Weak Clique–Covering Cycles and Paths

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Abstract

In this paper, we investigate the sufficient conditions for a graph to contain a cycle (path) C such that $G - V(C)$ is a disjoint union of cliques. In particular, sufficient conditions involving degree sum and neighborhood union are obtained.

1 Introduction A cycle C of a graph G is called a *covering cycle*, or *C-cycle*, if $V(G) - V(C)$ is an independent set of vertices in G. Covering cycles have many applications in the study of longest cycles and paths in graphs. Such cycles have also been called dominating cycles in the literature. Sufficient conditions for a graph to have a covering cycle begin with a result of Nash-Williams.

Theorem 1 (Nash-Williams[7]) *Let G be a 2-connected graph of order $n \geq 3$. If the minimum degree $\delta(G) \geq \frac{n+2}{3}$, then $G - V(C)$ is a union of independent vertices of G for every longest cycle C.*

Bondy generalized Nash-William's result by showing the following.

Theorem 2 (Bondy [1]) *Let G be a 2-connected graph of order n. If*

$$d(u) + d(v) + d(w) \geq n + 2$$

for every three independent vertices u, v, and w, then $G - V(C)$ is a union of independent vertices for every longest cycle C of G.

Moving further in this direction, Bondy made the following conjecture in the same paper.

Conjecture 1 (Bondy[1]) *Let G be a simple k-connected graph on n vertices. If the degree sum of any $k+1$ independent vertices is at least $n + k(k-1)$, and if C is a longest cycle of G, then $G - V(C)$ contains no path of length $k-1$.*

Let k be a positive integer. A *weak k-covering cycle* of a graph G is a cycle C such that each component of $G - V(C)$ has fewer than k vertices. Such a cycle has also been called a *k-dominating cycle.* The following result is due to Fraisse.

Theorem 3 (Fraisse[5]) *Let G be a simple k-connected graph on $n \geq 3$ vertices in which the degree sum of any $k+1$ independent vertices is at least $n + k(k-1)$. Then G has a weak k-covering cycle.*

In this paper we generalize the concept the covering cycle in another direction. A cycle C of a graph G is called a *weak clique-covering cycle*, or a *CC-cycle*, if each component of $G - V(C)$ is a clique. Clearly, if a cycle C is a C-cycle then it is a CC-cycle. We will investigate the sufficient conditions for a graph to have a CC-cycle. First notice that if G has a CC-cycle then G has a maximal cycle which is a CC-cycle. Let $p \geq 3$ be a positive integer. The graph $K_2 + 3(K_p - E(K_2))$ shows that there are infinitely many integers n such that there is a graph of order n with minimum degree $\delta \geq (n-2)/3$ and no CC-cycles. Thus, we do not expect much improvement in the minimum degree in Nash-William's theorem if we replace the C-cycle by CC-cycle. In this paper we will investigate the neighborhood union conditions and the mixed neighborhood union and degree sum conditions for graphs having CC-cycles.

Let G be an arbitrary graph. Throughout this paper we will use $NC_2(G)$ to denote the minimum value of the cardinality of the neighborhood union over every pair of nonadjacent vertices in G, that is,

$$NC_2(G) = \min\{|N(u) \cup N(v)| : u \neq v \text{ and } uv \notin E(G)\},$$

and $\sigma_2(G)$ to denote the minimum degree sum over every pair of nonadjacent vertices of G, that is,

$$\sigma_2(G) = \min\{d(u) + d(v) : u \neq v \text{ and } uv \notin E(G)\}.$$

The following results are obtained.

Theorem 4 *Let G be a 3-connected graph of order n. If*

$$d(u) + d(v) + 2|N(u) \cup N(v)| \geq n + 4$$

for every pair of nonadjacent vertices u and v, then G has a longest cycle which is also a CC-cycle.

Let $p \geq 2$ be a positive integer and let M_{2p} denote the graph obtained from K_{2p} by removing a perfect matching. Then the graph $G = K_3 + 4M_{2p}$ has $n = 8p + 3$ vertices and

$$d(u) + d(v) + 2|N(u) \cup N(v)| \geq 8p + 4 = n + 1$$

for every pair of nonadjacent vertices u and v. It is readily seen that G does not have a CC-cycle. Thus the gap between the theorem and the example for the lower bounds is the difference between $n+4$ and $n+2$. If we replace the condition of 3-connectedness by 2-connectedness, we obtain the following result.

