Domination and total domination critical trees with respect to relative complements

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Abstract

Let G be a spanning subgraph of $K_{s,s}$ and let H be the complement of G relative to $K_{s,s}$; that is, $K_{s,s} = G \oplus H$ is a factorization of $K_{s,s}$. For a graphical parameter $\mu(G)$, a graph G is *$\mu(G)$-critical* if $\mu(G+e) < \mu(G)$ for every e in the ordinary complement $\overline{G}$ of G, while G is *$\mu(G)$-critical relative to $K_{s,s}$* if $\mu(G+e) < \mu(G)$ for all $e \in E(H)$. We show that no tree T is $\mu(T)$-critical and characterize the trees T that are $\mu(T)$-critical relative to $K_{s,s}$, where $\mu(T)$ is the domination number and the total domination number of T.

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1 Introduction

A set $S \subseteq V(G)$ of a graph G is a *dominating set* if every vertex not in S is adjacent to a vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set. A *total dominating set* in a graph G is a subset S of $V(G)$ such that every vertex in $V(G)$ is adjacent to a vertex of S. Every graph G without isolated vertices has a total dominating set, since $S = V(G)$ is such a set. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set. A dominating set of G of cardinality $\gamma(G)$ is called a *$\gamma(G)$-set*, or just a *γ-set*, if the graph G is understood from the context. We use similar notation for the other domination parameters. Domination in graphs, with its many variations, is now well studied in graph theory. For a more thorough treatment of domination and for terminology not defined here, we refer the reader to [2, 6].

A graph is said to be *γ-domination critical*, or just *γ-critical*, if $\gamma(G) = \gamma$ and $\gamma(G+e) = \gamma - 1$ for every edge e in the complement $\overline{G}$ of G. This concept of γ-critical graphs has been studied by, among others, Blitch [1], Sumner [12], Sumner and Blitch [13], Sumner and Wojcicka [14], and Wojcicka [15]. Haynes, Mynhardt, and van der Merwe [8] - [11] defined a graph G to be *total domination edge critical*, or simply *k_t-critical*, if $\gamma_t(G+e) < \gamma_t(G) = k$ for any edge $e \in E(\overline{G})$. Whereas the addition of an edge from the complement $\overline{G}$ can change the domination number of G by at most one, it can change the total domination number by as much as two.

Proposition 1 *[8] For any edge $e \in E(\overline{G})$,*

$$\gamma_t(G) - 2 \leq \gamma_t(G+e) \leq \gamma_t(G).$$

In this paper we consider a different complement concept. If G is a spanning subgraph of F, then the graph $F - E(G)$ is the *complement of G relative to F* with respect to a fixed embedding of G into F. The idea of a relative complement of a graph was suggested by Cockayne [3] and is studied in [4]. We shall assume that the complete bipartite graph $K_{s,s}$ has partite sets $\mathcal{L}$ and $\mathcal{R}$ (representing "left" and "right"), and that $G \oplus H = K_{s,s}$ is a factorization of $K_{s,s}$. (If G and H are graphs on the same vertex set but with disjoint edge sets, then $G \oplus H$ denotes the graph whose edge set is the union of their edge sets.) Notice that if G is uniquely embeddable in

$K_{s,s}$, then H is unique. We henceforth consider only spanning subgraphs G of $K_{s,s}$ such that G is uniquely embeddable in $K_{s,s}$. For the remainder of this paper H will always denote the complement of G relative to $K_{s,s}$.

Haynes and Henning [7] studied *domination critical graphs with respect to the relative complement.* They defined a graph G to be *γ-critical relative to $K_{s,s}$* if $\gamma(G) = \gamma$ and $\gamma(G+e) = \gamma - 1$ for all $e \in E(H)$. Furthermore, if u and v are non-adjacent vertices in different partite sets of G, then $\gamma(G+uv) = \gamma - 1$ and so there exists a set W of cardinality $\gamma - 1$ that dominates $G+uv$. Since W does not dominate G, it must be that exactly one of u and v, say v, belongs to W and that W dominates all of G except u. Thus, $S = W - \{v\}$ is a set of cardinality $\gamma - 2$ such that $S \cup \{v\}$ dominates $G - u$ and we write $[v, S] \mapsto u$. In particular, when we write $[v, S] \mapsto u$ it is understood that u is not dominated by S. To distinguish between domination critical graphs relative to $K_{s,s}$ and domination critical graphs relative to ordinary complements, we say that a graph is *γ-critical* (RC) if it is domination critical relative to $K_{s,s}$ and just γ-critical if the ordinary complement is being considered.

