empty path). The path obtained by "tracing $P$ backward" is denoted by $P^{-1}$; that is to say, $P^{-1} = u_m\cdots u_1u_0$.

We conclude this section with propositions which we need in our proof of Theorem 1. Proposition C appears in [2] as Lemma 2.1; Proposition D appears in [4] as Corollary; and Proposition E appears in [3] as Theorem 1.

**Proposition C.** Let $d \ge 1$ be an integer, let $G$ be a nonseparable graph, and let $x, y, z$ and $w$ be vertices of $G$ with $x \ne z$. Suppose that

$$d_G(u) \ge d \text{ for all vertices } u \text{ in } V(G) - \{x, z, w\}$$

and, in the case where $y \in \{x, z, w\}$, suppose further that

$$d_G(y) \ge \min\{d,\ 3\}.$$

Then $G$ contains an $(x, y, z; d)$-path.

**Proposition D.** Let $d \ge 1$ be an integer, let $G$ be a nonseparable graph, and let $x, z$ and $y$ be vertices of $G$ with $x \ne z$. Suppose that

$$\max\{d_G(u), d_G(v)\} \ge d \text{ for any nonadjacent distinct vertices } u \text{ and } v \text{ in } V(G) - \{x, z\},$$

and

$$d_G(y) \ge d.$$

Then $G$ contains an $(x, y, z; d)$-path.

**Proposition E.** Let $d \geq 4$ be an integer, and let $G$ be a 4-connected graph with $|V(G)| \geq 2d - 1$. Let $x$ and $z$ be distinct vertices of $G$, and suppose that

$$\max\{d_G(u), d_G(v)\} \geq d \text{ for any nonadjacent}$$
$$\text{distinct vertices } u \text{ and } v \text{ in } V(G) - \{x, z\}.$$

Then $G$ contains an $(x, z; 2d - 2)$-path which passes through all vertices whose degree in $G$ is strictly less than $d$.

## 2. PROOF OF THEOREM 1.

Throughout this section, let $k, d, G, x, z, Y$ be as in Theorem 1. We proceed by induction on $|Y|$. Let $S$ denote the set of vertices in $V(G) - \{x, z\}$ whose degree in $G$ is strictly less than $d$. By (1.1),

$$\text{any two distinct vertices in } S \text{ are adjacent to each other.} \tag{2.1}$$

If $Y \subseteq S$, then the result follows from Proposition E. Thus we may assume that $Y \not\subseteq S$ (so $Y \neq \phi$). By the induction hypothesis, $G$ contains an $(x, Y', z; 2d - 2)$-path for any proper subset $Y'$ of $Y$. Let $P$ be a longest $(x, z)$-path which passes through at least $|Y| - 1$ vertices in $Y$and passes through all vertices in $Y \cap S$. Then

$$P \text{ is an } (x, z; 2d - 2)\text{-path.} \tag{2.2}$$

Hence if $Y \subseteq V(P)$, then there is nothing to be proved. Thus we may assume $Y \not\subseteq V(P)$. By our choice of $P$, $Y - V(P)$ consists of a single vertex whose degree is greater than or equal to $d$. Write $Y - V(P) = \{y\}$. If $d = k$, the $k$-connectedness of $G$ implies that $S = \phi$, and hence the result follows from Theorem A. Thus we may assume $d \geq k + 1$. Then $2d - 2 \geq k$, and hence

$$|V(P)| > k \tag{2.3}$$

by (2.2). Label the vertices in $(Y - \{y\}) \cup \{x, z\}$ along $P$ as $y_0, y_1, \cdots, y_{|Y|}$ with $x = y_0$ and $z = y_{|Y|}$. Let $I_i = P[y_i, y_{i+1}]$ for each $i$ with $0 \le i \le |Y|-1$, and let

$$\mathcal{I} = \{ I_i \mid 0 \le i \le |Y| - 1\}.$$

Let $H$ be the connected component of $G - V(P)$ which contains $y$. If $\alpha$ and $\beta$ are vertices in $N_P(V(H))$ such that

$$\alpha \text{ occurs before } \beta \text{ on } P, \quad \{\alpha, \beta\} \subseteq V(I_i) \text{ for some } I_i \in \mathcal{I},$$

and such that

$$\text{there exists an } (\alpha, y, \beta)\text{-path } Q \text{ whose inner vertices are in } V(H),$$

then the path

$$P' = P[x, \alpha]QP[\beta, z]$$

is an $(x, Y, z)$-path. In what follows, we show that $l(P') \ge 2d - 2$ for some choice of $\alpha$, $\beta$ and $Q$. For this purpose, we divide the proof into several cases. We here summarize the nested structure of the cases for the convenience of the reader.

- **Case 1.** $H$ is separable (Lemmas 2.1 through 2.23).
  - **Case 1.1.** $n_P(Z_B^*) \ge k$ (Lemmas 2.8 through 2.21).
    - **Case 1.1.1.** $y \notin V(B)$, or $y = c$ and $d_B(y) \le 2$ (Lemmas 2.9 through 2.20).
      - **Subcase 1.** $w_0 + w_1^* \ge 2$.
      - **Subcase 2.** $w_0 + w_1^* \le 1$ (Lemmas 2.16 through 2.20).
        - **Subcase 2.1** $w_0 + w_1^+ \ge 2$.
        - **Subcase 2.2.** $w_0 + w_1^+ \le 1$ (Lemmas 2.17 through 2.20).
          - **Subcase 2.2.1.** $w \ge k - 1$ (Lemma 2.17).
          - **Subcase 2.2.2.** $w \le k - 2$ (Lemmas 2.18 through 2.20).
    - **Case 1.1.2.** $y \in A$, or $y = c$ and $d_B(y) \ge 3$ (Lemma 2.21).
  - **Case 1.2.** $n_P(Z_B^*) \le k - 1$ (Lemmas 2.22 and 2.23).
- **Case 2.** $H$ is nonseparable.

**Case 1.** $H$ is separable :

Let $C$ be the set of cutvertices of $H$, let $\mathcal{E}$ be the set of endblocks of $H$, and set

$$\mathcal{E}^+ = \{B \in \mathcal{E} \mid (V(B) - C) \cap S = \phi\}.$$

In view of (2.1), we have $\mathcal{E}^+ \neq \phi$. Let $B \in \mathcal{E}$. Let $c = c_B$ denote the unique vertex in $V(B) \cap C$, and set $A = A_B = V(B) - \{c\}$. Define $c' = c'_B$ as follows: if $y \notin V(B)$, then let $c'$ be the (unique) vertex in $C$ with $c' \neq y$ such that $c'$ and $y$ are in the same block of $H$ and such that every $(c, y)$-path passes through $c'$; if $y \in V(B)$, then let $c' = c$. Define $K = K_B$ as follows: if $y \notin V(B)$, then let $K$ be the connected component of $H - c'$ which contains $y$; if $y \in V(B)$, then let $K = H - A$. Set $Z = Z_B = V(K)$, and $Z^* = Z^*_B = A \cup Z$.

Since $G$ is $k$-connected, it follows from (2.3) that

$$n_P(Z^*_B) \geq n_P(A_B) \geq k - 1 \text{ for every } B \in \mathcal{E}. \tag{2.4}$$

Define

$$\mathcal{E}^* = \{B \in \mathcal{E}^+ \mid n_P(Z^*_B) \geq k\}.$$

We henceforth fix $B \in \mathcal{E}^+$ (recall that $\mathcal{E}^+ \neq \phi$). In the case where $\mathcal{E}^* \neq \phi$, we choose $B$ so that $B \in \mathcal{E}^*$. Let $c, A, c', K, Z, Z^*$ be as in the preceding paragraph. Let $z'$ denote the last vertex in $N_P(Z^*)$ on $P$, and for a vertex $u \in N_P(Z^*) - \{z'\}$, let $u^+$ denote the vertex in $V(P(u, z]) \cap N_P(Z^*)$ closest to $u$ on $P$. First we give a lemma concerning vertices $a$ and $b$ in $Z^*$ such that

$$a \neq b \text{ and } \{a, b\} \cap A \neq \phi. \tag{2.5}$$

**Lemma 2.1.** Let $a$ and $b$ be vertices in $Z^*$ which satisfy (2.5). Then there exists an $(a, b; d - n_P(Z^*))$-path in $H$. Furthermore, the following hold.
(I) If $\{a, b\} \subseteq V(B)$, then in $B$, there exists an $(a, b; d - n_P(Z^*))$-path.
(II) If one of the following conditions (i), (ii) or (iii) holds, then there exists an $(a, y, b; d - n_P(Z^*))$-path in $H$:
(i) $y \in A$;
(ii) $y = c$ and $d_B(y) \geq 3$;
(iii) $\{a, b\} \cap Z \neq \phi$.

**Proof.** We may assume $a \in A$. Define $b'$ as follows: if $b \in V(B)$, then let $b' = b$; otherwise, let $b' = c$. Further, define $y'$ as follows: if (II)(i) or (ii) holds, let $y' = y$; otherwise, let $y' = a$. Applying Proposition C to $B$, we see that there exists an $(a, y', b'; d - n_P(Z^*))$-path $R$ in $B$. Then in the case where (II)(i) or (ii) holds, and in the case where $y = c = b'$, $R$ is an $(a, y, b'; d - n_P(Z^*))$-path. This completes the proof for the case where $b \in V(B)$. Thus assume that $b \notin V(B)$ (so $b' = c$). Let $Q$ be a $(c, b)$-path in $H$. In the case where $y \notin V(B)$, we may assume that $Q$ passes through $y$ by the definition of $K$. Then $RQ$ is an $(a, y, b; d - n_P(Z^*))$-path in $H$. □

Let $\mathcal{W}$ denote the set of those pairs $(u, v)$ of vertices in $N_P(Z^*)$ such that $u$ occurs before $v$ on $P$ and such that there exist $a \in N_{Z^*}(u)$ and $b \in N_{Z^*}(v)$ which satisfy (2.5). Further define $\mathcal{W}^*$ as follows: if $y \notin V(B)$, or if $y = c$ and $d_B(y) \leq 2$, then let $\mathcal{W}^*$ denote the set of those pairs $(u, v)$ in $\mathcal{W}$ such that there exist $a \in N_{Z^*}(u)$ and $b \in N_{Z^*}(v)$ such that they satisfy (2.5) and

$$\{a, b\} \cap Z \neq \phi; \tag{2.6}$$

otherwise, i.e., if $y \in A$, or if $y = c$ and $d_B(y) \geq 3$, then let $\mathcal{W}^* = \mathcal{W}$. The following two lemmas follow immediately from the definition of $\mathcal{W}$ and $\mathcal{W}^*$ and from Lemma 2.1.