Theorem 5 *Let G be a 2-connected graph of order n. If $NC_2(G) \geq \frac{1}{3}(n+10)$, then G has a CC-cycle.*

It is readily seen that if G has a weak clique-covering cycle C, then any cycle containing all vertices of C is also a clique-covering cycle. Let $C_4 = x_1y_1x_2y_2x_1$ be a 4-cycle and let $p_1 > p_2$ be two positive integers. Let

$$G = (2K_{p_1} + \{x_1, x_2\}) \cup (2K_{p_2} + \{y_1, y_2\}) \cup E(C_4).$$

It is not difficult to see that every longest cycle of G has the vertex set of $2K_{p_1} \cup \{x_1, x_2\}$. Hence G contains no longest cycle which is also a CC-cycle. Therefore, even though there is a CC-cycle, and hence a maximal CC-cycle, guaranteed by the above theorem, it need not be a longest cycle.

A path P of G is called a *weak clique-covering path*, or *CC-path*, if each component of $G - V(P)$ is a clique. We will consider the condition that for every pair of vertices u and v there is a CC-path joining u and v. In fact, we will use the following results to prove Theorem 5.

Theorem 6 *Let G be a 3-connected graph of order n. If*

$$d(u) + d(v) + |N(u) \cup N(v)| \geq n + 3$$

for every pair of nonadjacent vertices u and v, then for every pair of vertices x and y there is CC-path $P[x, y]$ which is also a longest path joining x and y.

Let $p \geq 3$ be a positive integer. The graph $3M_{2p} + K_3$ has $n = 6p + 3$ vertices and satisfies

$$d(u) + d(v) + |N(u) \cup N(v)| \geq 6p + 3 = n,$$

for every pair of nonadjacent vertices but fails to contain CC-paths between two vertices of the K_3. Thus, the gap between the bound given in the above theorem and the example is the difference between $n+3$ and $n+1$. The graph $2M_{2p}+K_2$ shows that the condition of 3-connected graphs is necessary in the above theorem. Next, weakening the hypothesis from 3-connected to 2-connected results in the loss of the longest path property obtained in the above theorem.

Theorem 7 *Let G be a 2-connected graph of order n. If $|N(u)\cup N(v)| \geq \frac{1}{2}(n+4)$ for every pair of nonadjacent vertices u and v, then for every pair vertices x and y there is a CC-path $P[x,y]$.*

Using the above result, we obtain the following theorem.

Theorem 8 *Let G be a 2-connected graph of order n, and let x_0, y_0 be two distinct vertices of G. If $|N(u)\cup N(v)| \geq \frac{1}{2}(n+6)$ for every pair of nonadjacent vertices u and v with*

$$\{u,\ v\}\cap\{x_0,y_0\}=\emptyset,$$

then G has a CC-path $P[x_0,y_0]$.

2 Preliminary Lemmas The first lemma, on hamiltonian graphs, and the second, on graph structure, will be used in the proofs.

Lemma 1 (Chen [4]) *Let G be a 2-connected graph of order n. If*

$$d(u)+d(v)+2|N(u)\cup N(v)|\geq 2n-1,$$

for every pair of nonadjacent vertices u and v, then G is hamiltonian.

For a 2-connected graph G, let $D(G)$ be the maximum integer S such that for any two distinct vertices u and v in G there is an u-v path of length at least S. If G is connected and has cut vertices, we set

$$D(G)=\max\{D(G^*) : G^* \text{ is an end block of } G\}.$$

Further, for G a connected graph with u and v distinct vertices, let uGv be any longest $u-v$ path contained in G.