We extend the concept of domination critical graphs with respect to the relative complement to total domination. A graph G is *total domination edge critical relative to $K_{s,s}$*, or just *k_t-critical* (RC), if $\gamma_t(G+e) < \gamma_t(G) = k$ for all $e \in E(H)$. Suppose u and v are non-adjacent vertices in different partite sets of a k_t-critical(RC) graph G. Then, $\gamma_t(G+uv) \leq k-1$ and so there exists a set W of cardinality at most $k-1$ that totally dominates $G+uv$. If W contains exactly one of u and v, say v, then W dominates all of G except u. Thus, $S = W - \{v\}$ is a set of cardinality at most $k-2$ such that $S \cup \{v\}$ totally dominates $G - u$ and we write $[v, S] \mapsto_t u$. In particular, when we write $[v, S] \mapsto_t u$ it is understood that u is not dominated by S.

In Section 2 we characterize the γ-critical(RC) trees and show that no tree is γ-critical. In Section 3 we charcterize the k_t-critical(RC) trees and show that no tree is k_t-critical.

We use the following notation. An *endvertex* is a vertex of degree 1 and its neighbor is called a *support vertex.* An endvertex of a tree is also called a *leaf.* For a graph $G = (V, E)$ and $X \subseteq V$, let L_X denote the set of leaves in G that belong to X, and let S_X denote the set of support vertices in G that belong to X. For sets $S, X \subseteq V$, if S dominates X, then we write $S \succ X$, while if S totally dominates X, we write $S \succ_t X$. If $S = \{s\}$ or $X = \{x\}$, we also write $s \succ X$, $S \succ_t x$, etc.

2 Domination Critical Trees

Our aim in this section is first to show that no tree is γ-critical and secondly to characterize domination critical trees with respect to the relative complement. We begin with two straightforward but useful results.

Observation 2 *For any γ-critical (γ-critical (RC)) graph G and edge $uv \in E(\overline{G})$ ($uv \in E(H)$), exactly one of u and v is in any γ-set of $G + uv$.*

Lemma 3 *No path is γ-critical or γ-critical(RC).*

Proof. First we note that if P_n is domination critical relative to $K_{s,s}$, then $n = 2s$ implying that its endvertices are in different partite sets (that is, the edge between the endvertices is in the relative complement). Adding the edge between the endvertices of a path P_n yields a cycle C_n. But $\gamma(P_n) = \gamma(C_n)$, so P_n is not γ-critical or γ-critical(RC). □

We show first that no tree is γ-critical.

Theorem 4 *No tree is γ-critical.*

Proof. Suppose T is a γ-critical tree. If a support vertex is adjacent to two leaves u and v, then $\gamma(T) = \gamma(T + uv)$, contradicting the fact that T is γ-critical. Hence, each support vertex is adjacent to only one leaf. If u and v are different support vertices, then there exists a γ-set of T containing both u and v. Observation 2 implies that u and v must be adjacent. Hence, T has can have at most two support vertices. Thus, T is a path, which contradicts Lemma 3. □

Next, we characterize those trees that are domination critical relative to $K_{s,s}$. For this purpose, we need the following notation. Recall that a star is the complete bipartite graph $K_{1,k}$. The *corona* $G \circ K_1$ is the graph formed from a copy of G by adding a new vertex v' for each $v \in V(G)$ such that v and v' are adjacent. We now characterize the trees that are γ-critical relative to $K_{s,s}$.