**Lemma 2.2.** Let $(u, v) \in \mathcal{W}$. Then there exists a $(u, v; d - n_P(Z^*) + 2)$-path $Q$ whose inner vertices lie in $V(H)$. Further, if $u \in N_P(a)$ and $v \in N_P(b)$ for some $a$ and $b$ in $A$ with $a \neq b$, then there exists a $(u, v; d - n_P(Z^*) + 2)$-path $Q$ whose inner vertices lie in $V(B)$. □

**Lemma 2.3.** For every $(u, v) \in \mathcal{W}^*$, there exists a $(u, y, v; d - n_P(Z^*) + 2)$-path whose inner vertices lie in $V(H)$. □

Let

$$\begin{aligned}
W &= \{\, u \in N_P(Z^*) - \{z'\} \mid (u, u^+) \in \mathcal{W} \,\}, \\
W^* &= \{\, u \in W \mid (u, u^+) \in \mathcal{W}^* \,\}, \\
W_0 &= \{\, u \in W \mid |V(P(u, u^+)) \cap Y| = 0 \,\}, \\
W_0^* &= W_0 \cap W^*, \\
W_1 &= \{\, u \in W \mid |V(P(u, u^+)) \cap Y| = 1 \,\}, \\
W_1^- &= \{\, u \in W_1 \mid V(P(u, u^+)) \cap Y \subseteq S \,\}, \\
W_1^+ &= W_1 - W_1^-, \\
W_1^* &= W_1^+ \cap W^*, \\
W_1' &= W_1^+ - W_1^*, \\
W_2 &= W - (W_0 \cup W_1).
\end{aligned}$$

Let $w = |W|$, $w_0 = |W_0|$, $w_1 = |W_1|$, $w_2 = |W_2|$, $w^* = |W^*|$, $w_0^* = |W_0^*|$, $w_1^- = |W_1^-|$, $w_1^+ = |W_1^+|$, $w_1^* = |W_1^*|$, $w_1' = |W_1'|$. The following lemma follows immediately from the definition of $w_1^+, w_1^-$ and $w_2$:

**Lemma 2.4.** $w_1^+ + w_1^- + 2w_2 \le |V(P) \cap Y| \le k - 2$. □

**Lemma 2.5.** For every $(u_0, v_0) \in \mathcal{W}$, $V(P[u_0, v_0)) \cap W \ne \phi$.

**Proof.** Choose $(u_1, v_1) \in \mathcal{W}$ with $u_1, v_1 \in V(P[u_0, v_0])$ so that $P[u_1, v_1]$ is minimal. We show that $u_1 \in W$. For this purpose, it suffices to show that $v_1 = u_1^+$. By way of contradiction, suppose that there exists $u \in V(P(u_1, v_1)) \cap N_P(Z^*)$. Take $a \in N_{Z^*}(u_1)$ and $b \in N_{Z^*}(v_1)$ so that $(a, b)$ satisfies (2.5). Assume first that $N_A(u) \ne \phi$, and take $a' \in N_A(u)$. Then since $a \ne b$, $(a, a')$ or $(a', b)$ satisfies (2.5), and hence we have $(u_1, u) \in \mathcal{W}$ or $(u, v_1) \in \mathcal{W}$, which contradicts the minimality of $P[u_1, v_1]$. Assume now that $N_A(u) = \phi$. Then since $u \in N_P(Z^*), N_Z(u) \ne \phi$, and hence we can take $a' \in N_Z(u)$. But then since $\{a, b\} \cap A \ne \phi$, $(a, a')$ or $(a', b)$ satisfies (2.5), which again contradicts the minimality of $P[u_1, v_1]$. □

**Lemma 2.6.** For every $(u_0, v_0) \in \mathcal{W}^*$, $V(P[u_0, v_0)) \cap W^* \ne \phi$.

**Proof.** In the case where $\mathcal{W}^* = \mathcal{W}$, the result is immediate from Lemma 2.5. Thus assume that $\mathcal{W}^* \subsetneq \mathcal{W}$ (so $y \notin V(B)$, or $y = c$ and $d_B(y) \le 2$). Let $(u_0, v_0) \in \mathcal{W}^*$. Choose $(u_1, v_1) \in \mathcal{W}^*$ with $u_1, v_1 \in V(P[u_0, v_0])$ so that

$P[u_1, v_1]$ is minimal. It suffices to show that $v_1 = u_1^+$. By the definition of $\mathcal{W}^*$,
(i) $N_A(u_0) \neq \phi$ and $N_Z(v_0) \neq \phi$ ; or
(ii) $N_Z(u_0) \neq \phi$ and $N_A(v_0) \neq \phi$.
Suppose that there exists $u \in V(P(u_0, v_0)) \cap N_P(Z^*)$. If $u \in N_P(A)$, then we have $(u, v_0) \in \mathcal{W}^*$ or $(u_0, u) \in \mathcal{W}^*$ according to whether (i) or (ii) holds. Similarly if $u \in N_P(Z)$, then we have $(u_0, u) \in \mathcal{W}^*$ or $(u, v_0) \in \mathcal{W}^*$ according to whether (i) or (ii) holds. Thus in any case, at least one of $(u_0, u)$ or $(u, v_0)$ belongs to $\mathcal{W}^*$, which contradicts the minimality of $P[u_1, v_1]$. □

**Lemma 2.7.**
(i) $l(P[u, u^+]) \geq 2$ for all $u \in N_P(Z^*) - \{z'\}$.
(ii) $l(P[u, u^+]) \geq d - n_P(Z^*) + 2$ for all $u \in W_0 \cup W_1^*$.
**Proof.** If $u \in W_1^*$, let $Q$ be a longest $(u, y, u^+)$-path whose inner vertices lie in $V(H)$; otherwise, let $Q$ be a longest $(u, u^+)$-path whose inner vertices lie in $V(H)$. We have $l(Q) \geq 2$ and, in the case where $u \in W_0 \cup W_1^*$, we have $l(Q) \geq d - n_P(Z^*) + 2$ by Lemmas 2.2 and 2.3. Consider the path

$$R = P[x, u]QP[u^+, z].$$

We first prove (i). If $V(P(u, u^+)) \cap Y \neq \phi$, then $V(P(u, u^+)) \neq \phi$, and hence we clearly have $l(P[u, u^+]) \geq 2$. Thus we may assume $V(P(u, u^+)) \cap Y = \phi$. Then since $R$ passes through all vertices in $V(P) \cap Y$, the maximality of $l(P)$ implies $l(P) \geq l(R)$, and hence $l(P[u, u^+]) \geq l(Q) \geq 2$, as desired. Thus (i) is proved. To prove (ii), assume that $u \in W_0 \cup W_1^*$. Then by the definition of $W_0$ and $W_1^*$ and by the choice of $Q$, $R$ passes through at least $|Y| - 1$ vertices in $Y$, and passes through all vertices in $Y \cap S$. Hence the maximality of $l(P)$ implies

$$l(P[u, u^+]) \geq l(Q) \geq d - n_P(Z^*) + 2,$$

as desired. □

**Case 1.1.** $n_P(Z_B^*) \geq k$ :

In addition to $\mathcal{W}$ and $\mathcal{W}^*$, we define another set $\mathcal{U}$ of pairs of vertices in $N_P(Z^*)$ and, in addition to $W$, $W_0$, etc., we define three other subsets $U, U_0, U_1$ of $N_P(Z^*)$. Recall that $K = \langle Z \rangle$. We consider the following condition for vertices $a, b$ in $Z$:

$$K \text{ contains an } (a, y, b)\text{-path} \tag{2.7}$$

(we allow the possibility that $a = y = b$). Define $\mathcal{U}$ as follows: if $y \notin A$, then let $\mathcal{U}$ denote the set of those pairs $(u, v)$ of vertices in $N_P(Z)$ such that $u$ occurs before $v$ on $P$ and such that there exist vertices $a \in N_Z(u)$ and $b \in N_Z(v)$ which satisfy (2.7); if $y \in A$, then let $\mathcal{U}$ denote the set of those pairs $(u, v)$ of vertices in $N_P(y)$ such that $u$ occurs before $v$ on $P$. Then by the definition of $\mathcal{W}$, $\mathcal{W}^*$ and $\mathcal{U}$,

$$\mathcal{W} \cap \mathcal{U} \subseteq \mathcal{W}^*. \tag{2.8}$$

Let

$$\begin{aligned} U &= \{u \in N_P(Z^*) - \{z'\} \mid (u, u^+) \in \mathcal{U}\}, \\ U_0 &= \{u \in U \mid |V(P(u, u^+)) \cap Y| = 0\}, \\ U_1 &= U - U_0. \end{aligned}$$