Lemma 2 (Fraisse and Jung[6]) *Assume G is a connected graph, but is not complete. Then, there exist non-adjacent vertices v_1 and v_2 in G such that v_i is not a cut vertex of G and $D(G) \geq d(v_i)$ $(i = 1, 2)$. Furthermore, by the definition of $D(G)$, $d(v) \leq D(G)$ for all $v \in V(G)$ if G is a complete graph. If G has cut vertices, let G_1 and G_2 be two end-blocks of G. In this case for any non-cut vertices $v_i \in V(G_i)$ $(i = 1, 2)$, we have*

$$|V(v_1Gv_2])| \geq D(G_1) + D(G_2) + 1.$$

Let H be a connected subgraph of a graph G and u and v be two vertices in H. We use uHv to denote a longest path in H from u to v. For the remainder of this paper, we assume every cycle (or path) X has an orientation. For any two vertices u and v in X we let $X[u, v]$ denote the segment from u to v along the orientation of X while $X^-[u, v]$ denotes the segment of X from u to v along the opposite direction. We define $X(u, v] = X[u, v] - u$ with similar definitions for $X[u, v)$ and $X(u, v)$. Note, only one of $X[u, v]$ and $X^-[u, v]$ is defined if X is a path and both $X[u, v]$ and $X^-[u, v]$ are defined if X is a cycle. For every $x \in V(X)$, we let x^+ denote the successor of x along the orientation of X and x^- denote the predecessor of x along the orientation of X. Furthermore, we define $x^{++} = (x^+)^+$, etc.

3 Proofs of Theorems We will prove the theorems according to the following order: Theorem 6, Theorem 7, Theorem 8, Theorem 4, and Theorem 5.

3.1 Proof of Theorem 6 Let x_0 and y_0 be any two vertices of G and $P[x_0, y_0]$ be a longest path joining x_0 and y_0. To the contrary, assume that there is a connected component H in $G - V(P[x_0, y_0])$ which is not a clique. Since G is 3-connected there are two subintervals of $P[x_0, y_0]$, say, $P[x_1, y_1]$ and $P[x_2, y_2]$, which may share at most one vertex such that

- For each $i = 1, 2$, there are two distinct vertices u_i, v_i in H such that $u_i x_i$, $v_i y_i \in E(G)$ and one of them is in an end block H^* of H if H is not 2-connected;
- $N(H) \cap P(x_i, y_i) = \emptyset$ for each $i = 1, 2$.

Since $P[x_0, y_0]$ is a longest path joining x_0 and y_0,

$$|V(P(x_i, y_i))| \geq |u_i H v_i| \geq D(H) + 1.$$

for each $i = 1,2$. By Lemma 2.2, there are two nonadjacent vertices in u_0 and $v_0 \in V(H)$ such that

$$D(H) \geq \max\{d_H(u_0), d_H(v_0)\} \geq \frac{1}{2}|(N(u_0) \cup N(v_0)) \cap H|.$$

Further, without loss of generality, assume that

$$d_H(u_0) + d_H(v_0) = \min\{d_H(u) + d_H(v) : u,\ v \in V(H), uv \notin E(H)\}.$$

We will show that

$$d(u_0) + d(v_0) + |N(u_0) \cup N(v_0)| \leq n + 2.$$

In the proof we will use the obvious fact that

$$d(u_0) + d(v_0) + |N(u_0) \cup N(v_0)| \leq 2|N(u_0) \cup N(v_0)| + |N(u_0) \cup N(v_0)|.$$

Using the property that $P[x_0, y_0]$ is a longest path joining x_0 and y_0, we have the following:

- x^+, x^- (one of which may not be defined) are not in $N(u_0) \cup N(v_0)$ if $x \in (N(u_0) \cup N(v_0)) \cap V(P[x_0, y_0])$;
- $x^+, x^{++}, x^{+++}, x^-, x^{--}, x^{---}$ (some of which may not exist) are not in $N(u) \cup N(v)$ if $x \in (N(u_0) \cap N(v_0)) \cap V(P[x_0, y_0])$.

Thus, we have

$$\begin{aligned} 2|(N(u_0) \cup N(v_0)) \cap V(P[x_0, y_0])| + |N(u_0) \cap N(v_0) \cap V(P[x_0, y_0]))| &\leq \\ (|V(P[x_0, y_0]| + 2) + (2 - |V(P(x_1, y_1))|) + (2 - |V(P(x_2, y_2))|)) &= \\ n - |V(H)| - |V(P(x_1, y_1))| - |V(P(x_2, y_2))| + 6. \end{aligned}$$

