On A Characterization of Balanced Matroids *

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Abstract

Balance has played an important role in the study of random graphs and matroids. A graph is balanced if its average degree is at least as large as the average degree of any of its subgraphs. The density of a non-empty loopless matroid is the number of elements of the matroid divided by its rank. A matroid is balanced if its density is at least as large the density of any of its submatroids. Veerapdiyan and Arumugan obtained a characterization of balanced graphs; we extend their result to give a characterization of balanced matroids.

The idea of a balanced graph originated with the work of Erdős and Rényi [1] in the 1950's. If G is a graph, define the **density** of G, $d'(G)$, to be the average degree of G. A graph is **balanced** if $d'(H) \leq d'(G)$ for all subgraphs H of G. Veerapdiyan and Arumugan [3] proved the following characterization of balanced graphs.

Theorem 1 *A simple graph G is balanced if and only if for every component H of G, H is balanced and $d(H) = d(G)$.*

The notion of balance was extended to matroids by Kelly and Oxley [4] in the 1980's in their work on random matroids. For matroid notation, we follow Oxley [2]. Let $M = (E, \mathcal{F})$ be a matroid with ground set E and flats $\mathcal{F}$. The matroids we consider here are loopless and non-empty. Define the density, $d(M)$, of such a matroid by $d(M) := |M|/\rho(M)$ where ρ is

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the rank function of the matroid, M. A matroid M is **balanced** if for all submatroids H of M, $d(H) \leq d(M)$. The matroid is **strictly balanced** if the inequality is strict for all proper submatroids.

In order to shorten the determination whether a matroid M is balanced, we describe the submatroids of M which are in some sense the most dense; this is possible through the use of the closure operator. If H is a submatroid of M, then denote the **closure** of H by $\overline{H}$. One can easily show $d(H) \leq d(\overline{H})$. Thus, a matroid is balanced if and only if $d(F) \leq d(M)$ for all flats F of M.

Theorem 2 *A matroid M is balanced if and only if for every component N of M, N is balanced and $d(N) = d(M)$.*

Proof: If M is balanced, then clearly $d(N) \leq d(M)$, for every submatroid N of M. One can easily show that if N is a component of a matroid M with $d(N) \leq d(M)$ and $d(M \backslash N) \leq d(M)$, then equality must hold in both cases. Now, since M is balanced and $d(N) = d(M)$, it follows that N is balanced. Conversely, suppose that each component N of M is balanced and $d(N) = d(M)$. We show $d(F) \leq d(M)$ for all flats F of M. Each flat of M can be written as $F = \oplus F_N$ where $F_N = F \cap N$ and the sum is taken over all components N of M. Let $|F_N| = a_N$ and $\rho(F_N) = b_N$. Since each component N is balanced, $d(F_N) \leq d(N) = d(M)$, thus for all N $a_N \rho(M) \leq b_N |M|$. Hence,

$$d(F) = \frac{\sum_N a_N}{\sum_N b_N} = \frac{\rho(M) \sum_N a_N}{\rho(M) \sum_N b_N} \leq \frac{|M| \sum_N b_N}{\rho(M) \sum_N b_N} = d(M).$$

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Corollary 3 *The direct sum of the balanced matroids M_1 and M_2 is balanced if and only if $d(M_1) = d(M_2) = d(M_1 \oplus M_2)$. The direct sum is never strictly balanced.*

References

[1] P. Erdős and A. Rényi, On random graphs I, *Publ. Math. Debrecen*,**6** (1959), pp. 290-297.

[2] J.G. Oxley, *Matroid Theory*, Oxford University Press, Oxford, 1992.

[3] N. Veerapandiyan and S. Arumugam, On balanced graphs, *Ars Combinatoria*,**32** (1991), pp. 221-223.

[4] D. Kelly and J. Oxley, Threshold functions for some properties of random subsets of projective spaces, *Quart. J. Math. Oxford,* (2) **33** (1982), pp. 463-469.