Then by (2.8),

$$W_0 \cap U_0 \subseteq W_0^*. \tag{2.9}$$

**Lemma 2.8.** For every $(u_0, v_0) \in \mathcal{U}$, $V(P[u_0, v_0)) \cap (W^* \cup U) \neq \phi$.

**Proof.** If there exist $u_0', v_0' \in V(P[u_0, v_0])$ such that $(u_0', v_0') \in \mathcal{W}^*$, the desired conclusion follows from Lemma 2.6. Thus we may assume that no two vertices $u_0', v_0' \in V(P[u_0, v_0])$ satisfy $(u_0', v_0') \in \mathcal{W}^*$. Choose $(u_1, v_1) \in \mathcal{U}$ with $u_1, v_1 \in V(P[u_0, v_0])$ so that $P[u_1, v_1]$ is minimal. It suffices to show that $v_1 = u_1^+$. By way of contradiction, suppose that there exists $u \in V(P(u_1, v_1)) \cap N_P(Z^*)$. Take $a' \in N_{Z^*}(u)$. Assume first that $y \notin A$. By the definition of $\mathcal{U}$, there exist $a \in N_Z(u_1)$ and $b \in N_Z(v_1)$ such that they satisfy (2.7), and hence $a' \in Z$ by the assumption that no two vertices $u_0', v_0' \in V(P[u_0, v_0])$ satisfy $(u_0', v_0') \in \mathcal{W}^*$. Let $R$ be an $(a, y, b)$-path in $K$, let $Q$ be an $(a', V(R))$-path in $K$, and let $t$ be the terminal vertex of $Q$. If $t \in V(R[a, y])$, then the path $QR[t, b]$ is an $(a', y, b)$-path in $K$, and hence

$(u, v_1) \in \mathcal{U}$, which contradicts the minimality of $P[u_1, v_1]$. Similarly, if $t \in V(R[y, b])$, then considering the path $R[a, t]Q^{-1}$, we get $(u_1, u) \in \mathcal{U}$, which again contradicts the minimality of $P[u_1, v_1]$. Assume now that $y \in A$. In this case, $\{u_1, v_1\} \subseteq N_P(y)$ by the definition of $\mathcal{U}$, and hence $a' = y$ by the assumption that no two vertices $u_0', v_0' \in V(P[u_0, v_0])$ satisfy $(u_0', v_0') \in \mathcal{W}^*$. But then $(u_1, u) \in \mathcal{U}$, which again contradicts the minimality of $P[u_1, v_1]$. □

**Case 1.1.1.** $y \notin V(B)$, or $y = c$ and $d_B(y) \leq 2$ :

**Lemma 2.9.** There exist $(y, V(P))$-paths $R_i, 1 \leq i \leq k$, which satisfy the following conditions.
(i) For any $i, j$ with $i \neq j$, $\quad V(R_i) \cap V(R_j) = \{y\}$.
(ii) For each $i$, let $t_i$ denote the terminal vertex of $R_i$, and let $b_i$ denote the vertex which occurs immediately before $t_i$ on $R_i$. Then $\{b_1, \cdots, b_k\} \subseteq Z^*$, $|\{i \mid b_i \in Z\}| \geq k-1$, and $V(R_i[y, b_i]) \subseteq Z$ for each $i$ with $b_i \in Z$.
**Proof.** First we consider the case where $y \neq c$ (so $y \neq c'$). By Menger's Theorem and (2.3), we can find $(y, V(P))$-paths $Q_i, 1 \leq i \leq k$, pairwise disjoint except at $y$. For each $1 \leq i \leq k$, let $\tau_i$ be the terminal vertex of $Q_i$, and let $a_i$ be the vertex which occurs immediately before $\tau_i$ on $Q_i$. Let $X = \{\tau_i \mid 1 \leq i \leq k\}$. If $V(Q_i[y, a_i]) \subseteq Z$ for all $i$, then we obtain paths with the desired properties by simply letting $R_i = Q_i$ (so $t_i = \tau_i$ and $b_i = a_i$) for each $i$. Thus assume that $V(Q_i[y, a_i]) \not\subseteq Z$ for some $i$. We may assume $V(Q_1[y, a_1]) \not\subseteq Z$. Then $Q_1$ passes through $c'$, and hence $V(Q_i[y, a_i]) \subseteq Z$ for all $2 \leq i \leq k$. Assume for the moment that there exists $\tau \in N_P(A) - (X - \{\tau_1\})$. Take $a \in N_A(\tau)$ and let $Q$ be a $(c', a)$-path in $H$. Now if we let $R_1 = Q_1[y, c']Qa\tau$ and $R_i = Q_i$ for $2 \leq i \leq k$, then all requirements are satisfied. Thus we may assume that $N_P(A) - (X - \{\tau_1\}) = \phi$. Since $n_P(A) \geq k-1$ by (2.4), and since $|X - \{\tau_1\}| = k-1$, we have $N_P(A) = X - \{\tau_1\}$ and $n_P(A) = k-1$. Since $n_P(Z^*) \geq k$ by the assumption of Case 1.1, this implies that there exists $\tau \in N_P(Z) - (X - \{\tau_1\})$. Take $a \in N_Z(\tau)$, let $Q$ be an $\left( a,\ V(Q_1[y, c')) \bigcup \left( \bigcup_{i=2}^{k} V(Q_i[y, a_i]) \right) \right)$-path in $K$, and let $a'$

be the terminal vertex of $Q$ ( thus if $a \in V(Q_1[y,c')) \bigcup \Big( \bigcup_{i=2}^{k} V(Q_i[y,a_i]) \Big)$, then $Q = a$ and $a' = a$ ). If $a' \notin \bigcup_{i=2}^{k} V(Q_i(y,a_i])$, then $a' \in V(Q_1[y,c'))$, and hence we obtain paths with the desired properties by letting $R_1 = Q_1[y,a']Q^{-1}a\tau$ and $R_i = Q_i$ for $2 \le i \le k$. Thus assume that $a' \in V(Q_i(y,a_i])$ for some $i$, $2 \le i \le k$. We may assume that $a' \in V(Q_2(y,a_2])$. Take $b \in N_A(\tau_2)$, and let $Q'$ be a $(c',b)$-path in $H$. Now if we let $R_1 = Q_1[y,c']Q'b\tau_2$, $R_2 = Q_2[y,a']Q^{-1}a\tau$ and $R_i = Q_i$ for $3 \le i \le k$, then all requirements are satisfied.

We now consider the case where $y = c$ and $d_B(y) \le 2$. Take two new vertices $u_0, v_0$ with $u_0 \ne v_0$ and $u_0, v_0 \notin V(G)$, and define a graph $G_0$ by

$$\begin{aligned} V(G_0) &= V(G) \cup \{u_0, v_0\}, \\ E(G_0) &= E(G) \cup \{u_0u \mid u \in N_{Z \cup V(P)}(y) \cup \{y\}\} \cup \{v_0u \mid u \in V(P)\}. \end{aligned}$$

Then $d_{G_0}(u_0) = d_G(y) - d_B(y) + 1 \ge d - 1 \ge k$ and $d_{G_0}(v_0) > k$. Consequently, $G_0$ is $k$-connected, and hence there exist $k$ pairwise internally disjoint $(u_0,v_0)$-paths $Q_i$, $1 \le i \le k$. Let $t_i$ be the vertex which occurs immediately before $v_0$ on $Q_i$. Since $N_{G_0}(v_0) = V(P)$, we may assume that

$$V(P) \cap V(Q_i) = \{t_i\} \text{ for all } 1 \le i \le k. \tag{2.10}$$

We may also assume that for each $1 \le i \le k-1$, $y \notin V(Q_i)$. Then by (2.10), $V(Q_i) \cap A = \phi$ for all $1 \le i \le k-1$. Now for $1 \le i \le k$, let $a_i$ be the vertex which occurs immediately after $u_0$ on $Q_i$, and let

$$R_i = \; ya_iQ_i[a_i,t_i] \qquad \text{for } 1 \le i \le k-1,$$

$$R_k = \begin{cases} ya_kQ_k[a_k,t_k] & (\text{ if } y \notin V(Q_k)\ ) \\ Q_k[y,t_k] & (\text{ if } y \in V(Q_k)\ ). \end{cases}$$

Then all requirements are satisfied. □

**Lemma 2.10.** $W_0^* \cup U_0 \ne \phi$.

**Proof.** Let $t_1, \cdots, t_k$ be as in (ii) of Lemma 2.9. Since $|\mathcal{I}| = |Y| \le k-1$, there exists $I_i \in \mathcal{I}$ such that $|\{t_1, \cdots t_k\} \cap V(I_i)| \ge 2$. At the cost of relabeling, we may assume $t_1, t_2 \in V(I_i)$, and $t_1$ occurs before $t_2$ on $P$. Then since $(t_1, t_2) \in \mathcal{W}^* \cup \mathcal{U}$ by Lemma 2.9,

$$\begin{aligned} \phi &\neq V(P[t_1, t_2)) \cap (W^* \cup U) && \text{(by Lemmas 2.6 and 2.8)} \\ &= V(P[t_1, t_2)) \cap (W_0^* \cup U_0) && \text{(by the definition of } W_0^* \text{ and } U_0\text{)}, \end{aligned}$$

as desired. □

We choose $\alpha \in W_0^* \cup U_0$ as follows. First assume that $W_0^* \neq \phi$. In this case, we let $\alpha \in W_0^*$ (any element of $W_0^*$ will do). Next assume that $W_0^* = \phi$. Then $U_0 \neq \phi$ by Lemma 2.10. In this case, we let $\alpha \in U_0$ (any element of $U_0$ will do). Let $P_0$ be a longest $(\alpha, y, \alpha^+)$-path whose inner vertices lie in $V(H)$, and let

$$P' = P[x, \alpha]P_0P[\alpha^+, z].$$

Then $P'$ is an $(x, Y, z)$-path passing through all vertices in $N_P(Z^*)$. We show that $l(P') \geq 2d - 2$. For this purpose, we give estimates of the length of subpaths of $P'$ (Lemmas 2.11 through 2.15).

