# A Krasnosel'skii Theorem for Permissible Paths Whose Edges Are Parallel to Three Given Vectors in the Plane

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ABSTRACT. Let $\mathcal{S}$ be the set of vectors $\left\{e^{i\theta}: \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}\right\}$, and let S be a nonempty simply connected union of finitely many convex polygons whose edges are parallel to vectors in $\mathcal{S}$. If every three points of S see a common point via paths which are permissible (relative to $\mathcal{S}$), then S is starshaped via permissible paths. The number three is best possible.

**1. Introduction.** We begin with some definitions. For vectors $s$, $t$ in the plane with $s = \alpha t$, we say that the parallel vectors $s$ and $t$ have the *same direction* if $\alpha > 0$, *opposite direction* if $\alpha < 0$. Let $\mathcal{S} = \{s_1, \dots s_k\}$ be a set of vectors in the plane with $s_i$ and $s_j$ nonparallel for $i \neq j$. Let $\lambda$ be a simple polygonal path whose edges $[v_{i-1}, v_i]$, $1 \leq i \leq m$, are parallel to the vectors in $\mathcal{S}$. Path $\lambda$ is called *permissible relative to* $\mathcal{S}$ if and only if no two associated vectors $v_{i-1} v_i$ have opposite direction. For S a set in the plane and $x, y$ points in S, we say $x$ *sees* $y$ ($x$ is *visible* from $y$) *via permissible paths* if and only if there is a path in S which is permissible relative to $\mathcal{S}$ and which contains both $x$ and $y$. Set S is *starshaped via permissible paths* if and only if for some point $p$ in S, $p$ sees each point of S via permissible paths, and the set of all such points $p$ is the *permissible kernel* of S.

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In case set $\mathcal{S}$ above contains exactly two vectors, one parallel to each of the coordinate axes, then a permissible path relative to $\mathcal{S}$ is called a *staircase path.* Staircase paths have been useful in studying orthogonal polygons (i.e., polygons whose edges are parallel to the coordinate axes), and in fact analogues of the familiar Krasnosel'skii theorem [7] have been obtained by replacing the usual notion of visibility via straight line segments with the related idea of visibility via staircase paths. (See [9], [4], [3], [2].) The planar version of Krasnosel'skii's theorem states that for S nonempty and compact, S is starshaped (via segments) if and only if every three points of S are visible (via segments) from a common point. Analogously, for S a nonempty simply connected orthogonal polygon in $\mathbb{R}^2$, S is starshaped via staircase paths if and only if every two points of S are visible via staircase paths from a common point [2].

An interesting question which arises is the following: Can the results for staircase paths and orthogonal polygons be extended to permissible paths and polygons whose edges are parallel to vectors in set $\mathcal{S} = \{s_1, \ldots, s_k\}$ for $k > 2$? This paper investigates the problem when $k = 3$, replacing the vector set $\left\{e^{i\theta} : \theta = 0, \frac{\Pi}{2}\right\}$ by the analogous set $\left\{e^{i\theta} : \theta = 0, \frac{\Pi}{3}, \frac{2\Pi}{3}\right\}$. It turns out that although the staircase number two no longer works as the Krasnosel'skii number, the usual Krasnosel'skii number three produces the desired result.

The following familiar terminology will be used: cl S, int S, and bdry S will denote the closure, interior, and boundary, respectively, for set S. The distance between points $x$ and $y$ will be denoted dist $(x, y)$. When $x \neq y, R(x, y)$ will be the ray from $x$ emanating through $y$, and L $(x, y)$

will be the corresponding line. If $\lambda$ is a simple path containing points $x$ and $y$, then $\lambda(x,y)$ will denote the subpath of $\lambda$ from $x$ to $y$. The reader may refer to Valentine [10], to Lay [8], to Danzer, Grünbaum, Klee [5], and to Eckhoff [6] for discussions concerning visibility via segments and Krasnosel'skii - type theorems.

**2. The Results.** We will establish the following theorem.

Theorem 1. Let $\mathcal{S}$ be the set of vectors $\left\{e^{i\theta}:\theta=0,\frac{\Pi}{3},\frac{2\Pi}{3}\right\}$, and let S be a nonempty simply connected union of finitely many convex polygons whose edges are parallel to vectors in $\mathcal{S}$. If every three points of S see a common point via paths which are permissible (relative to $\mathcal{S}$), then S is starshaped via permissible paths. The number three is best possible.

