The rank partition and the covering number of the elements of the dual matroid

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Abstract

We present results that characterize the covering number and the rank partition of the dual of a matroid M using properties of M. We prove, in particular, that the elements of covering number 2 in M^* are the elements of the closure of the maximal 2-transversals of M.

From the results presented it can be seen that every matroid M is a weak map image of a transversal matroid with the same rank partition.

1 Introduction

Let S be a nonempty finite set with cardinality m and M a matroid on S. By M^* we mean the dual of M. If B is a basis of M then the basis of

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M^*, $S \setminus B$, will be called the *dual basis of* B. The rank function of M is denoted by rk_M (briefly by rk). We use rk^* to denote the rank function of M^*. The closure operator of M is denoted by cl_M (briefly by cl). Let X be a subset of S. The *restriction* of M to X is denoted by $M|X$. The restriction of M to $S \setminus X$ is usually denoted by $M \setminus X$. An element $x \in S$ is a *loop of* M if x does not belong to any basis of M, and x is a *coloop* of M if x belongs to every basis of M. The set of loops of M will be denoted by $\mathcal{L}(M)$ and the set of coloops of M is denoted by $\mathcal{CL}(M)$.

Let $\mathcal{A} = (H_1, \ldots, H_q)$ be a family of nonempty subsets of the set S. We denote by $M[\mathcal{A}]$ the *transversal matroid* associated with $\mathcal{A}$, i.e. the matroid on S whose independent sets are the partial transversals of $\mathcal{A}$.

Let $M_1, \ldots, M_k$ be matroids on S. The family of subsets of S,

$$\{I_1 \cup \ldots \cup I_k \ : \ I_i \text{ is an independent set of } M_i,\ i = 1, \ldots, k\}$$

is the family of independent sets of a matroid on S, called the union of $M_1, \ldots, M_k$ and denoted by $\bigvee_{i=1}^{k} M_i$, [6]. It is well known that

$$\mathrm{rk}_{M_1 \vee \cdots \vee M_k}(A) = \min_{X \subseteq A} \left(\sum_{i=1}^{k} \mathrm{rk}_{M_i}(X) + |A \setminus X| \right) \tag{1}$$

The union of k copies of a matroid M on S, is called the *kth power of* M. We denote this matroid by $M^{(k)}$ and its rank by $\rho_k(M)$ or briefly by ρ_k. We denote by ρ_k^* the rank of $(M^*)^{(k)}$. The rank function in $M^{(k)}$ will be denoted by rk_k. In particular $\mathrm{rk}_k = \mathrm{rk}_{M^{(k)}}$. We use rk_k^* to denote the rank function of $(M^*)^{(k)}$.

By convention $\rho_0 = 0$. The sequence [2]

$$\rho(M) = (\rho_1 - \rho_0, \rho_2 - \rho_1, \ldots, \rho_m - \rho_{m-1})$$

is a partition of $|S \setminus \mathcal{L}(M)|$ and is called the *rank partition of* M.

It is known from [2] that if B is a basis of $M^{(k)}$ there exists independent subsets of M, $B_1, \ldots, B_k$ such that $B = B_1 \dot\cup \cdots \dot\cup B_k$ and

$$\rho_t = \sum_{i=1}^{t} |B_i|, \quad t = 1, \ldots, k.$$

We say that $B_1 \cup \cdots \cup B_k$ is a *k-factorization* of B.

It is easy to see that if $B_1 \cup \cdots \cup B_k$ is a k-factorization of a basis B of $M^{(k)}$ then B_1 is a basis of M. However not all the bases of M occur as factors in a k-factorization of a basis of $M^{(k)}$, as we can see in the following example:

Example

Let V be a real vector space and let $(e_1,\ e_2)$ be a family of linearly independent vectors of V. Let

$$x_1 = e_1,\ x_2 = e_2,\ x_3 = e_1,\ x_4 = e_1 + e_2$$

Let M be the vectorial matroid on $\mathbb{L}\mathrm{in}(x_1,\ x_2,\ x_3,\ x_4)$ (matroid on $\{1,2,3,4\}$ such that J is independent in $\mathbb{L}\mathrm{in}(x_1,\ x_2,\ x_3,\ x_4)$ if $\{x_i\ :\ i \in J\}$ is linearly independent).