Theorem 5 *A tree T is γ-critical(RC) if and only if T is the corona $K_{1,\gamma-1} \circ K_1$ for $\gamma \geq 3$.*

Proof. First, we show that the corona $T = K_{1,\gamma-1} \circ K_1$ for $\gamma \geq 3$ is γ-critical(RC). It is easy to see that $\gamma(K_{1,\gamma-1} \circ K_1) = \gamma$ and the set S consisting of all the support vertices is a γ-set. Note that without loss of generality, $\mathcal{L}$ consists of one support vertex and $\gamma - 1$ leaves, while $\mathcal{R}$ consists of $\gamma - 1$ support vertices and one leaf. Consider first adding an edge from a support vertex v to a leaf u. Let u' be the support vertex of u. Then $S - \{u'\} \succ T + uv$. Next consider adding an edge between two leaves $u \in \mathcal{L}$ and $v \in \mathcal{R}$, where $v' \in \mathcal{L}$ and $u' \in \mathcal{R}$ are the support vertices of v and u, respectively. Then $(S - \{u', v'\}) \cup \{u\} \succ T + uv$ and hence T is γ-critical.

Conversely, we show that any γ-critical(RC) tree T is the corona $K_{1,\gamma-1} \circ K_1$. Assume that T is γ-critical relative to $K_{s,s}$. Then T has partite sets $\mathcal{L}$ and $\mathcal{R}$ such that $|\mathcal{L}| = |\mathcal{R}| = s$. By Lemma 3, no path is γ-critical implying that T has at least three leaves. First we show that no vertex is adjacent to two or more leaves.

Claim 6 *No vertex is adjacent to two or more leaves.*

Proof. Suppose $v \in \mathcal{L}$ is adjacent to two leaves, say v_1 and v_2. Since no star is γ-critical(RC), v has at least one neighbor u in $\mathcal{R}$ that is not a leaf. Let $u_1 \in N(u) - \{v\}$. Then $u_1 \in \mathcal{L}$. Consider $T + u_1v_1$. Then $[u_1, S] \mapsto v_1$ or $[v_1, S] \mapsto u_1$. In either case, we may assume that $v \in S$ to dominate v_2. Since S dominates v_1, it is not the case that $[u_1, S] \mapsto v_1$. Thus, $[v_1, S] \mapsto u_1$. Since $v \in S$, v_1 is in S just to dominate u_1. It follows that $(S - \{v_1\}) \cup \{u\}$ dominates T and has cardinality less than $\gamma(T)$, a contradiction. Thus, each vertex is adjacent to at most one leaf. □

Claim 7 *If $u \in \mathcal{L}$ and $v \in \mathcal{R}$ are support vertices, then $uv \in E(T)$.*

Proof. Suppose $u \in \mathcal{L}$ and $v \in \mathcal{R}$ are support vertices. Then there exists a γ-set of T containing both u and v. Therefore it follows, from Observation 2, that u and v must be adjacent. □

Claim 8 *One partite set, $\mathcal{L}$ say, contains at most one support vertex and $\mathcal{R}$ contains at most one leaf.*

Proof. Suppose each of $\mathcal{L}$ and $\mathcal{R}$ contains two support vertices. By Claim 7, support vertices in different partite sets are adjacent, so these four vertices induce a cycle, contradicting the fact that T is a tree. Hence one of the partite sets, $\mathcal{L}$ say, has at most one support vertex. Then Claim 6 implies that $\mathcal{R}$ has at most one leaf. □

¿From Claim 8, we may assume that $\mathcal{L}$ has at most one support vertex and $\mathcal{R}$ has at most one leaf. If $\mathcal{R}$ has no leaf, then each vertex in $\mathcal{R}$ has degree at least 2 in T, and so T has at least $2s$ edges, which contradicts the fact that T is a tree of order $2s$. Thus, $\mathcal{R}$ has exactly one leaf, say y', and $\mathcal{L}$ has exactly one support vertex, say y.