Proof. For each point $x$ in set S, define set $A_x = \{y : x$ sees $y$ via a permissible path in S$\}$. The proof of the theorem will be accomplished by a sequence of lemmas.

Lemma 1. For each point $x$ in set S, the corresponding set $A_x$ is closed.

Proof of Lemma 1. We use a variation of a technique from [3, Lemma 1]. For convenience of notation, let $\mathcal{S}=\{s_i : 1 \le i \le 3\}$. Consider the finite family of lines determined by edges of polygons which contribute to set S, and let $\mathcal{V}$ denote the set of points which belong to at least two of these lines. To each point v in $V$, we associate three lines $L_1, L_2, L_3$, where $L_i$ contains $v$ and is parallel to the vector $s_i$ in $S$, $1 \le i \le 3$. The corresponding family of lines $L$ gives rise to a collection T of nondegenerate closed polygonal regions such that

1) No member of T contains any other nondegenerate closed polygonal region determined by $L$, and

2) $\cup\{T: T \ in \ \mathrm{T}\}$=cl (int S).

Let B be the family $\{\text{int } T: T \ in \ \mathrm{T}\} \cup \{(s,t) : [s,t]$ an edge of $T, T$ in $\mathrm{T}\} \cup \{(s,t) : [s,t]$ an edge of S and $(s,t) \cap \mathrm{cl}(\mathrm{int}\, \mathrm{S}) = \varnothing\}$. Clearly B is finite and $\cup \{\mathrm{cl}\ \mathrm{B} : \mathrm{B} \text{ in } \mathrm{B}\} = \mathrm{S}$.

The following result will be useful in finishing the proof of Lemma 1.

Proposition 1. For points $x, y$ in set S and set B in B, if $y \in \mathrm{B} \cap A_x$, then cl $\mathrm{B} \subseteq A_x$.

Proof of Proposition 1. Assume for the moment that B is fully two dimensional. For each vector $s_i$ in $S$, there are two lines parallel to $s_i$ which support B, $1 \le i \le 3$. Let $U_i$ denote the open strip bounded by these two parallel lines. Certainly no parallel member of L lies in $U_i$ and hence no point of V lies in $U_i$.

Let $\lambda$ be a permissible $x - y$ path ordered from $x$ to $y$. Without loss of generality, assume that $\lambda$ has as few segments as possible. There is a first segment of $\lambda$ which meets the open set B, and for an appropriate labeling, this first segment is parallel to vector $s_1$. Assume that its vector is in the direction of $s_1$ as well. Observe that this segment necessarily lies in the open strip $U_1$ defined above.

There are two cases to consider.

Case 1. Assume that $\lambda \subseteq U_1$. For any $z$ in cl B, a permissible $x - z$ path may be obtained by using a vector parallel to $s_1$ from $x$ to a suitable point in cl B, followed by a vector parallel to $s_2$ or $s_3$ to point $z$. (See Figure 1.) Thus cl B $\subseteq A_x$.

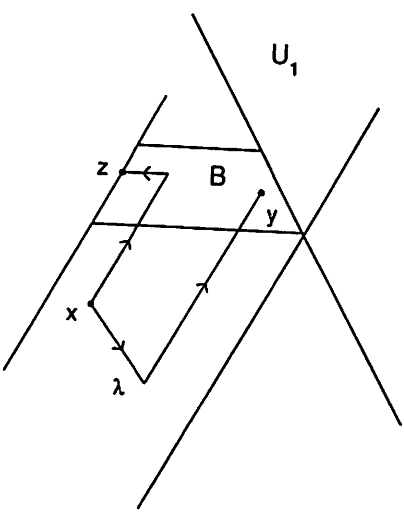


Figure 1.