Then $B = \{2,4\}$ is a basis of M, which is not the first factor of a 2-factorization of a basis of $M^{(2)}$.

A basis of M that occurs, as first factor, in a factorization of a basis of $M^{(k)}$ is called *k-special*. We say that a basis B of M is *special* if it is m-special. We can easily see that a special basis is k-special for every k, $1 \leq k \leq m$.

The *covering number* of $x \in (S \setminus \mathcal{L}(M))$, is the smallest positive integer t such that $x \in C\mathcal{L}(M^{(t)})$, [3] (i.e. $\mathrm{rk}_t(S \setminus x) < \mathrm{rk}_t(S)$).

We denote this integer by $s_x(M)$ and we extend this concept to elements $x \in \mathcal{L}(M)$ by defining $s_x(M) = m + 1$.

The element $x \in M$ is a coloop of M (i.e. $s_x(M) = 1$) if and only if $x \in \mathcal{L}(M^*)$ (i.e. $s_x(M^*) = m + 1$) and $x \in \mathcal{L}(M)$ (i.e. $s_x(M) = m + 1$) if and only if $x \in C\mathcal{L}(M^*)$ (i.e. $s_x(M^*) = 1$).

Let M be a matroid on S. A set $T \subseteq S$ is *k-transversal* of M if T is independent in $M^{(k)}$ and there exists pairwise disjoint independents subsets of T, $I_1, \ldots, I_k$ satisfying:

1) I_i is a basis of $M|T$, $i = 1, \ldots, k$;

2) $I_1 \cup \cdots \cup I_k = T$.

The study of the k-transversals has been done in [1]. In particular, it is proved in [1] that T is a k-transversal if and only if T is an independent set of $M^{(k)}$ satisfying

$$|T| = k\,\mathrm{rk}_{M^{(k)}}(T)$$

It is proven in the above referred article that if C is a circuit of $M^{(k)}$ and $y \in C$ then $C \setminus y$ is a k-transversal. It is also proved that the maximal k-transversals, by inclusion, have the same closure in M, denoted by $D_k(M)$, or briefly by D_k, and that there is a maximal k-transversal contained in each basis of $M^{(k)}$. Furthermore, if T is the maximal k-transversal contained in the basis B of $M^{(k)}$ then

$$S \setminus B = \mathrm{cl}_{M^{(k)}}(T) \setminus T = \mathrm{cl}_M(T) \setminus T. \qquad (2)$$

In this article we are going to study the covering number of the elements of M^* and the rank partition of M^*.

2 The relations between the matroid M^* and the matroid $M[\mathcal{A}_{F^*}]$

Let M be a matroid on S, and let the q-set $F^* = \{y_1, \dots, y_q\}$ be a basis of M^* and F be the dual basis of F^*. We define $\mathcal{A}_{F^*} = (H_1, \dots, H_q)$, where H_i is the fundamental circuit of y_i in F, $i = 1, \dots, q$. We denote by $M[\mathcal{A}^*_F]$ the matroid whose independent sets are the partial transversals of $\mathcal{A}_{F^*}$.

Given a subset X of S we are going to denote by R_X the set of integers that index the circuits H_i which have a nonempty intersection with X, i.e.

$$R_X = \{i \in \{1, \dots, q\} \, : \, H_i \cap X \neq \emptyset\}.$$

2.1 Theorem

Every independent set of M^* is an independent set of $M[\mathcal{A}_{F^*}]$, i.e. a partial transversal of $(H_1, \dots, H_q)$

Proof

We start by proving the following claim:

Claim

Let G^* be a basis of M^* and G the dual basis of G^*. Let Δ be the subset of $\{1, \dots, q\}$ such that $G \cap F^* = \{y_j \, : \, j \in \Delta\}$. Then there exits a bijection

$$\psi : \Delta \to G^* \cap F$$

such that $\psi(j) \in H_j, \quad j \in \Delta$.

Proof

We are going to prove this claim by induction on the cardinality of $G \cap F^*$. We start by remarking that if $x \in G^* \cap F$ and C is the fundamental circuit of x in G there exists $k \in \Delta$ such that $y_k \in C$ and $x \in H_k$.