¿From Claim 7 we know that y is adjacent to every vertex in $S_{\mathcal{R}}$. Let $A = \mathcal{L} - L_{\mathcal{L}} - \{y\}$ and $B = \mathcal{R} - S_{\mathcal{R}} - \{y'\}$. Since $|L_{\mathcal{L}}| = |S_{\mathcal{R}}|$ by Claim 6, $|A| = |B|$. Suppose, $A \neq \emptyset$. Then every vertex in A (B, respectively) has degree at least 2 in T. If each vertex of A is adjacent only to vertices of B, then $\langle A \cup B \rangle$ contains a cycle. Hence, at least one vertex of $S_{\mathcal{R}}$, x say, is adjacent to a vertex of A. Let $T' = \langle A \cup B \cup \{x, y\} \rangle$. Then, T' is a bipartite graph with partite sets $A \cup \{y\}$ and $B \cup \{x\}$ where $|A \cup \{y\}| = |B \cup \{x\}|$. However, every vertex of $B \cup \{x\}$ has degree at least 2 in T', and so T' contains a cycle, a contradiction. Hence, $A = \emptyset$ and $B = \emptyset$. Thus, $T = K_{1,\gamma-1} \circ K_1$ for some $\gamma \geq 3$, completing the proof of Theorem 5. □

3 Total Domination Critical Trees

We have two aims in this section. Our first aim is to show that no tree is k_t-critical. Our second aim is to characterize total domination critical trees with respect to the relative complement. We again begin with a useful observation.

Observation 9 *For any k_t-critical (k_t-critical(RC)) graph G and edge $uv \in E(\overline{G})$ ($uv \in E(H)$), at least one of u and v is in any γ_t-set of $G + uv$.*

Since $\gamma_t(P_n) = \gamma_t(C_n)$, a proof similar to that of Lemma 3 yields the following result.

Lemma 10 *No path is k_t-critical or k_t-critical(RC).*

We show first that no tree is k_t-critical.

Theorem 11 *No tree is k_t-critical.*

Proof. Suppose T is a k_t-critical tree. If a support vertex is adjacent to two leaves u and v, then $\gamma_t(T) = \gamma_t(T+uv)$, contradicting the fact that T is k_t-critical. Hence, each support vertex is adjacent to only one leaf. We show next that no two support vertices are adjacent.

Claim 12 *No two support vertices are adjacent.*

Proof. Suppose that u and v are support vertices of u' and v', respectively, and that u and v are adjacent. Consider $T' = T + u'v'$ and let S' be a total dominating set of T'. If both u' and v' are in S', then $(S' - \{u', v'\}) \cup \{u, v\} \succ_t T$, a contradiction since $|S'| < \gamma_t(T)$. Hence we may assume that $u' \in S'$ and $v' \notin S'$, implying that $u \in S'$ and u' is the only neighbor of v' in T' that belongs to S'. But then $(S - \{u'\}) \cup \{v\} \succ_t T$, again a contradiction. □

Let u' and v' be the leaves on a longest path in T, and let u and v be the support vertices of u' and v', respectively. Since each support vertex is adjacent to only one leaf, it follows that each of u and v has degree exactly 2. We show now that u and v have a common neighbor.

Claim 13 *The vertices u and v have a common neighbor.*

Proof. Suppose that u and v do not have a common neighbor. Let $N(u) = \{u', w\}$ and let $N(v) = \{v', z\}$. If $wz \notin E(T)$, then $\gamma_t(T) = \gamma_t(T + wz)$, contradicting the fact that T is k_t-critical. Hence, w and z must be adjacent.

Since T is not a path, by Lemma 10, at least one of w and z, say w, has degree at least 3. Let x be a neighbor of w different from u and z. Since no two support vertices are adjacent, w cannot be a support vertex of T. Hence, x is not a leaf. We now consider a longest path from w to a leaf that contains x as its internal vertex. Let y' be the leaf on such a path, and let y be the neighbor of y'. Then, y is necessarily a support vertex of degree exactly 2 (possibly, $y = x$). If the neighbor of y different from y' is distinct from w, then adding an edge between that neighbor of y and z produces a graph with the same total domination number as T, a contradiction. Hence, $x = y$, and so x is a remote vertex of degree exactly 2. (In fact, we have shown that every neighbor of w, different from z, is a remote vertex of degree exactly 2.) But now $\gamma_t(T) = \gamma_t(T + uz)$, contradicting the fact that T is k_t-critical. □