**Lemma 2.11.**

(i) $l(P'[u, u^+]) \geq 2$ for all $u \in N_P(Z^*) - \{z'\}$.

(ii) $l(P'[u, u^+]) \geq d - n_P(Z^*) + 2$ for all $u \in W_0 \cup W_1^*$.

**Proof.** If $u \neq \alpha$, then $P'[u, u^+] = P[u, u^+]$, and hence the results follow from Lemma 2.7. Thus we may assume $u = \alpha$. Since $P'[\alpha, \alpha^+] = P_0$, we immediately see that $l(P'[\alpha, \alpha^+]) \geq 2$. Now asssume that $\alpha \in W_0 \cup W_1^*$. Then $\alpha \in W_0 \cap (W_0^* \cup U_0) \subseteq W_0^*$ by the choice of $\alpha$ and (2.9). Hence by Lemma 2.3, the maximality of $l(P_0)$ implies that

$$l(P'[\alpha, \alpha^+]) = l(P_0) \geq d - n_P(Z^*) + 2. \quad \square$$

In Lemmas 2.12 through 2.14, we are mainly concerned with $l(P'[u, u^+])$ for $u \in W_1 - W_1^* (= W_1' \cup W_1^-)$. Let

$$\begin{aligned} W' &= U_0 - (W_0 \cup \{\alpha\}) \ \ (\subseteq N_P(Z^*) - (W \cup \{z'\})\ ), \\ w' &= |W'|. \end{aligned}$$

**Lemma 2.12.** For any $u_1 \in W_1'$ and for any $u_2 \in W'$,

$$l(P'[u_1, u_1^+]) + l(P'[u_2, u_2^+]) \geq d - n_P(Z^*) + 4.$$

**Proof.** Since $u_i \neq \alpha$ for $i = 1, 2$, we have

$$P'[u_i, u_i^+] = P[u_i, u_i^+] \quad \text{for } i = 1, 2.$$

Thus it suffices to show that

$$l(P[u_1, u_1^+]) + l(P[u_2, u_2^+]) \geq d - n_P(Z^*) + 4.$$

By the definition of $W_1'$ and by Lemma 2.2, we can find a $(u_1, u_1^+; d - n_P(Z^*) + 2)$-path $Q_1$ whose inner vertices lie in $V(B)$, and by the definition of $W'$, we can find a $(u_2, y, u_2^+; 2)$-path $Q_2$ whose inner vertices lie in $Z$. For these paths $Q_1$ and $Q_2$, we have $V(Q_1) \cap V(Q_2) \subseteq \{y\}$ (the case where $V(Q_1) \cap V(Q_2) = \{y\}$ is possible when $y = c$). First assume that $V(Q_1) \cap V(Q_2) = \phi$. We may assume that $u_1$ occurs before $u_2$ on $P$. Consider the path

$$R = P[x, u_1]Q_1P[u_1^+, u_2]Q_2P[u_2^+, z].$$

Then the maximality of $l(P)$ implies that $l(P) \geq l(R)$, and hence

$$\begin{aligned} l(P[u_1, u_1^+]) + l(P[u_2, u_2^+]) &\geq l(Q_1) + l(Q_2) \\ &\geq d - n_P(Z^*) + 4, \end{aligned}$$

as desired. Next assume that $V(Q_1) \cap V(Q_2) = \{y\}$. Then the maximality of $l(P)$ implies

$$l(P[u_1, u_1^+]) \geq l(Q_1) \geq d - n_P(Z^*) + 2,$$

and hence

$$\begin{aligned} l(P[u_1, u_1^+]) + l(P[u_2, u_2^+]) &\geq (d - n_P(Z^*) + 2) + 2 \\ &= d - n_P(Z^*) + 4 \end{aligned}$$

by Lemma 2.7.(i). □

**Lemma 2.13.** Let $u_1$ and $u_2$ be distinct vertices in $W_1^-$. Then

$$l(P'[u_1,u_1^+]) + l(P'[u_2,u_2^+]) \geq 6.$$

Suppose further that $\{(u_1,u_2),(u_1^+,u_2^+)\} \cap \mathcal{W} \neq \phi$. Then

$$l(P'[u_1,u_1^+]) + l(P'[u_2,u_2^+]) \geq d - n_P(Z^*) + 6.$$

**Proof.** We may assume $u_1$ occurs before $u_2$ on $P$. Write $V(P(u_i,u_i^+)) \cap Y = \{s_i\}$ for each $i = 1,2$. Then $s_1, s_2 \in S$ by the definition of $W_1^-$, and hence $s_1 s_2 \in E(G)$ by (2.1). Let $Q$ be a longest $(u_1,u_2)$-path whose inner vertices are in $V(H)$, and let $Q'$ be a longest $(u_1^+,u_2^+)$-path whose inner vertices are in $V(H)$. Consider the path

$$R = P[x,u_1]QP^{-1}[u_2,s_1]s_1s_2P[s_2,z].$$

Then $R$ is an $(x,z)$-path passing through all vertices in $V(P) \cap Y$. Consequently, the maximality of $l(P)$ implies that

$$\begin{aligned} l(P[u_1,u_2^+]) &\geq l(R[u_1,u_2^+]) \\ &= l(Q) + l(P[s_1,u_2]) + l(R[s_1,s_2]) + l(P[s_2,u_2^+]), \end{aligned}$$

and hence

$$l(P[u_1,s_1]) + l(P[u_2,s_2]) \geq l(Q) + l(R[s_1,s_2]). \tag{2.11}$$

Similarly, considering the path

$$R' = P[x,s_1]s_1s_2P^{-1}[s_2,u_1^+]Q'P[u_2^+,z],$$

we get

$$l(P[s_1,u_1^+]) + l(P[s_2,u_2^+]) \geq l(Q') + l(R'[s_1,s_2]). \tag{2.12}$$

By adding (2.11) and (2.12), we obtain

$$\begin{aligned} l(P[u_1,u_1^+]) + l(P[u_2,u_2^+]) &\geq l(Q) + l(Q') + l(R[s_1,s_2]) + l(R'[s_1,s_2]) \\ &= l(Q) + l(Q') + 2 \end{aligned} \tag{2.13}$$

On the other hand, for each $i = 1,2$, $u_i \neq \alpha$ by the choice of $\alpha$, and hence $P[u_i,u_i^+] = P'[u_i,u_i^+]$. Hence $l(P'[u_1,u_1^+]) + l(P'[u_2,u_2^+]) \geq l(Q) + l(Q') + 2$ by (2.13). Since $l(Q) \geq 2$ and $l(Q') \geq 2$, this implies that

$$l(P'[u_1,u_1^+]) + l(P'[u_2,u_2^+]) \geq 6.$$

Moreover, in the case where $(u_1, u_2) \in \mathcal{W}$ or $(u_1^+, u_2^+) \in \mathcal{W}$, we have

$$l(Q) \geq d - n_P(Z^*) + 2 \quad \text{or} \quad l(Q') \geq d - n_P(Z^*) + 2,$$

respectively, and hence we obtain

$$l(P'[u_1, u_1^+]) + l(P'[u_2, u_2^+]) \geq d - n_P(Z^*) + 6. \quad \square$$

**Lemma 2.14.** Let $X$ be a subset of $W_1^-$. Then

$$\sum_{u \in X} l(P'[u, u^+]) \geq 3|X| - \varepsilon_X,$$

where

$$\varepsilon_X = \begin{cases} 1 & (\text{if } |X| = 1) \\ 0 & (\text{otherwise}). \end{cases}$$

**Proof.** In the case where $|X| = 0$, there is nothing to be proved and in the case where $|X| = 1$, the result is immediate from Lemma 2.11. Thus we may assume $|X| \geq 2$. Write

$$X = \{u_1, \cdots, u_m\}, \quad \text{where } m = |X|.$$

Then by Lemma 2.13, we obtain

$$\sum_{1 \leq i \leq m-1} \{l(P'[u_i, u_i^+]) + l(P'[u_{i+1}, u_{i+1}^+])\}$$
$$+\{l(P'[u_m, u_m^+]) + l(P'[u_1, u_1^+])\} \geq 6m,$$

$$\text{i.e.,} \qquad 2\sum_{u \in X} l(P'[u, u^+]) \geq 6|X|,$$

as desired. $\square$

By letting $X = W_1^-$ in Lemma 2.14, we get

$$\left.\begin{aligned} &\sum_{u \in W_1^-} l(P'[u, u^+]) \geq 3w_1^- - \varepsilon, \\ &\text{where} \quad \varepsilon = \begin{cases} 1 & (\text{if } w_1^- = 1) \\ 0 & (\text{otherwise}). \end{cases} \end{aligned}\right\} \tag{2.14}$$

For $u \in W_2$, we have $P'[u, u^+] = P[u, u^+]$. Hence by the definition of $W_2$, we get:

**Lemma 2.15.** $l(P'[u, u^+]) \geq 3$ for all $u \in W_2$. □

Using the lemmas above, we now proceed to show $l(P') \geq 2d - 2$. By Lemma 2.11.(i), $l(P') \geq 2(n_P(Z^*) - 1)$. Hence if $n_P(Z^*) \geq d$, then $l(P') \geq 2d - 2$, as desired. Thus we may assume

$$n_P(Z^*) \leq d - 1. \tag{2.15}$$

**Subcase 1.** $w_0 + w_1^* \geq 2$ :

By Lemma 2.11, we obtain

$$l(P') \geq 2(d - n_P(Z^*) + 2) + (n_P(Z^*) - 3) \cdot 2 = 2d - 2,$$

as desired.