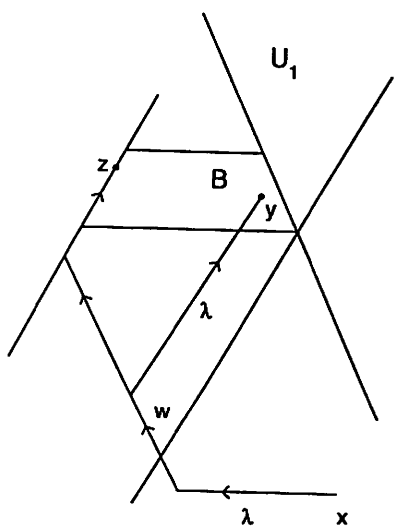


Figure 2.

Case 2. Assume that $\lambda \not\subseteq U_1$. Then there is a first segment $w$ of $\lambda$ such that $w$ meets $U_1$ and all successive segments of $\lambda$ lie in $U_1$. Clearly $w$ is not parallel to $s_1$, and for an appropriate labeling, $w$ is parallel to $s_2$. Assume also that its vector is in the direction of $s_2$. Since $\lambda$ has fewest possible segments, $w$ cannot be extended to meet B. Hence $w$ can be extended to cross $U_1$ (i.e., to meet both lines bounding $U_1$) without entering B. For $z$ in cl B, a permissible $x - z$ path may be obtained this way: Use $\lambda$ from $x$ to the first point of segment $w$, followed by a vector in the direction of $s_2$ to a suitable point of cl $U_1$, followed by a vector in the direction of $s_1$ to point $z$. (See Figure 2.) Again cl B $\subseteq A_z$ .

It remains to consider the case in which B is a segment. Again let $\lambda$ be a permissible $x - y$ path. Then either

1) $\lambda$ enters B by way of some set int $T$, $T$ in T, where B is an edge of $T$, or

2) $\lambda$ enters B along the edge B itself from an end point of B.

If 1) occurs, then using the earlier part of the proof, cl B $\subseteq T \subseteq A_x$. If 2) occurs, clearly cl B $\subseteq A_x$. This finishes the proof of Proposition 1.

Finally, using Proposition 1, it is easy to see that set $A_x$ is closed, for $A_x$ is a finite union of appropriate sets from{ cl B:B in B}.

Note: The proof of Lemma 1 may be extended to any set of vectors $S = \{s_1, \ldots, s_k\}$ in the plane.

Lemma 2. If $a,b \in A_x$ and $[a,b] \subseteq$ S, then $[a,b] \subseteq A_x$.

Proof of Lemma 2. Let $p \in (a,b)$ to show that $p \in A_x$. Select permissible paths $\mu_a, \mu_b$ in S from $x$ to $a$, from $x$ to $b$, respectively, and let W denote the simply connected subset of S determined by $\mu_a \cup \mu_b \cup [a,b]$. Observe that if one of $\mu_a$ or $\mu_b$ contains point $p$, then $p \in A_x$, and the argument is finished. Otherwise, $\mu_a \cup \mu_b \cup [a,b]$ bounds a full hemisphere $H \subseteq$ W at $p$ along $[a,b]$. Let $S = \{s_1, s_2, s_3\}$ and let $t_1, t_2, t_3$ be three rays emanating from point $p$ such that $t_i$ is parallel to vector $s_i$ and $t_i$ meets $H \sim \{p\}$, $1 \le i \le 3$. For convenience of notation, assume that $s_i$ and $t_i$ have the same direction, $1 \le i \le 3$, and that $t_1, t_2, t_3$ are labeled in a clockwise direction from ray $R(p,a)$ to ray $R(p,b)$.

For the moment, assume that neither $\mu_a \cup [a,b]$ nor $\mu_b \cup [a,b]$ alone determines hemisphere $H$. Let $u_i, v_i$ denote the first point of ray $t_i$ (i.e. the point of $t_i$ nearest $p$) on $\mu_a, \mu_b$ respectively, if such a point exists, $1 \le i \le 3$. (At least one of $u_i, v_i$ will exist for each $t_i$.) Without loss of generality, assume that $u_2$ exists on $t_2$ and that, if $t_2$ meets $\mu_b$, the corresponding order on $t_2$ is $p < u_2 \le v_2$. Then $[p, u_2] \subseteq$ W $\subseteq S$, and it is not hard to see that $u_1$ exists and $[p, u_1] \subseteq H \subseteq$ W $\subseteq S$ also. (See Figure 3.) If either $\mu_a(x, u_1) \cup [u_1, p]$ or $\mu_a(x, u_2) \cup [u_2, p]$ is a permissible $x - p$ path, then the argument is finished. Otherwise, $\mu_a$ must contain both a vector in the direction of $s_1$ (and $t_1$) and a vector in the direction of $s_2$ (and $t_2$).