By Claim 13, u and v must have a common neighbor, w say. Since T is not a path, by Lemma 10, w has degree at least 3. Furthermore, since no two support vertices are adjacent, w cannot be a support vertex of T. However, by our choice of u' and v', $diam\,T = d(u', v') = 4$. It follows that T is a obtained from a star $K_{1,k-1}$, $k \geq 4$, with center w by subdividing each edge exactly once. But then we can add an edge between any two support vertices of T to produce a tree T' with $\gamma_t(T) = \gamma_t(T')$, contradicting the fact that T is k_t-critical. We deduce, therefore, that no tree is k_t-critical. □

Next, we characterize those trees that are k_t-critical relative to $K_{s,s}$. We will use the concept of an internal private neighbor. A vertex u is said to be an *internal private neighbor (ipn)* of a vertex v with respect to a set S if $u \in S$ and $N(u) \cap S = \{v\}$.

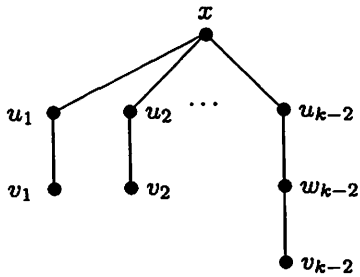

Figure 1: A k_t-critical(RC) tree $K^*_{1,k-2}$.

Theorem 14 *A tree T is k_t-critical(RC) if and only if T is a subdivided star $K^*_{1,k-2}$, for $k \geq 5$, with exactly one edge subdivided twice.*

Proof. Let T be the subdivided star $K^*_{1,k-2}$ for $k \geq 5$ with one edge subdivided twice as shown in Figure 1. Then, $\gamma_t(T) = k$ and the set S consisting of the nonleaf vertices, that is, $S = \{x, w_{k-2}, u_i \mid 1 \leq i \leq k-2\}$, is a γ_t-set for T.

To see that T is k_t-critical(RC), we consider each edge in the relative complement of T and demonstrate that its addition to T decreases the total domination number. The set $S - \{w_{k-2}\} \succ_t T + xv_{k-2}$. For $i < k-2$, the set $(S - \{u_i, u_{k-2}, w_{k-2}\}) \cup \{v_i, v_{k-2}\} \succ_t T + v_i v_{k-2}$. Furthermore, for $i < k-2$, the set $S - \{u_{k-2}\} \succ_t T + w_{k-2}u_i$. Finally, $S - \{u_i\} \succ_t T + v_i u_\ell$ for $1 \leq i \leq k-3$, $1 \leq \ell \leq k-2$, and $i \neq \ell$. Thus, T is k_t-critical(RC).

Conversely, we show that if T is a k_t-critical(RC) tree, then T is a subdivided star $K^*_{1,k-2}$ for $k \geq 5$ with exactly one edge subdivided twice. Assume that T is k_t-critical with respect to $K_{s,s}$. Then T is bipartite with partite sets $\mathcal{L}$ and $\mathcal{R}$ such that $|\mathcal{L}| = |\mathcal{R}| = s$. By Lemma 10, no path is k_t-critical(RC) implying that T has at least three leaves.

To complete the proof, we proceed with a series of claims. The proof of the first claim is identical to the proof of Claim 12.

Claim 15 *No two support vertices are adjacent.*

Claim 16 *No vertex is adjacent to two or more leaves.*

Proof. Suppose $v \in \mathcal{L}$ is adjacent to two leaves, say v_1 and v_2. Since no star is k_t-critical(RC), v has at least one neighbor u in $\mathcal{R}$ that is not a leaf. Let $u_1 \in N(u) - \{v\}$. Then $u_1 \in \mathcal{L}$. Consider $T' = T + u_1v_1$ and let S' be a γ_t-set of T'. Then $v \in S'$ to dominate v_2. If both u_1 and v_1 are in S', then $(S' - \{v_1\}) \cup \{u\} \succ_t T$, a contradiction since $|S'| < \gamma_t(T)$. Hence exactly one of u_1 and v_1 is in S. Thus, $[u_1, S] \mapsto_t v_1$ or $[v_1, S] \mapsto_t u_1$. If $[u_1, S'] \mapsto_t v_1$, then, since $v \in S$, v_1 is dominated by S, a contradiction. Thus, $[v_1, S] \mapsto_t u_1$. Since $v \in S$, v_1 is in S just to dominate u_1. It follows that $(S - \{v_1\}) \cup \{u\}$ totally dominates T and has cardinality less than $\gamma_t(T)$, a contradiction. Thus, each vertex is adjacent to at most one leaf. □