**Subcase 2.** $w_0 + w_1^* \leq 1$ :

Recall that

$$\begin{aligned} W' &= U_0 - (W_0 \cup \{\alpha\}) \ (\subseteq N_P(Z^*) - (W \cup \{z'\})\ ), \\ w' &= |W'|. \end{aligned}$$

**Lemma 2.16.** $w' \geq w_1'$.

**Proof.** Since $w_0^* \leq 1$, $W_0^* \subseteq \{\alpha\}$ by the choice of $\alpha$, and hence

$$|W_0^* \cup \{\alpha\}| = 1. \tag{2.16}$$

Let $t_1, \cdots, t_k$; $b_1, \cdots, b_k$ be as in (ii) of Lemma 2.9. We may assume $b_1, \cdots, b_{k-1} \in Z$. Set $T = \{t_1, \cdots, t_{k-1}\}$. Since $b_j \in N_Z(t_j)$ for each $1 \leq j \leq k-1$, we clearly have

$$N_Z(t) \neq \phi \text{ for each } t \in T. \tag{2.17}$$

We may assume that $t_1, \cdots, t_{k-1}$ occur on $P$ in this order. Set

$$\begin{aligned} T_0 &= \{ t_j \mid 1 \le j \le k-2, \ |V(P(t_j, t_{j+1})) \cap Y| = 0 \}, \\ T_1 &= T - T_0, \\ \mathcal{J} &= \{ I_i \in \mathcal{I} \mid V(I_i) \cap T \ne \phi \}. \end{aligned}$$

Then $|T_1| \le |\mathcal{J}|$, and hence $|T_0| \ge k-1-|\mathcal{J}|$. Since $(t_j, t_{j+1}) \in \mathcal{U}$ for all $1 \le j \le k-2$, it follows from Lemma 2.8 that $V(P[t_j, t_{j+1}]) \cap (W_0^* \cup U_0) \ne \phi$ for each $j$ with $t_j \in T_0$. With this in mind, take $u_j \in V(P[t_j, t_{j+1}]) \cap (W_0^* \cup U_0)$ for each $j$ with $t_j \in T_0$, and set $U_0' = \{u_j \mid t_j \in T_0\}$. Then

$$U_0' \subseteq W_0^* \cup U_0 \tag{2.18}$$

and

$$|U_0'| = |T_0| \ge k - 1 - |\mathcal{J}|. \tag{2.19}$$

Set

$$\begin{aligned} V_1 &= W_1' \cup \{ u^+ \mid u \in W_1' \}, \\ \mathcal{J}' &= \{ I_i \in \mathcal{I} \mid V(I_i) \cap V_1 \ne \phi \}. \end{aligned}$$

By the definition of $W_1'$,

$$N_A(u) \ne \phi \text{ and } N_Z(u) = \phi \text{ for each } u \in V_1, \tag{2.20}$$

and hence

$$T \cap V_1 = \phi \tag{2.21}$$

by (2.17). Now if $w_1' = 0$, then there is nothing to be proved. Thus we may assume $w_1' > 0$. Then

$$|\mathcal{J}'| \ge w_1' + 1. \tag{2.22}$$

In what follows, we separate some points of the proof of Lemma 2.16, and present them as claims.

**Claim 1.** Let $0 \le i \le |Y| - 1$, and suppose that $I_i \in \mathcal{J} \cap \mathcal{J}'$. Take $t \in V(I_i) \cap T$ and $u \in V(I_i) \cap V_1$. Then $t \ne u$, and we have $V(P[u, t)) \cap W_0^* \ne \phi$ or $V(P[t, u)) \cap W_0^* \ne \phi$, according to whether $u$ occurs before $t$ or $t$ occurs before $u$.

**Proof.** We clearly have $t \neq u$ by (2.21). Now if $u$ occurs before $t$ on $P$, then $(u,t) \in \mathcal{W}^*$ by (2.17) and (2.20), and hence $V(P[u,t)) \cap W_0^* \neq \phi$ by Lemma 2.6; similarly, if $t$ occurs before $u$, then $(t,u) \in \mathcal{W}^*$ and hence $V(P[t,u)) \cap W_0^* \neq \phi$. □

**Claim 2.** Let $t_j \in T_0$. Then $V(P[t_j, t_{j+1}]) \cap V_1 = \phi$.
**Proof.** Suppose that there exists $u \in V(P[t_j, t_{j+1}]) \cap V_1$. Then by Claim 1, $w_0^* \geq |V(P[t_j,u)) \cap W_0^*| + |V(P[u,t_{j+1})) \cap W_0^*| \geq 2$, a contradiction. □

**Claim 3.** $|\mathcal{J} \cap \mathcal{J}'| \leq w_0^* \leq 1$.
**Proof.** By Claim 1, we have $V(P[y_i, y_{i+1})) \cap W_0^* \neq \phi$ for each $i$ with $I_i \in \mathcal{J} \cap \mathcal{J}'$. Hence $|\mathcal{J} \cap \mathcal{J}'| \leq w_0^* \leq 1$, as desired. □

**Claim 4.** $|\mathcal{J} \cap \mathcal{J}'| + |(W_0^* \cup \{\alpha\}) \cap U_0'| \leq 1$.
**Proof.** Suppose that $|\mathcal{J} \cap \mathcal{J}'| + |(W_0^* \cup \{\alpha\}) \cap U_0'| \geq 2$. Then by Claim 3 and (2.16),

$$|\mathcal{J} \cap \mathcal{J}'| = w_0^* = 1 \tag{2.23}$$

and

$$|(W_0^* \cup \{\alpha\}) \cap U_0'| = 1. \tag{2.24}$$

Since $W_0^* \neq \phi$ by (2.23), $W_0^* = \{\alpha\}$ by the choice of $\alpha$. Consequently,

$$|W_0^* \cap U_0'| = 1 \tag{2.25}$$

by (2.24). Write $\mathcal{J} \cap \mathcal{J}' = \{I_i\}$, and take $u \in V(I_i) \cap V_1$. If $V(P[y_i,u)) \cap T \neq \phi$, let $v_1$ be the vertex in $V(P[y_i,u)) \cap T$ closest to $u$ on $P$, and let $v_2 = u$; if $V(P(u, y_{i+1}]) \cap T \neq \phi$, let $v_1 = u$, and let $v_2$ be the vertex in $V(P(u, y_{i+1}]) \cap T$ closest to $u$ on $P$. Then

$$V(P(v_1, v_2)) \cap T = \phi, \tag{2.26}$$

and we get

$$V(P[v_1, v_2)) \cap W_0^* \neq \phi \tag{2.27}$$

by Claim 1. Since it follows from Claim 2 and (2.26) that $V(P[v_1, v_2)) \cap V(P[t_j, t_{j+1})) = \phi$ for each $t_j \in T_0$, we get $V(P[v_1, v_2)) \cap U_0' = \phi$ by the definition of $U'_0$. Consequently, it follows from (2.25) and (2.27) that $w_0^* \geq |W_0^* \cap U'_0| + |V(P[v_1, v_2)) \cap W_0^*| \geq 2$, a contradiction. □

Returning to the proof of the lemma, we now obtain

$$\begin{aligned}
w' &= |U_0 - (W_0 \cup \{\alpha\})| \\
&= |U_0 - (W_0^* \cup \{\alpha\})| \qquad \text{(by (2.9))} \\
&\geq |U'_0 - (W_0^* \cup \{\alpha\})| \qquad \text{(by (2.18))} \\
&= |U'_0| - |(W_0^* \cup \{\alpha\}) \cap U'_0| \\
&\geq k - 1 - |\mathcal{J}| - |(W_0^* \cup \{\alpha\}) \cap U'_0| \qquad \text{(by (2.19))} \\
&\geq k - 1 - (k - 1 - |\mathcal{J}'| + |\mathcal{J} \cap \mathcal{J}'|) - |(W_0^* \cup \{\alpha\}) \cap U'_0| \\
&\qquad\qquad \text{(since } |\mathcal{J} \cup \mathcal{J}'| \leq |\mathcal{I}| = |Y| \leq k - 1) \\
&\geq |\mathcal{J}'| - 1 \qquad \text{(by Claim 4)} \\
&\geq w'_1 \qquad \text{(by (2.22)).}
\end{aligned}$$

This completes the proof of Lemma 2.16. □

**Subcase 2.1.** $w_0 + w_1^+ \geq 2$ :

First assume $w_0 + w_1^* = 1$. Then $w'_1 \geq 1$, and hence $w' \geq w'_1 \geq 1$ by Lemma 2.16. Let

$$\{u\} = W_0 \cup W_1^*,$$
$$v_1 \in W'_1, \quad v_2 \in W'.$$

Then by Lemmas 2.11 and 2.12,

$$\begin{aligned}
l(P') &\geq l(P'[u, u^+]) + l(P'[v_1, v_1^+]) + l(P'[v_2, v_2^+]) + (n_P(Z^*) - 4) \cdot 2 \\
&\geq (d - n_P(Z^*) + 2) + (d - n_P(Z^*) + 4) + 2n_P(Z^*) - 8 \\
&= 2d - 2,
\end{aligned}$$

as desired. Next assume that $w_0 + w_1^* = 0$. Then $w' \geq w'_1 \geq 2$. Hence we can find $u_1, u_2, v_1, v_2$ such that

$$\{u_1, u_2\} \subseteq W'_1, \quad u_1 \neq u_2,$$
$$\{v_1, v_2\} \subseteq W', \quad v_1 \neq v_2.$$

Consequently, we obtain

$$\begin{aligned} l(P') &\geq l(P'[u_1,u_1^+]) + l(P'[v_1,v_1^+]) \\ &\qquad + l(P'[u_2,u_2^+]) + l(P'[v_2,v_2^+]) + (n_P(Z^*) - 5)\cdot 2 \\ &\geq 2(d - n_P(Z^*) + 4) + 2n_P(Z^*) - 10 \\ &= 2d - 2. \end{aligned}$$

**Subcase 2.2.** $w_0 + w_1^+ \leq 1$ :

**Subcase 2.2.1.** $w \geq k-1$ :

By the assumption of this case,

$$w_0 + w_1^+ + w_1^- + w_2 \geq k - 1. \tag{2.28}$$

Combining Lemma 2.4 and (2.28), we get

$$w_2 \leq w_0 - 1.$$

Consequently,

$$w_0 = 1, \quad w_2 = 0, \quad w_1^+ = 0,$$

and hence again by Lemma 2.4 and (2.28),

$$w_1^- = k-2, \quad |Y| = k-1,$$

which implies that

$$V(I_i) \cap W_1^- \neq \phi \text{ for every } 0 \leq i \leq |Y| - 2 = k - 3.$$

**Lemma 2.17.** There exist $u_0 \in W_1^-$ and $u_1 \in W_1^-$ such that $u_0$ occurs before $u_1$ on $P$ and such that $\{(u_0,u_1),(u_0^+,u_1^+)\} \cap \mathcal{W} \neq \phi$.