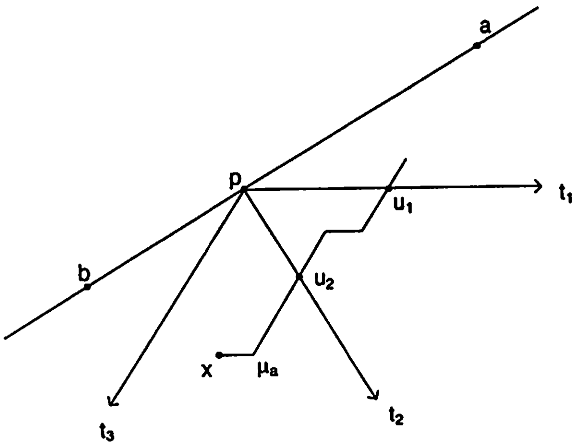


Figure 3.

Clearly vectors in the directions of $s_1$ and $s_2$ alone cannot comprise the permissible $x-a$ path $\mu_a$. A similar statement holds for the triple $s_1$, $s_2$, and $s_3$. Therefore, $\mu_a$ must contain vectors in the directions of $s_1$, $s_2$, and $-s_3$. Observe that $x$ and $t_3$ must lie in the same closed halfplane determined by the line of $t_2$. There are two cases to consider.

Case 1. Suppose that $x$ lies in the convex region bounded by rays $t_2$ and $t_3$. Observe that $[p,u_2]\cup \mu_a \ (u_2,a)\cup[p,a]$ bounds a simply connected subset $W' \subseteq W \subseteq S$. The vector at $x$ in the direction of $-s_3$ necessarily meets $[p,u_2]$ at some point $w$, and since $\mu_a$ consists exclusively of vectors of type $s_1$, $s_2,-s_3$, $\mu_a$ lies in one of the closed halfplanes determined by $L(x,w)$.

If $[x,w] \subseteq S$, then $[x,w] \cup [w,p]$ is a permissible $x-p$ path, finishing Case 1. If $[x,w] \not\subseteq S$, then path $\mu_b$ must cross $[x,w]$. Consider path $\mu_b$. Certainly $\mu_b$ meets $t_3$ at point $v_3$. If $\mu(x,v_3)$ employs no vector in the direction of $s_3$, then $(x,v_3) \cup [v_3,p]$ comprises a permissible $x-p$ path, again finishing the argument. Otherwise, $\mu_b\,(x,v_3)$ contains vectors of type $s_3$. To reach point $b$, path $\mu_b$ must contain either a vector of type $-s_1$ or a vector of type $-s_2$, and since $\mu_b$ crosses $[x,w]$, $\mu_b$ must contain vectors of type $s_1$ and $-s_2$. Hence $\mu_b$ employs vectors of type $s_1, -s_2$ and $s_3$.

Observe that there is a first segment in $\mu_b$ which meets $R(p,b) \sim \{p\}$, and this segment must be in the direction of $-s_2$. Consider the first point $q$ of $\mu_b$ such that a vector at $q$ in the direction of $-s_2$ lies in S and meets $R(p,b)$, say at $b'$. (See Figure 4.) (Of course $(q,b']$ need not meet $\mu_b$.) Then $\mu_b\,(x,q) \cup [q,b']$ is a permissible path in S.

If $b' = p$, we have a permissible $x-p$ path, the desired result. Otherwise, the ray from $b'$ in the direction of $s_1$ will meet ray $t_2$, say at $c$, and by our choice of $b', [b',c] \subseteq W \subseteq S$. Then $\mu_b(x,q) \cup [q,b'] \cup [b',c] \cup [c,p] \subseteq S$ consists exclusively of vectors of type $s_1$, $-s_2$, and $s_3$, producing the required permissible $x-p$ path. This completes Case 1.