If there is no leaf in $\mathcal{R}$, then each vertex in $\mathcal{R}$ has degree at least 2 in T, and so T has at least $2s$ edges, which contradicts the fact that T is a tree of order $2s$. Hence, $L_{\mathcal{R}} \neq \emptyset$, and so $S_{\mathcal{L}} \neq \emptyset$. Similarly, $L_{\mathcal{L}} \neq \emptyset$ and $S_{\mathcal{R}} \neq \emptyset$. By Claim 16, $|L_{\mathcal{L}}| = |S_{\mathcal{R}}|$ and $|L_{\mathcal{R}}| = |S_{\mathcal{L}}|$. Let $O_{\mathcal{L}} = \mathcal{L} - L_{\mathcal{L}} - S_{\mathcal{L}}$ and let $O_{\mathcal{R}} = \mathcal{R} - L_{\mathcal{R}} - S_{\mathcal{R}}$. Since $|\mathcal{L}| = |\mathcal{R}| = s$, it follows that $|O_{\mathcal{L}}| = |O_{\mathcal{R}}|$. By Claim 15, there are no edges between vertices in the sets $S_{\mathcal{L}}$ and $S_{\mathcal{R}}$. Moreover, since T is connected, each vertex in $S_{\mathcal{L}}$ ($S_{\mathcal{R}}$, respectively) has a neighbor in $O_{\mathcal{R}}$ ($O_{\mathcal{L}}$, respectively) implying that $O_{\mathcal{L}}$ and $O_{\mathcal{R}}$ are not empty. Hence, $L_{\mathcal{L}}$, $S_{\mathcal{L}}$, and $O_{\mathcal{L}}$ (respectively, $L_{\mathcal{R}}$, $S_{\mathcal{R}}$, and $O_{\mathcal{R}}$) is a partition of $\mathcal{L}$ (respectively, $\mathcal{R}$).

Lemma 10 implies that at least one of $S_{\mathcal{L}}$ and $S_{\mathcal{R}}$ has cardinality two or more. We may assume that $|S_{\mathcal{R}}| \geq 2$. We show, then, that $|S_{\mathcal{L}}| = 1$.

Claim 17 $|S_{\mathcal{L}}| = |L_{\mathcal{R}}| = 1$.

Proof. Suppose to the contrary that $|S_{\mathcal{L}}| \geq 2$. Consider $T' = T + uv$ where $u \in S_{\mathcal{L}}$ and $v \in S_{\mathcal{R}}$. Now u and v are still support vertices in T', so they are in some γ_t-set of T'. Since T is k_t-critical, without loss of generality, there must be a vertex $u' \in O_{\mathcal{R}}$ that is in a γ_t-set S of T only to totally dominate u. In other words, u' is an ipn of u and $S - \{u'\} \succ T$. Note that S can be chosen to contain no leaves, that is, $S_{\mathcal{L}} \cup S_{\mathcal{R}} \subseteq S$. It follows that $S \cap O_{\mathcal{L}} \neq \emptyset$ and $S \cap O_{\mathcal{R}} \neq \emptyset$. Let $x \in S \cap O_{\mathcal{L}}$ and assume that u' is not adjacent to x. Since S is minimal, $\gamma_t(T + u'x) = \gamma_t(T)$, contradicting the fact that T is k_t-critcal. Thus, $u'x \in E(T)$. Since u' and x are arbitrary, it follows that each such ipn in $O_{\mathcal{R}}$ ($O_{\mathcal{L}}$, respectively) is adjacent to every vertex in $O_{\mathcal{L}} \cap S$ ($O_{\mathcal{R}} \cap S$, respectively). In particular, if there are at least two ipn with respect to S in both $O_{\mathcal{R}}$ and $O_{\mathcal{L}}$, a cycle is formed, contradicting the fact that T is a tree. Hence we may assume, without loss of generality, that at most one vertex in $S_{\mathcal{L}}$ has an ipn in $O_{\mathcal{R}}$. Since $|S_{\mathcal{L}}| \geq 2$, there is a vertex in $S_{\mathcal{L}}$ with at least two neighbors in $O_{\mathcal{R}} \cap S$. Furthermore, since T is k_t-critical and there are no edges between $S_{\mathcal{L}}$ and $S_{\mathcal{R}}$, each vertex in $S_{\mathcal{R}}$ has an ipn in $O_{\mathcal{L}}$. But then since $|S_{\mathcal{R}}| \geq 2$, there are at least two such ipn in $O_{\mathcal{L}}$ and they must be adjacent to every vertex in $O_{\mathcal{R}} \cap S$. We have just seen that $|O_{\mathcal{R}} \cap S| \geq 2$, so again a cycle is formed contradicting that T is a tree. We deduce, therefore, that $|S_{\mathcal{L}}| = 1$, and so $|L_{\mathcal{R}}| = 1$. □