Note that

$$d_H(u_0) + d_H(v_0) \leq 2D(H) \leq |V(P(x_1, y_1))| + |V(P(x_2, y_2))| - 2$$

and

$$|(N(u_0) \cup N(v_0)) \cap V(H)| \leq |V(H)| - 2.$$

Hence

$$2|N(u_0) \cup N(v_0)| + |N(u_0) \cap N(v_0)| \leq n + 2,$$

a contradiction. □

3.2 Proof of Theorem 7 By Theorem 6, the result holds for 3-connected graphs. We assume that G has a cut-set $\{u_0, v_0\}$. Clearly, $G - \{u_0, v_0\}$ has at most three connected components by the neighborhood union condition. Furthermore, if $G - \{u_0, v_0\}$ has three connected components, then each of them must be a clique. It is readily seen that the result holds. Assume that $G - \{u_0, v_0\}$ has exactly two connected components H_1 and H_2. If both are cliques, the result clearly holds. It is not difficult seeing that one of them must be complete. Assume that H_2 is a clique. Let u and v be any two nonadjacent vertices in H_1. Then

$$|(N(u) \cup N(v)) \cap V(H_1)| \geq \frac{1}{2}(n + 4) - 2 \geq \frac{1}{2}\left(|V(H_1)| + 3\right).$$

If H_1 itself is 3-connected, by Theorem 6, for every pair of nonadjacent vertices in H_1 there is a path connecting them in H_1 such that removing all vertices of this path leaves only cliques, establishing the result. If H_1 is not 3-connected, in an argument for H_1, similar to the argument for G above, let $\{u_1, v_1\}$ be a cut-set for H_1. Further, let $H_{1,1}$ and $H_{1,2}$ be any subgraphs of $H_1 - \{u_1, v_1\}$ with no edges between. It can be shown is this case that $H_{1,1}$ and $H_{1,2}$, as well as H_2 are cliques. It is then possible to show that for given nonadjacent vertices $u, v \in V(G)$, there exists a path $P[u, v]$ such that $G - V(P[u, v])$ is the union of disjoint cliques. □

3.3 Proof of Theorem 8 The proof of this theorem is similar to the above and is left to the reader.

3.4 Proof of Theorems 4 and 5 Let G be a 2-connected graph such that

$$d(u) + d(v) + 2|N(u) \cup N(v)| \geq n + 4$$

for every pair of nonadjacent vertices u and v. Note that

$$d(u) + d(v) + 2|N(u) \cup N(v)| \leq 3|N(u) \cup N(v)| + |N(u) \cap N(v)|.$$

Let C be a longest cycle of G such that the number of the connected components of $G - V(C)$ is as large as possible. Suppose, to the contrary, there is a connected component H of $G - V(C)$ which is not a clique. Let $m = |V(H)|$. Clearly, $m \geq 3$. We will break the proof into the following sequence of claims.

Claim 1 *If x is a vertex in $V(C)$ such that x^- and $x^+ \in N_C(H)$, then there is a $x^* \in V(G) - (V(C) \cup V(H))$ such that $xx^* \in E(G)$.*

Proof: Let u, $v \in V(H)$ such that ux^-, $vx^+ \in E(G)$. If $u \neq v$, then $C[x^+, x^-]uHvx^+$ is a cycle longer than C, a contradiction. And, $N_H(x^+) = N_H(x^-)$ and $|N_H(x^+)| = |N_H(x^-)| = 1$. We denote $N_H(x^+) = N_H(x^-) = \{u\}$. Note, $C^* = C[x^+, x^-]ux^+$ is also a longest cycle of G and $G - V(C^*)$ has at least one more component than $G - V(C)$, unless x is adjacent to some vertex x^* in $G - V(C)$. Also, x^* cannot be in H, since this would imply a cycle longer than C. □

Claim 2 *H is 2-connected.*

Proof: To the contrary, suppose H is not 2-connected. Let H_1 be the end-block such that $D(H_1) = D(H)$ and H_2 be another end-block of H and w_i be the only cut-vertex of H in H_i for each $i = 1, 2$. Note, w_1 and w_2 may be the same vertex. Since G is 2-connected, there are two disjoint subintervals $C[x_1, y_1]$ and $C^-[x_2, y_2]$ with at most end vertices in common, such that x_i

is adjacent to one of the non-cut vertices in H_j and y_i is adjacent to at least one non-cut vertex in H_ℓ for each $i = 1$, 2 and $\{j, \ell\} = \{1, 2\}$. Since C is a longest cycle of G, we have $|C(x_i, y_i)| \geq D(H_1) + D(H_2) + 1$. Furthermore, by the maximality of the number of components of $G - V(C)$, the inequality holds only if H has a hamiltonian path with end vertices neighbors of x_i and y_i respectively.