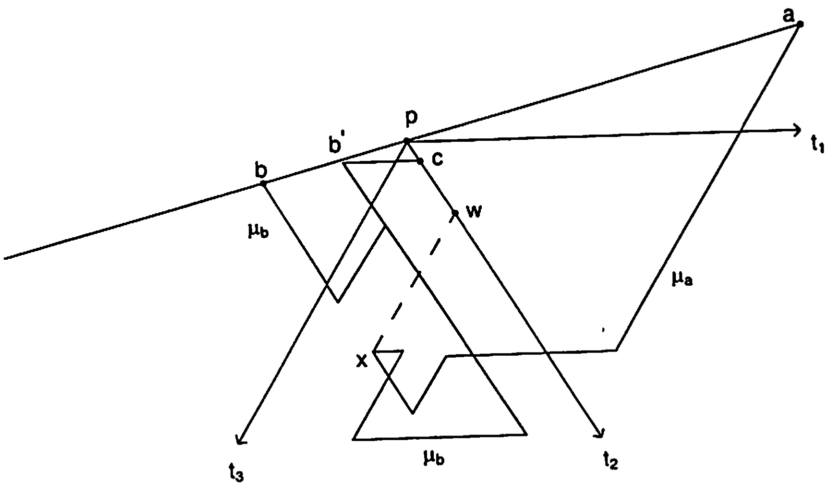


Figure 4.

Case 2. Suppose that point $x$ fails to lie in the convex region determined by rays $t_2$ and $t_3$. Then $\mu_a$ necessarily meets $t_3$ before meeting $t_2$. Hence $u_3 \in \mu_a \cap t_3$, and $\mu_a(x,u_3) \cup [u_3,p]$ is a permissible $x-p$ path, finishing Case 2.

Finally, if one of $\mu_a \cup [a,b]$ or $\mu_b \cup [a,b]$ alone determines hemisphere $H \subseteq W$, assume that $\mu_a \cup [a,b]$ determines $H$. Then vectors of type $s_1, s_2, s_3$ cannot comprise path $\mu_a$, so for one of the $s_i$ vectors, say $s_1, \mu_a$ employs vectors of type $-s_1$. Moreover, $\mu_a$ meets each ray $t_i$ at point $u_i$ closest to $p, 1 \le i \le 3$. Thus $\mu_a(x,u_1) \cup [u_1,p]$ is a permissible $x-p$ path, finishing the proof of Lemma 2.

Lemma 3. For $x, y$ in $S$, the corresponding set $A_x \cup A_y$ is simply connected.

Proof of Lemma 3. Let $\lambda$ be a simple closed curve in $A_x \cup A_y$, with point $p$ interior to the closed bounded region $U \subseteq S$ determined by $\lambda$. We will show that $p \in A_x \cup A_y$. For each line $L$ through $p$, there is at least one pair $a,b$ in $\lambda \cap L$ for which $p \in (a,b) \subseteq U$. To each $L$ we may associate such pairs $a_L, b_L$ and $a_L', b_L'$ with $a_L \leq a_L' < p < b_L' \leq b_L$ for which dist $(a_L, b_L)$ is maximal and dist $(a_L', b_L')$ is minimal. If one of $a_L, a_L'$ is in $A_x$ and one of $b_L, b_L'$ is in $A_x$, then by Lemma 2, $p \in A_x$, finishing the proof. Hence we assume that for an appropriate labeling, $a_L, a_L' \in A_x$ and $b_L, b_L' \in A_y$. Then by Lemma 2 $\left[a_L, a_L'\right] \subseteq A_x$ and $\left[b_L, b_L'\right] \subseteq A_y$.

Define set $U' = \cup \{[a,b] : a,b \text{ in } \lambda \text{ and } p \in (a,b) \subseteq U\}$. Certainly $U'$ is starshaped via straight line segments at $p$, and $p \in \operatorname{int} U'$. Moreover, it is easy to see that set $U'$ is closed. (A converging sequence of appropriate segments $[a_n, b_n]$ will have as its limit a segment $[a_o, b_o]$ at $p$ and in $U$, with $a_o, b_o \in \lambda$.) Notice that for any line $L$ at $p$ and for the associated pairs $a_L, b_L$ and $a_L', b_L'$ defined previously, if $r \in \left(a_L, b_L'\right)$, then $[r,p] \subseteq U \sim \lambda = \operatorname{int} \mathrm{U}$, and clearly $[r,p] \subseteq \operatorname{int} \mathrm{U}'$. Hence each boundary point of $U'$ on line L must belong to $\left[a_L, a_L'\right] \cup \left[b_L', b_L\right] \subseteq A_x \cup A_y$, and therefore bdry $U' \subseteq A_x \cup A_y$.