By Claim 17, $|S_{\mathcal{L}}| = 1$. Let $S_{\mathcal{L}} = \{u\}$ and $L_{\mathcal{R}} = \{w\}$, and so w is the leaf adjacent to u.

Claim 18 $|O_{\mathcal{L}}| = |O_{\mathcal{R}}| = 1$.

Proof. Suppose $|O_{\mathcal{L}}| = |O_{\mathcal{R}}| \geq 2$. We show first that u has an ipn in $O_{\mathcal{R}} \cap S$. Suppose u has two or more neighbors in $O_{\mathcal{R}} \cap S$. Then each of these neighbors is necessary to dominate a vertex in $O_{\mathcal{L}}$. Now since u is not adjacent to any vertex in $S_{\mathcal{R}}$, every vertex in $S_{\mathcal{R}}$ must have an ipn in $O_{\mathcal{L}} \cap S$. Since $|S_{\mathcal{R}}| \geq 2$, it follows that there are at least two such ipns. Furthermore, each of these ipn must be adjacent to all the neighbors of u in $O_{\mathcal{R}} \cap S$. Hence a cycle is formed, producing a contradiction. Thus, u has an ipn in $O_{\mathcal{R}} \cap S$, say u'.

Now u' is adjacent to every vertex in $O_{\mathcal{L}} \cap S$. Since $|S_{\mathcal{R}}| \geq 2$, $|O_{\mathcal{L}} \cap S| \geq 2$, and so u' is adjacent to at least two vertices of $O_{\mathcal{L}} \cap S$. Let $T' = \langle O_{\mathcal{L}} \cup O_{\mathcal{R}} \cup \{u, w\} \rangle$. Then, T' is a bipartite graph with partite sets $O_{\mathcal{L}} \cup \{u\}$ and $O_{\mathcal{R}} \cup \{w\}$. Furthermore, $|O_{\mathcal{L}} \cup \{u\}| = |O_{\mathcal{R}} \cup \{w\}|$. However, every vertex of $O_{\mathcal{R}} - \{u'\}$ has degree at least 2 in T', while u' has degree at least 3 in T' and w has degree 1 in T'. Thus, T' has at least $2|O_{\mathcal{R}} \cup \{w\}|$ edges, and so T'–a graph of order $|O_{\mathcal{R}} \cup \{w\}|$–contains a cycle, contradicting the fact that T is a tree. □

By Claim 18, $|O_{\mathcal{L}}| = |O_{\mathcal{R}}| = 1$. Since T is connected, the vertex of $O_{\mathcal{L}}$ is adjacent to every vertex of $S_{\mathcal{R}} \dot{\cup} O_{\mathcal{R}}$ and the vertex of $O_{\mathcal{R}}$ is adjacent to u. Hence, T is the subdivided star $K^*_{1,k-2}$ with exactly one edge subdivided twice as shown in Figure 1. □

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