Let $u_i \in V(H_i)$ such that $D(H_i) \geq d(u_i)$ for each $i = 1$, 2. Since u_1 and u_2 belong to different blocks of H, it follows that $d_H(u_1) + d_H(u_2) \leq |V(H)| - 1$. Thus, we have, by Claim 3.1,

$$\begin{aligned} & 3|(N(u_1) \cup N(u_2)) \cap V(C)| + |N(u_1) \cap N(u_2) \cap V(C)| \\ \leq \; & |V(C)| + 6 - |C(x_1, y_1)| - |C(x_2, y_2)| \\ \leq \; & n + 6 - |V(H)| - 2(d_H(u_1) + d_H(u_2) + 1) \\ \leq \; & n + 3 - 3(d_H(u_1) + d_H(u_2)). \end{aligned}$$

Therefore,

$$3|N(u_1) \cup N(u_2)| + |N(u_1) \cap N(u_2)| \leq n + 3,$$

a contradiction. □

An interval $C[x_i, y_i]$ is called an H-interval if there are two distinct vertices u_i and $v_i \in V(H)$ such that $x_i u_i \in E(G)$ and $y_i v_i \in E(G)$. Note, if $C[x_i, y_i]$ is an H-interval, then $|V(C(x_i, y_i))| \geq D(H) + 1$. Furthermore, if the equality holds, replacing $C(x_i, y_i)$ by $u_i H v_i$, gives us another longest cycle C^*. By the maximality of the number of components of $G - V(C)$, it follows that $u_i H v_i$ is a hamiltonian path in H. Thus, it is true that

$$|V(C(x_i, y_i))| \geq d(u) + 2$$

if $D(H) \geq d(u)$ and u has a nonadjacent vertex in H.

Claim 3 *There are at most three distinct H-intervals on C.*

Proof: To the contrary, suppose that there exist 4 distinct H-intervals. Let u_0 and v_0 be two nonadjacent vertices in H such that

$$D(H) \geq d(u_0) \quad \text{and} \quad D(H) \geq d(v_0).$$

Without loss of generality, assume that

$$d_H(u_0) + d_H(v_0) = \min\{d_H(u) + d_H(v) : u,\, v \in V(H) \text{ and } uv \notin E(H)\}.$$

Clearly, $|(N(u_0) \cup N(v_0)) \cap V(H)| \leq |V(H)| - 2$. Then

$$\begin{aligned} & 3|(N(u_0) \cup N(v_0)) \cap V(C)| + |(N(u_0) \cap N(v_0)) \cap V(C)| \\ \leq\ & |V(C)| + 12 - 4\max\{d(u_0) + 2,\, d(v_0) + 2\} \\ \leq\ & n + 4 - |V(H)| - 2(d_H(u_0) + d_H(v_0)) \\ \leq\ & n + 2 - 2(d_H(u_0) + d_H(v_0)) - |(N(u_0) \cup N(v_0)) \cap V(H)|, \end{aligned}$$

which gives us that

$$3|N(u_0) \cup N(v_0)| + |N(u_0) \cap N(v_0)| \leq n + 2,$$

a contradiction. □

Note, if there are three independent edges between H and C then, there are three H-intervals, which implies that

$$3|(N(u) \cup N(v)) \cap V(C)| + |(N(u) \cap N(v)) \cap V(C)| \leq |V(C)| + 9 - 3(D(H) + 1).$$
(1)

Claim 4 *If there are three independent edges between H and C, then H is hamiltonian.*

Proof: Suppose, to the contrary, that H is not hamiltonian. Then, by Lemma 1, there are two nonadjacent vertices u_0^* and v_0^* such that

(2) $$d_H(u_0^*) + d_H(v_0^*) + 2|N(u_0^*) \cup N(v_0^*)| \leq 2|V(H)| - 2.$$

Let u_0 and v_0 be two nonadjacent vertices of H such that

$$d_H(u_0) + d_H(v_0) = \min\{d_H(u) + d_H(v) : u,\ v \in V(H) \text{ and } uv \notin E(H)\}.$$