The following observations will be useful: For points s, t in bdry $U'$ and on a common ray at $p$, clearly $[s,t] \subseteq U'$. Futhermore, $[s,t] \cap \operatorname{int} U' = \varnothing$, for if there existed a point $u$ in $\operatorname{int} U'$ with $s < u < t < p$, then $p$ would see via straight line segments in $U'$ all points in some neighborhood of $u$, and $t \in (u,p) \subseteq \operatorname{int} U'$, impossible by our choice of $t$. It follows that $[s,t] \subseteq$ bdry $U'$.

Certainly bdry $U'$ is closed. We assert that bdry $U'$ is connected as well. Suppose on the contrary that bdry $U'$ has two or more components. Let $C$ be such a component, and define $D = (\text{bdry} U') \sim C$. Set $C$ is open and closed in bdry $U'$; hence $D$ is closed in bdry $U'$, and both $C$ and $D$ are closed in the plane.

Notice that for $c$ in $C$ and $d$ in $D$, $d \notin R(p,c)$. Otherwise, by a previous observation, $[d,c] \subseteq$ bdry $U'$, forcing $d$ and $c$ to belong to the same component of bdry $U'$, impossible. Therefore, no ray at $p$ meets both $C$ and $D$.

Since $C \cup D =$ bdry $U'$, for at least one of these sets, say $C$, set $C$ contains points $c_1$ and $c_2$ with $c_2 \notin R(p,c_1)$. Corresponding rays $R(p,c_1)$ and $R(p,c_2)$ determine two closed subsets of the plane, say $V$ and $W$. Moreover, at least one $\operatorname{int} V$ or $\operatorname{int} W$ is disjoint from $D$, for if $d_1 \in (\operatorname{int} V) \cap D$ and $d_2 \in (\operatorname{int} W) \cap D$, then $R(p,d_1) \cup R(p,d_2) \subseteq \mathbb{R}^2 \sim C$ would separate $c_1$ from $c_2$, impossible. Assume that $\operatorname{int} V$ is disjoint from $D$, and select an angle at $p$ of minimal measure such that one of the corresponding closed subsets of the plane contains set $C$ (and hence contains $V$). Let $R_1, R_2$ (not necessarily distinct) denote the associated rays. Since $C$ is closed, each of $R_1$ and $R_2$ meets $C$. Observe that the remaining closed set (possibly degenerate) determined by $R_1$ and $R_2$ contains $D$. If $R_1 \neq R_2$, then every ray at $p$ which meets the corresponding open region must intersect $(\text{bdry } U') \sim C = D$, and since $D$ is closed, $R_i \cap D \neq \varnothing$ for $i = 1, 2$. If $R_1 = R_2$, then trivially $R_i \cap D \neq \varnothing$, $i = 1, 2$. Either way, we have the ray $R_1$ from $p$ which meets both $C$ and $D$. However, we noticed earlier that no ray from $p$ has this property. We have a contradiction, our supposition is false, and bdry $U'$ is connected, the desired result.

The rest of the argument is easy. By Lemma 1, sets $A_x$ and $A_y$ are closed. Hence $A_x \cap$ (bdry $U'$) and $A_y \cap$ (bdry $U'$) are closed subsets of bdry $U'$, and their union is exactly bdry $U'$. Since bdry $U'$ is connected, it cannot be a union of separated sets. Therefore, either one of the set $A_x \cap$ (bdry $U'$) or $A_y \cap$ (bdry $U'$) is empty, or $A_x \cap A_y \cap (\text{bdry} U') = \varnothing$. If $A_x \cap (\text{bdry} U') = \varnothing$, then for any line $L$ at $p$, the endpoints of $L \cap U'$ belong to bdry $U' \subseteq A_y$, and (by Lemma 2) $p \in A_y$. If $A_x \cap A_y \cap (\text{bdry} U') \neq \varnothing$, choose a point $z$ in this intersection. The ray emanating from $z$ through $p$ meets bdry $U'$ at a last point $z'$ with $z < p < z'$, and $z' \in A_x \cup A_y$. Assume that $z' \in A_x$. Then (again by Lemma 2) $p \in [z, z'] \subseteq A_x$.