**Proof.** Write $W_0 = \{v_0\}$, and let $v_0 \in V(P[y_m, y_{m+1}))$. We first consider the case where $k \geq 5$ or $m \neq 1$. In this case, there exists $l \in \{i \in \mathbb{N} \mid 1 \leq i \leq k-3\} - \{m\}$. For this $l$ and for $j = 0, 1$, let $V(I_{l-1+j}) \cap W_1^- = \{u_j\}$. Suppose that $(u_0, u_1) \notin \mathcal{W}$. Then $N_A(u_0) = N_A(u_1)$ and $n_A(u_0) \leq 1$ and, in the case where $n_A(u_0) = 1$, we have $N_Z(u_0) = N_Z(u_1) = \phi$. Since $(u_0, u_0^+) \in \mathcal{W}$, this implies that $u_0^+ \neq u_1$ and $(u_0^+, u_1) \in \mathcal{W}$, and hence by

Lemma 2.5, there exists $u \in V(P[u_0^+, u_1)) \cap W_0 \subseteq V(P[y_l, y_{l+1}))$, which contradicts the fact that $w_0 = 1$. Thus $(u_0, u_1) \in \mathcal{W}$.

We now consider the case where $k = 4$ and $m = 1$. For $j = 0, 1$, let $V(I_j) \cap W_1^- = \{u_j\}$. Since $v_0 \in V(P[y_1, y_2)) \cap W_0$, $|V(I_1) \cap N_P(Z^*)| \geq 2$ by the definition of $W_0$, and hence

$$u_0^+ \neq u_1. \tag{2.29}$$

Since $w_0 = 1$, we also get

$$V(P[y_0, y_1)) \cap W_0 = V(P[y_2, y_3]) \cap W_0 = \phi. \tag{2.30}$$

First assume $|A| \geq 3$. Then since $G - c$ is 3-connected,

$$n_A(V(P)) \geq 3. \tag{2.31}$$

By Lemma 2.5, (2.29) and the assumption that $w_0 = 1$,

$$N_A(V(I_1)) = N_A(\{u_0^+, u_1\}).$$

By Lemma 2.5 and (2.30),

$$N_A(V(I_0)) = N_A(u_0), \quad N_A(V(I_2)) = N_A(u_1^+).$$

Consequently,

$$N_A(V(P)) = N_A(\{u_0, u_1\}) \cup N_A(\{u_0^+, u_1^+\}),$$

and hence by (2.31), $n_A(\{u_0, u_1\}) \geq 2$ or $n_A(\{u_0^+, u_1^+\}) \geq 2$, which implies that $(u_0, u_1) \in \mathcal{W}$ or $(u_0^+, u_1^+) \in \mathcal{W}$, respectively. Next assume $|A| \leq 2$, and take $a \in A$. Then

$$n_P(a) = d_G(a) - d_B(a) \geq d - 2 \geq n_P(Z^*) - 1$$

by (2.15), and hence $a \in N_A(u_0) \cap N_A(u_0^+)$ or $a \in N_A(u_1) \cap N_A(u_1^+)$ by (2.29). Now if $a \in N_A(u_0) \cap N_A(u_0^+)$, then since we have $N_{Z^*}(u_1) - \{a\} \neq \phi$, or $N_{Z^*}(u_1^+) - \{a\} \neq \phi$ by the fact that $(u_1, u_1^+) \in \mathcal{W}$, we get $(u_0, u_1) \in \mathcal{W}$ or $(u_0^+, u_1^+) \in \mathcal{W}$; similarly, if $a \in N_A(u_1) \cap N_A(u_1^+)$, the desired conclusion follows from the fact that $(u_0, u_0^+) \in \mathcal{W}$. □

By the fact that $w_0 = 1$ and by Lemmas 2.17, 2.11 and 2.13,

$$l(P') \geq (d - n_P(Z^*) + 2) + (d - n_P(Z^*) + 6) + (n_P(Z^*) - 4) \cdot 2 = 2d,$$

as desired.

**Subcase 2.2.2.** $w \leq k - 2$ :

**Lemma 2.18.** $|V(B)| \geq d - n_P(Z^*) + 3$.
**Proof.** Take $a \in A$. We first show that

$$n_P(a) \leq n_P(Z^*) - 2. \tag{2.32}$$

By way of contradiction, suppose that $n_P(a) \geq n_P(Z^*) - 1$. Then $u \in W \cup \{z'\}$ for every $u \in N_P(Z^* - \{a\})$. Consequently,

$$N_P(Z^* - \{a\}) \subseteq W \cup \{z'\}, \tag{2.33}$$

and hence

$$n_P(Z^* - \{a\}) \leq w + 1 \leq k - 1 \tag{2.34}$$

by the assumption of Subcase 2.2.2. On the other hand, since $G - c'$ is $(k-1)$-connected, $n_P(Z) \geq k - 1$, and hence

$$n_P(Z^* - \{a\}) \geq k - 1. \tag{2.35}$$

By (2.33), (2.34) and (2.35),

$$N_P(Z^* - \{a\}) = W \cup \{z'\}. \tag{2.36}$$

Since $n_P(Z^*) \geq k$ by the assumption of Case 1.1, (2.34) implies that

$$N_P(a) - N_P(Z^* - \{a\}) \neq \phi.$$

By (2.36), $z' \notin N_P(a) - N_P(Z^* - \{a\})$. Hence there exists $u \in N_P(a) - N_P(Z^* - \{a\})$ such that $u^+ \in N_P(Z^* - \{a\})$. Then $u \in W$, which contradicts (2.36). Thus (2.32) is proved. We now obtain

$$n_P(Z^*) - 2 \geq n_P(a) = d_G(a) - d_B(a) \geq d - (|V(B)| - 1),$$

and hence

$$|V(B)| \geq d - n_P(Z^*) + 3,$$

as desired. □

**Lemma 2.19.** $4 \leq |V(B)| \leq w + 1$.

**Proof.** The first inequality follows immediately from Lemma 2.18 and (2.15). To prove the second inequality, we let $m = \min\{\, |V(B)|,\ k\,\}(\geq 4)$, and show that $m \leq w + 1$ and $m = |V(B)|$. We first prove the following claim.

**Claim.** There exist $m$ independent edges $e_i = a_i u_i \quad (1 \leq i \leq m)$ with $a_i \in Z^*$ and $u_i \in V(P)$ such that $|A \cap \{a_1, \cdots, a_m\}| \geq m - 1$.

**Proof.** Since $G - c$ is $(k-1)$-connected and since $|V(P)| > k - 1 \geq m - 1$, there exist $m - 1$ independent edges $f_i = c_i v_i \quad (1 \leq i \leq m - 1)$ with $c_i \in A$ and $v_i \in V(P)$. If $N_P(Z^* - \{c_1, \cdots, c_{m-1}\}) - \{v_1, \cdots, v_{m-1}\} \neq \phi$, then by letting $e_m$ be an edge joining $Z^* - \{c_1, \cdots, c_{m-1}\}$ and $V(P) - \{v_1, \cdots, v_{m-1}\}$, and letting $e_i = f_i$ for each $1 \leq i \leq m-1$, we get edges with the desired properties. Thus we may assume $N_P(Z^* - \{c_1, \cdots, c_{m-1}\}) \subseteq \{v_1, \cdots, v_{m-1}\}$. Since we get $n_P(Z) \geq k - 1$ from the fact that $G - c'$ is $(k-1)$-connected, this implies that ($m = k$ and)

$$N_P(Z^* - \{c_1, \cdots, c_{m-1}\}) = \{v_1, \cdots, v_{m-1}\}. \tag{2.37}$$

Since $n_P(Z^*) \geq k$ by the assumption of Case 1.1, we also see that one of the $c_i$, say $c_1$, is adjacent to a vertex $u$ in $V(P) - \{v_1, \cdots, v_{m-1}\}$. By (2.37), $v_1$ is adjacent to a vertex $a$ in $Z^* - \{c_1, \cdots, c_{m-1}\}$. Now if we let $e_1 = c_1 u$ and $e_m = a v_1$, and let $e_i = f_i$ for each $2 \leq i \leq m - 1$, then the $e_i \quad (1 \leq i \leq m)$ satisfy the desired properties. □

Returning to the proof of the lemma, let $u_i \ (1 \leq i \leq m)$ be as in the claim. We may assume that $u_i$ occurs before $u_j$ on $P$ if $i < j$. Then $(u_i, u_{i+1}) \in W$ for each $1 \leq i \leq m - 1$. Consequently,

$$w \geq m - 1 \tag{2.38}$$

by Lemma 2.5, and hence $m \leq k-1$ by the assumption of Subcase 2.2.2. By the definition of $m$, this implies that $m = |V(B)|$, and hence $|V(B)| \leq w+1$ by (2.38). This completes the proof of Lemma 2.19. □