Then,

$$3D(H) \geq \frac{3}{2}(d_H(u_0) + d_H(v_0) + 4) \geq$$

$$\frac{1}{2}(d_H(u_0) + d_H(v_0) + 2|(N(u_0) \cup N(v_0)) \cap V(H)|) + 6.$$

Without loss of generality, we assume that

$$d_H(u_0) + d_H(v_0) + 2|(N(u_0) \cup N(v_0)) \cap V(H)| \geq$$

$$d_H(u_0^*) + d_H(v_0^*) + 2|(N(u_0^*) \cup N(v_0^*)) \cap V(H)|.$$

Combining this inequality with (2), it follows that

$$\begin{aligned} & d_H(u_0^*) + d_H(v_0^*) + 2|(N(u_0^*) \cup N(v_0^*)) \cup V(H)|) \\ \leq\ & (|V(H)| - 1) + (3D(H) - 6) \\ =\ & |V(H)| + 3D(H) - 7. \end{aligned}$$

By (1),

$$\begin{aligned} & 3|(N(u_0^*) \cup N(v_0^*)) \cap V(C)| + |N(u_0^*) \cap N(v_0^*) \cap V(C)| \\ \leq\ & n - |V(H)| - 3D(H) + 6 \\ =\ & n - 1 - (|V(H)| + 3D(H) - 7) \\ \leq\ & n - 2 - d_H(u_0^*) - d_H(v_0^*) - 2|N(u_0^*) \cap N(v_0^*) \cap V(C)|, \end{aligned}$$

which gives us that

$$3|N(u_0^*) \cup N(v_0^*)| + |N(u_0^*) \cap N(v_0^*)| \leq n + 2,$$

a contradiction. □

Claim 5 *If there are three independent edges between H and C, then*

$$d_H(u) + d_H(v) \geq |V(H)| + 1,$$

for every pair of nonadjacent vertices of u and v in H. In particular, H is hamiltonian connected.

Proof: Suppose, to the contrary, that u_0 and v_0 are two nonadjacent vertices in H such that

$$d_H(u_0) + d_H(v_0) \leq |V(H)|.$$

By Claim 4, let C^* be a hamiltonian cycle in H, and u_1, u_2, u_3 be three vertices along the cycle C^* such that each of them is an end of one the three independent edges between H and C. Since there are exactly three H-intervals on C. These three H-intervals must be in the form $C[x_1, y_1]$, $C[x_2, y_2]$, $C[x_3, y_3]$ along the orientation of C such that

$$\begin{array}{ccc} x_1u_1 \in E(G) & \text{and} & y_1u_2 \in E(G), \\ x_2u_2 \in E(G) & \text{and} & y_2u_3 \in E(G), \\ x_3u_3 \in E(G) & \text{and} & y_3u_1 \in E(G). \end{array}$$

Since C is a longest cycle in G, we have

$$|V(C(x_1, y_1))| + |V(C(x_2, y_2))| + |V(C(x_3, y_3))| \geq 2|V(H)| + 6.$$

Therefore, we have

$$\begin{aligned} & 3|(N(u_0) \cup N(v_0)) \cap V(C)| + |(N(u_0) \cap N(v_0)) \cap V(C)| \\ \leq \ & n + 9 - |V(H)| - \sum_{i=1}^{3} |V(C(x_1, y_1))| \\ \leq \ & n + 3 - 3|V(H)| \\ \leq \ & n + 3 - 3(d_H(u_0) + d_H(v_0)), \end{aligned}$$

which implies that

$$3|N(u_0) \cup N(v_0)| + |N(u_0) \cap N(v_0)| \leq n + 3,$$

a contradiction. □

The following claim completes the proof of Theorem 4.