We conclude that $p \in A_x \cup A_y$ and $A_x \cup A_y$ is simply connected, finishing the proof of Lemma 3.

Lemma 4. For points $x, y, z$ in set $S$, the corresponding union $A_x \cup A_y \cup A_z$ is simply connected.

Proof of Lemma 4. Certainly $\{A_x, A_y, A_z\}$ is a family of compact connected sets in the plane. By our original hypothesis, $A_x \cap A_y \cap A_z \neq \emptyset$. Certainly since every two of these sets meet, every two have a connected union. By Lemma 3, every two have a simply connected union. Hence we may apply [1, Proposition 1] to conclude that $A_x \cup A_y \cup A_z$ is simply connected.

We use the lemmas above to prove Theorem 1. Let $\mathcal{A}$ denote the family of nonempty compact sets $A_x$, $x$ in $S$. By a comment in Lemma 4, every two members of $\mathcal{A}$ have a connected union, and by Lemma 4 every three members of $\mathcal{A}$ have a simply connected union.

Therefore, by [1, Theorem 1], $\cap \{A_x : x \text{ in } S\}$ is nonempty (as well as simply connected and connected). Certainly any point $p$ in $\cap \{A_x : x \text{ in } S\}$ sees every point of $S$ via permissible paths, so $S$ is starshaped via such paths, finishing the proof of the theorem.

To see that the number three in Theorem 1 is best possible, consider the following example.

Example 1. Let $S$ denote the compact simply connected set in Figure 5, and let $\mathcal{S} = \{e^{i\theta} : \theta = 0, \frac{\Pi}{3}, \frac{2\Pi}{3}\}$. Observe that points $a$ and $c$ see $b'$, $b$ and $c$ see $a'$, $a$ and $b$ see $c'$ via paths which are permissible relative to $\mathcal{S}$. In fact, every two points of $S$ see via permissible paths a common point. However, set

$A_a \cup A_b \cup A_c$ is not simply connected, and points $a,b,c$ see via permissible paths no common point.

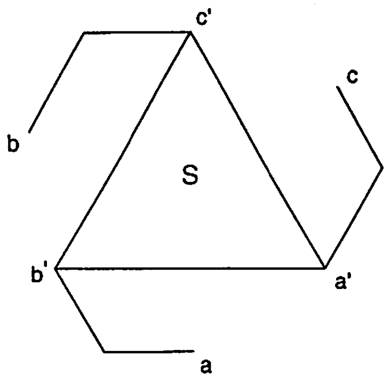


Figure 5.

Finally, we close with an interesting observation. It is well known that for set $A$ starshaped via straight line segments, the kernel of $A$ is convex. Likewise, for simply connected orthogonal polygon $B$ starshaped via staircase paths, the staircase kernel of $B$ is both horizontally and vertically convex. (See [3 , Theorem 1].) Hence we might expect an analogous result to hold for sets which are starshaped via paths permissible relative to $\mathcal{S} = \{s_1, s_2, s_3\}$. However, the following example shows that such a result fails.

Example 2. Let $\mathcal{S} = \{e^{i\theta} : \theta = 0, \frac{\Pi}{3}, \frac{2\Pi}{3}\}$, and let S be the compact simply connected set in Figure 6. Observe that path

$\cup \{[x_{i-1}, x_i] : 2 \leq i \leq 5\}$ is permissible relative to $S$. In fact, for any pair $y$, $z$ in S, $y$ sees $z$ via permissible paths. However, set S is not convex in any direction. That is, for any line $L$, some translate of $L$ contains points of S whose segment fails to lie in S.

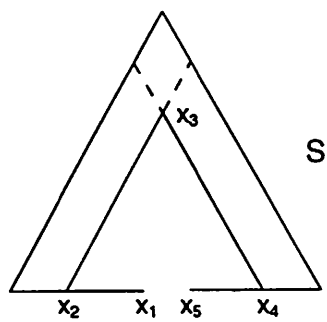

Figure 6.

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