By the assumption of Subcase 2.2 and by Lemmas 2.18 and 2.19,

$$w_1^- + w_2 \geq w - 1 \geq |V(B)| - 2 \geq d - n_P(Z^*) + 1. \tag{2.39}$$

Now assume for the moment that $w_0 + w_1^+ = 1$. Then

$$w - (w_1^- + w_2) = 1. \tag{2.40}$$

We show that

$$l(P') \geq d + n_P(Z^*) + (w_1^- + w_2) - 2\{w - (w_1^- + w_2)\} - 1. \tag{2.41}$$

If $w_0 + w_1^* = 1$, then by Lemmas 2.11, 2.15 and (2.14),

$$l(P') \geq (d - n_P(Z^*) + 2) + (3w_1^- - 1) + 3w_2 + 2(n_P(Z^*) - w - 1),$$

and hence (2.41) holds. Thus we may assume $w_0 + w_1^* = 0$. Then $w_1' = 1$. Consequently, by Lemmas 2.11, 2.16, 2.12, 2.15 and (2.14),

$$l(P') \geq (d - n_P(Z^*) + 4) + (3w_1^- - 1) + 3w_2 + 2(n_P(Z^*) - w - 2),$$

and hence (2.41) again holds. Thus (2.41) is proved. Combining (2.41), (2.39) and (2.40), we obtain

$$\begin{aligned} l(P') &\geq d + n_P(Z^*) + (d - n_P(Z^*) + 1) - 2 \cdot 1 - 1 \\ &= 2d - 2, \end{aligned}$$

as desired.

Thus we may assume $w_0 + w_1^+ = 0$. Then

$$w_1^- + w_2 = w. \tag{2.42}$$

We first consider the case where there exist $u \in W_1^-$ and $v \in W_1^-$ such that $(u, v) \in \mathcal{W}$. Write $W_1^- = \{u_1, \cdots, u_n\}$, where $n = w_1^-$. At the cost of relabeling, we may assume $(u_1, u_2) \in \mathcal{W}$. Then by letting $X = W_1^- - \{u_1, u_2\}$ in Lemma 2.14, we see that

$$\begin{aligned} \sum_{u \in W_1^- - \{u_1, u_2\}} l(P'[u, u^+]) &\geq 3(w_1^- - 2) - 1 \\ &= 3w_1^- - 7. \end{aligned} \tag{2.43}$$

On the other hand,

$$l(P'[u_1, u_1^+]) + l(P'[u_2, u_2^+]) \geq d - n_P(Z^*) + 6 \tag{2.44}$$

by Lemma 2.13. Adding (2.43) and (2.44), we obtain

$$\sum_{u \in W_1^-} l(P'[u, u^+]) \geq d - n_P(Z^*) + 3w_1^- - 1,$$

and hence

$$\begin{aligned} l(P') &\geq (d - n_P(Z^*) + 3w_1^- - 1) + 3w_2 + 2(n_P(Z^*) - w - 1) \\ &= d - n_P(Z^*) + (w_1^- + w_2) + 2(w_1^- + w_2 + n_P(Z^*) - w) - 3 \\ &\geq d - n_P(Z^*) + (d - n_P(Z^*) + 1) + 2n_P(Z^*) - 3 \\ &\qquad\qquad \text{(by (2.39) and (2.42))} \\ &= 2d - 2, \end{aligned}$$

as desired.

Finally, we consider the case where no two vertices $u$ and $v$ in $W_1^-$ satisfy $(u, v) \in \mathcal{W}$. Since $w_0 + w_1^+ = 0$, we see that for every $a \in A$,

$$\begin{aligned} n_{W_1^-}(a) + n_{N_P(Z^*) - W}(a) &= d_G(a) - d_B(a) - n_{W_2}(a) \\ &\geq d_G(a) - (|V(B)| - 1) - w_2 \\ &= d_G(a) - (|V(B)| + w_2) + 1, \end{aligned} \tag{2.45}$$

and hence

$$n_{W_1^-}(a) + n_{N_P(Z^*) - W}(a) \geq d - (|V(B)| + w_2) + 1. \tag{2.46}$$

Since $w_0 + w_1^+ = 0$, it follows from Lemmas 2.19 and 2.4 that

$$\begin{aligned} k + 1 - (|V(B)| + w_2) &\geq k + 1 - (w + 1 + w_2) \\ &= k - (w_1^- + 2w_2) \\ &\geq k - (k - 2) \\ &= 2. \end{aligned} \tag{2.47}$$

Since $d \geq k+1$, this implies

$$d - (|V(B)| + w_2) \geq 2. \tag{2.48}$$

**Lemma 2.20.** $n_P(Z^*) - w \geq d - w_2 - 1 - \varepsilon$, where

$$\varepsilon = \begin{cases} 1 & (\text{if } w_1^- = 1) \\ 0 & (\text{otherwise}), \end{cases}$$

as in (2.14).

**Proof.** Set $X = N_P(Z^*) - W$. Since $|\{ a \in A \mid u \in N_P(a) \}| \leq 1$ for every $u \in X - \{z'\}$,

$$|X - \{z'\}| \geq \sum_{a \in A} n_{X - \{z'\}}(a) \geq \sum_{a \in A} (n_X(a) - 1),$$

and hence

$$n_P(Z^*) - w = |X| \geq \sum_{a \in A} (n_X(a) - 1) + 1. \tag{2.49}$$

If $w_1^- = 1$, then by (2.46) and (2.48),

$$n_X(a) \geq d - (|V(B)| + w_2) \geq 2 \quad \text{for each } a \in A,$$

and hence by (2.49), we obtain

$$\begin{aligned} n_P(Z^*) - w &\geq (|V(B)| - 1)\{d - (|V(B)| + w_2) - 1\} + 1 \\ &\geq (|V(B)| - 2) \cdot 1 + \{d - (|V(B)| + w_2) - 1\} + 1 \\ &= d - w_2 - 2. \end{aligned}$$

Thus we may assume $w_1^- \neq 1$. In this case, by the assumption that no two vertices $u$, $v$ in $W_1^-$ satisfy $(u, v) \in \mathcal{W}$, we have

$$|\{a \in A \mid N_{W_1^-}(a) \neq \phi\}| \leq 1.$$

Hence by (2.46),

$$\begin{aligned} &\text{at least } |A| - 1 \text{ vertices } a \text{ in } A \text{ satisfy} \\ &n_X(a) \geq d - (|V(B)| + w_2) + 1. \end{aligned} \tag{2.50}$$

Since $|V(B)| \geq 4$ by Lemma 2.19, we now obtain

$$\begin{aligned} n_P(Z^*) - w &\geq (|V(B)| - 2)\{d - (|V(B)| + w_2)\} + 1 \\ &\qquad \text{(by (2.49) and (2.50))} \\ &\geq (|V(B)| - 3) \cdot 2 + \{d - (|V(B)| + w_2)\} + 1 \\ &\qquad \text{(by (2.48))} \\ &\geq (|V(B)| - 3) + 1 + \{d - (|V(B)| + w_2)\} + 1 \\ &= d - w_2 - 1. \quad \square \end{aligned}$$

Now since $w \geq 3$ by Lemma 2.19, we see that

$$\begin{aligned} l(P') &\geq (3w_1^- - \varepsilon) + 3w_2 + 2(n_P(Z^*) - w - 1) \\ &\geq 3w_1^- + 3w_2 + 2(d - w_2 - 2 - \varepsilon) - \varepsilon \\ &= (2d - 2) + (w - 3) + (2w_1^- + 1 - 3\varepsilon) \\ &\geq 2d - 2, \end{aligned}$$

as desired. This completes the proof for Case 1.1.1.

**Case 1.1.2.** $y \in A$, or $y = c$ and $d_B(y) \geq 3$ :

**Lemma 2.21.** $W_0^* \cup U_0 \neq \phi$.
**Proof.** Since $G$ is $k$-connected, there are $k$ $(y, V(P))$-paths pairwise disjoint except at $y$. Let $t_1, \cdots, t_k$ be the endvertices of these paths. Then for any $i, j$ with $i \neq j$ such that $t_i$ occurs before $t_j$ on $P$, $(t_i, t_j) \in \mathcal{W}^* \cup \mathcal{U}$. Thus, arguing exactly as in the proof of Lemma 2.10, we get $W_0^* \cup U_0 \neq \phi$. $\square$

Take $\alpha \in W_0^* \cup U_0$, and let $P_0$ be a longest $(\alpha, y, \alpha^+)$-path whose inner vertices lie in $V(H)$. Then the path $P' = P[x, \alpha]P_0P[\alpha^+, z]$ is an $(x, Y, z)$-path passing through all vertices in $N_P(Z^*)$. We can now argue as in Case 1.1.1 to obtain $l(P') \geq 2d - 2$ (note that $\mathcal{W}^* = \mathcal{W}$, $W_0^* = W_0$ and $W_1^* = W_1^+$ in this case, and thus Subcase 2.1 and the case where $1 = w_0 + w_1^+ > w_0 + w_1^* = 0$ in Subcase 2.2.2 do not occur and we do not need Lemmas 2.12 and 2.16). This is the end of Case 1.1.2, and concludes the discussion for Case 1.1.