Claim 6 *There are no three independent edges between H and C.*

Proof: To the contrary, assume that there are three independent edges between H and C. Then C has three H-intervals. By the above claim, H is hamiltonian connected, that is, $D(H) = |V(H)| - 1$. Let u_0 and v_0 be any two nonadjacent vertices in H. Clearly,

$$|(N(u_0) \cup N(v_0)) \cap V(H)| \leq |V(H)| - 2.$$

Then

$$\begin{aligned} & 3|(N(u_0) \cup N(v_0)) \cap V(C)| + |(N(u_0) \cap N(v_0)) \cap V(C)| \\ \leq\ & n + 9 - |V(H)| - 3|V(H)| \\ \leq\ & n + 3 - (d_H(u_0) + d_H(v_0)) - 2|(N(u_0) \cup N(v_0)) \cap V(C)|. \end{aligned}$$

Thus,

$$3|(N(u_0) \cup N(v_0)) \cap V(C)| + |(N(u_0) \cap N(v_0)) \cap V(C)| \leq n + 3,$$

a contradiction. □

3.4.1 The Proof of Theorem 5 With C and H as defined in the previous proof, now consider the stronger assumption that $NC_2(G) \geq \frac{1}{3}(n+10)$. Since there are no three independent edges between H and C, it follows that there are two vertices x_0, y_0 in $V(C) \cup V(H)$ such that their removal will leave H and C in distinct components. For convenience, let $H_1 = H$ and $H_2 = G - V(H_1)$, and $m_i = |V(H_i)|$ and $h_i = |V(H_i) \cap \{x_0, y_0\}|$ for each $i = 1, 2$.

By Claim 3.2 we know that H_1 is 2-connected. Since H_1 is a component in $G - V(C)$, $H_2 = G - V(H_1)$ is also 2-connected.

Claim 7 *The following inequalities hold.*

$$\frac{1}{3}(n+16-3h_2) \leq |V(H_1)| \leq \frac{1}{3}(2n-16+3h_1)$$
$$\frac{1}{3}(n+16-3h_1) \leq |V(H_2)| \leq \frac{1}{3}(2n-16+3h_2).$$

Proof: Note that every vertex v in $H_i - \{x_0, y_0\}$ has $N(v) \subseteq V(H_i) \cup \{x_0, y_0\}$. Let x_1 and y_1 be two distinct vertices in $(N_C(H))^+$. Then

$$\{x_0, y_0\} \cap \{x_1, y_1\} = \emptyset,$$

and

$$N(x_1) \cup N(y_1) \subseteq V(H_2) \cup \{x_0, y_0\}.$$

Thus, $|V(H_2)| \geq \frac{1}{3}(n+10) - h_1 + 2$, which implies

$$|V(H_1)| \leq \frac{1}{3}(2n-16+3h_1).$$

If there are two nonadjacent vertices u_0, $v_0 \in V(H_1)$ such that

$$\{u_0, v_0\} \cap \{x_0, y_0\} = \emptyset,$$

then

$$|V(H_2)| \leq \frac{1}{3}(2n-16+3h_2).$$

On the other hand, if there no such two vertices u_0 and v_0, it is readily seen that there exist two independent edges $u_1^* u_1$, $v_1^* v_1$ between H and C with u_1, $v_1 \in V(H)$, and there is a hamiltonian path of H joining u_1 and v_1. Thus, C has two H-intervals, $C[x_i, y_i]$ with $i = 1, 2$, such that $|V(C(x_i, y_i))| \geq |V(H_1)|$. In this case, it is very easy to see that for every two nonadjacent vertices u and v in H the following inequalities hold,

$$3|(N(u) \cup N(v)) \cap V(C)| \leq |V(C)| - 2|V(H)| \leq n - 3(|(N(u) \cup N(v)) \cap V(H)|),$$

which implies $|N(u) \cup N(v)| \leq n-2$, a contradiction. □

For each $i = 1, 2$, by Claim 3.7, the following inequality holds,

$$n \geq \frac{1}{2}(3|V(H_i)| + 16 - 3h_i).$$

For each $i = 1, 2$ and every two nonadjacent vertices $u, v \in H_i$ with $\{u, v\} \cap \{x, y\} = \emptyset$, by Claim 3.7, it follows that,

$$\begin{aligned} |(N(u) \cup N(v)) \cap V(H_i)| &\geq \frac{1}{3}(n + 10) - h_j, \text{ for } j \neq i \\ &\geq \frac{1}{3}(\frac{1}{2}(3m_i + 16 - 3h_2) + 10) - h_j, \text{ for } j \neq i \\ &\geq \frac{1}{2}(m_i + 12 - 3h_j, \text{ for } j \neq i \\ &\geq \frac{1}{2}(m_i + 6) \end{aligned}$$

By Theorem 8, we see that G has a cycle C^* containing x_0, y_0 such that $G - V(C^*)$ is the disjoint union of cliques. □

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