**Case 1.2.** $n_P(Z^*_B) \leq k-1$:

By the choice of $B$, the assumption of this case implies that $\mathcal{E}^* = \phi$ (see the paragraph preceding Lemma 2.1). By (2.4), the assumption of this case also implies that

$$n_P(Z^*) = n_P(A) = k-1, \tag{2.51}$$

where $Z^* = Z^*_B$ and $A = A_B$ as in Case 1.1. Since $n_P(V(H)) \geq k$ by the $k$-connectedness of $G$, we see from (2.51) that there exist $u_0$ and $a_0$ such that

$$u_0 \in V(P) - N_P(Z^*),\ a_0 \in V(H) - Z^*, \text{ and } a_0u_0 \in E(G). \tag{2.52}$$

This in particular implies that $V(H) \neq Z^*$, and hence $y \notin V(B)$. Let $H' = \langle Z \cup \{c'\}\rangle$, where $Z = Z_B$ and $c' = c'_B$. We define $\mathcal{E}'$ as follows: if $H'$ is nonseparable, then let $\mathcal{E}' = \{H'\}$; if $H'$ is separable, then let $\mathcal{E}'$ be the set of endblocks of $H'$. We have $\mathcal{E} \cap \mathcal{E}' \neq \phi$. Take $B' \in \mathcal{E} \cap \mathcal{E}'$. Then $Z_{B'} \supseteq V(H) - Z$, and hence

$$a_0 \in Z_{B'}. \tag{2.53}$$

For simplicity, set $A' = A_{B'} = V(B') - C$. Since $A' \subseteq Z$ and since $n_P(A') \geq k-1$, it follows from (2.51) that

$$N_P(Z^*) = N_P(A) = N_P(A') = N_P(Z). \tag{2.54}$$

By (2.54), (2.53), (2.52) and (2.51), $n_P(Z^*_{B'}) \geq |N_P(Z^*) \cup \{u_0\}| = k$. Since $\mathcal{E}^* = \phi$, this implies that

$$A' \cap S \neq \phi. \tag{2.55}$$

By (2.55), (2.1) and (2.54), $V(P) \cap Y \cap S \subseteq N_P(A') = N_P(Z^*)$, and hence

$$W_1^- = \phi \tag{2.56}$$

by the definition of $W_1^-$. Since $B' \in \mathcal{E} \cap \mathcal{E}'$ was arbitrary, (2.55) together with (2.1) also implies that

$$|\mathcal{E} \cap \mathcal{E}'| = 1. \tag{2.57}$$

Since (2.54) implies $N_P(Z^*) = W \cup \{z'\}$ and $\mathcal{W} = \mathcal{W}^*$, we get the following lemma by (2.56):

**Lemma 2.22.** $N_P(Z^*) = W_0^* \cup W_1^* \cup W_2 \cup \{z'\}$. □

**Lemma 2.23.** For every $a \in A$ and for every $a' \in A'$, there exists an $(a, y, a'; 2(d-k+1))$-path in $H$.

**Proof.** By Lemma 2.1 and (2.51), $B$ contains an $(a, c; d-(k-1))$-path. By extending this path, we obtain an $(a, c'; d-k+1)$-path $P_1$ in $H$ (recall that $c = c_B$ and $c' = c'_B$). Denote by $B_1$ the block which contains $c'$ and $y$. We first consider the case where $B_1 = B'$ (so $B_1 = H'$). In this case, applying Proposition D to $B'$, we see that there exists a $(c', y, a'; d-k+1)$-path $P_2$ in $B'$. Then $P_1P_2$ is an $(a, y, a'; 2(d-k+1))$-path in $H$. We now consider the case where $B_1 \neq B'$. By (2.57), $|V(B_1) \cap C| = 2$. Write $(V(B_1) \cap C) - \{c'\} = \{c_1\}$ (it is possible that $c_1 = y$). Note that $(V(B_1) - \{c', c_1\}) \cap S = \phi$ by (2.55) and (2.1). Assume for the moment that $|V(B_1)| \geq 3$. If $y = c_1$, let $y'$ be a vertex in $V(B_1) - \{c', c_1\}$; if $y \neq c_1$, let $y' = y$. Then $d_{B_1}(y') \geq d-k+1$. Now applying Proposition C to $B_1$, we can find a $(c', y', c_1; d-k+1)$-path $P_2$ in $B_1$. Then $P_2$ is a $(c', y, c_1; d-k+1)$-path. Extending the path $P_1P_2$, we obtain an $(a, y, a'; 2(d-k+1))$-path in $H$. Thus we may assume that $|V(B_1)| = 2$ (so $y = c_1$ and $V(B_1) = \{c', y\}$). By (2.57), there exists exactly one block $B_2$ of $H$ such that $B_2$ contains $y$ and $B_2 \neq B_1$. We have

$$d_{B_2}(y) = d_G(y) - \{(k-1) + 1\} \geq d - k.$$

First assume $B_2 = B'$. Then we can apply Proposition D to $B_2$ to obtain a $(y, a'; d-k)$-path in $B_2$, and hence a $(c', y, a'; d-k+1)$-path $P_2$ in $H$. Thus $P_1P_2$ is an $(a, y, a'; 2(d-k+1))$-path in $H$. Next assume $B_2 \neq B'$. Then again by (2.57), we can write $V(B_2) \cap C = \{y, c_2\}$. Applying Proposition C to $B_2$, we obtain a $(y, c_2; d-k)$-path in $B_2$, which we can extend to a $(c', y, a'; d-k+2)$-path $P_2$ in $H$. Then $P_1P_2$ is an $(a, y, a'; 2(d-k+1)+1)$-path in $H$. □

We first consider the case where $w_0^* \geq 1$. Take $\alpha \in W_0^*$. Let $P_0$ be a longest $(\alpha, y, \alpha^+)$-path whose inner vertices lie in $V(H)$, and define a path $P'$ by $P' = P[x, \alpha]P_0P[\alpha^+, z]$. By (2.54), we can find $a \in A$ and $a' \in A'$ such that $\alpha \in N_P(a)$ and $\alpha' \in N_P(a')$, and hence $l(P_0) \geq 2(d-k+1) + 2$

by Lemma 2.23. Hence

$$l(P') \geq 2(d-k+1)+2+2(k-3) = 2d-2,$$

as desired. We now consider the case where $w_0^* = 0$. Then by (2.51) and Lemma 2.22,

$$w_1^* + w_2 + 1 = k-1. \tag{2.58}$$

On the other hand,

$$w_1^* + 2w_2 \leq |V(P) \cap Y| \leq k-2 \tag{2.59}$$

by the definition of $W_1$ and $W_2$. By (2.58) and (2.59), $w_1^* = k-2$, $w_2 = 0$ and $|V(P) \cap Y| = k-2$. Hence $|\mathcal{I}| = k-1$, and $|V(I_i) \cap N_P(Z^*)| = 1$ for each $I_i \in \mathcal{I}$. Let $u_0$ and $a_0$ be as in (2.52). Choose $I_{i_0} \in \mathcal{I}$ so that $u_0 \in V(I_{i_0})$, and write $V(I_{i_0}) \cap N_P(Z^*) = \{u\}$. Write $\{u_0, u\} = \{\alpha, \beta\}$ so that $\alpha$ occurs before $\beta$ on $P$. Take $a' \in N_{A'}(u)$. By (2.53), there is an $(a_0, y, a')$-path in $H$, and hence there is an $(\alpha, y, \beta)$-path $P_0$ whose inner vertices are in $V(H)$. Consider the path $P' = P[x, \alpha]P_0P[\beta, z]$. Since $w_1^* = k-2 \geq 2$, there exists $v \in W_1^*$ such that $u_0 \notin V(P[v, v^+])$. Since $N_A(v) \neq \phi$ and $N_{A'}(v^+) \neq \phi$, $l(P[v, v^+]) \geq 2(d-k+1)+2$ by Lemma 2.23 and by the maximality of $l(P)$. Since $P[v, v^+] = P'[v, v^+]$ by the choice of $v$, we obtain

$$l(P') \geq 2(d-k+1)+2+2(k-3) = 2d-2,$$

as desired. This completes the proof for Case 1.2, and this is the end of Case 1.

**Case 2.** $H$ is nonseparable :

In this case, we argue as in Case 1.1.2 (hence as in Case 1.1.1 as well) with some alterations:

Let

$$B = H, \qquad A = Z^* = V(H).$$

We get $n_P(V(H)) \geq k$ by the assumption that $G$ is $k$-connected. Define $P'$ as in Case 1.1.2. If $n_P(V(H)) \geq d$, then we clearly have $l(P') \geq 2(n_P(V(H))-1) \geq 2(d-1)$. Thus we may assume $n_P(V(H)) \leq d-1$, and hence $|V(H)| \geq 2$. Applying Proposition D to $H$, we see that Lemma 2.1 holds. Consequently, if $w_0 + w_1^+ \geq 2$, then we can argue as in Case 1.1.2. Thus we may assume $w_0 + w_1^+ \leq 1$. Only the proofs of Lemmas 2.18 and 2.20 need modification. The proof of Lemma 2.18 works if we let $a = y$. Thus consider Lemma 2.20. We see that (2.47) and (2.48) hold in this case as well. Further, (2.45) holds for every $a \in V(H)$, and (2.46) holds for every $a \in V(H) - S$. Consequently, if $V(H) \cap S = \phi$, then we can argue as in the proof in Case 1.1.1. Thus we may assume $V(H) \cap S \neq \phi$. Then by (2.1) and the definition of $W_1^-$, $w_1^- = 0$, and hence it follows from (2.45) that

$$n_{N_P(Z^*) - W}(a) \geq k - (|V(H)| + w_2) + 1 \quad \text{for all } a \in V(H) \tag{2.60}$$

and

$$n_{N_P(Z^*) - W}(y) \geq d - (|V(H)| + w_2) + 1. \tag{2.61}$$

Now we can argue as in Case 1.1.1, using (2.60) and (2.61) in place of (2.50), and (2.47) in place of (2.48), to get

$$\begin{aligned} n_P(Z^*) - w &\geq (|V(H)| - 1) \cdot 1 + \{d - (|V(H)| + w_2)\} + 1 \\ &= d - w_2. \end{aligned}$$

This completes the proof of Theorem 1.

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