Let Γ be the subset of Δ such that

$$C \cap G \cap F^* = \{y_j \, : \, j \in \Gamma\}.$$

Since $x \in \mathrm{cl}\,(C)$ we have

$$\begin{aligned} x \in \mathrm{cl}\,(C \cap G) &= \mathrm{cl}\,((C \cap G) \cap (F \cup F^*)) \\ &\subseteq \mathrm{cl}\,((C \cap G \cap F) \cup (\{y_j : j \in \Gamma\}) \\ &\subseteq \mathrm{cl}\,((C \cap G \cap F) \cup (\bigcup_{j \in \Gamma} H_j)) \\ &\subseteq \mathrm{cl}\,((C \cap G \cap F) \cup [(\bigcup_{j \in \Gamma} H_j)) \setminus \{y_j : j \in \Gamma\}]). \end{aligned}$$

Since $x \in G^*$ then $x \notin C \cap G \cap F$. If

$$x \notin \bigcup_{j \in \Gamma} H_j$$

(bear in mind the third equation before) we get

$$x \notin (C \cap G \cap F) \cup [(\bigcup_{j \in \Gamma} H_j)) \setminus \{y_j : j \in \Gamma\}].$$

Therefore

$$x \cup (C \cap G \cap F) \cup [(\bigcup_{j \in \Gamma} H_j)) \setminus \{y_j : j \in \Gamma\}] \subseteq F$$

is an independent set. Contradiction. Therefore $\Gamma \neq \emptyset$ and $x \in H_k$ for some $k \in \Gamma$.

$|G \cap F^*| = 1$.
Define $\psi : \Delta = \{k\} \to G^* \cap F$ by setting $\psi(k) = x$. Obviously $\psi(k) \in H_k$.
This proves the case $|G \cap F^*| = 1$.

$|G \cap F^*| > 1$ Since there exists $k \in \Gamma$ such that $y_k \in C$ and $x \in H_k$, $G' = (G \setminus y_k) \cup x$ is a basis of M. Denote by $(G')^*$ the dual basis of G'. It can be easily seen that

$$|G' \cap F^*| = |G \cap F^*| - 1.$$

Let Λ be the subset of $\{1, \ldots, q\}$ such that

$$G' \cap F^* = \{y_j : j \in \Lambda\}$$

$(\Delta = \Lambda \cup k)$. By induction hypothesis there exists a bijection

$$\varphi : \Lambda \to (G')^* \cap F$$

such that $\varphi(j) \in H_j$, $j \in \Lambda$. Define ψ as the extension of φ satisfying $\psi(k) = x$. It is easy to see that ψ satisfies the requirements of the claim.

■

Let X be an independent set of M^*. Let $X_1 = X \cap F$ and $X_2 = X \cap F^*$. Then we have $X = X_1 \dot{\cup} X_2$. Since X is independent in M^* there exists a basis G^* of M^* that contains X and such that $G^* \setminus X \subseteq F^*$. By the claim we know that there is a bijective map, ψ from the index set of $G \cap F^*$, Δ, on X_1. Since $G^* \cap F^* \supseteq X_2$ we have that $G \cap F^*$ is disjoint of X_2. Then if Δ' is the subset of $\{1, \ldots, q\}$ such that

$$\{y_j \ : \ j \in \Delta'\} = X_2$$

and

$$\chi : \Delta \cup \Delta' \to X$$

defined by $\chi_{|\Delta} = \psi$ and $\chi(j) = y_j, \quad j \in \Delta'$ is a bijection and obviously $\chi(j) \in H_j, \quad j \in \Delta \cup \Delta'$. Therefore X is a partial transversal.

■

The following result is very well known

2.2 Proposition

If C is a circuit of M and C^* is a circuit of M^* then $|C \cap C^*| \neq 1$

2.3 Proposition

Let A be an independent set of M^* and C a circuit of M disjoint of A. Then for every $y \in C$, $A \cup y$ is independent in M^*.

Proof

Assume that $A \cup y$ is not an independent set of M^*. Let C^* be the circuit of M^* such that

$$C^* \subseteq A \cup y.$$

Then $C \cap C^* = \{y\}$. Contradiction (remind Proposition 2.2).

■

Remark

It is not difficult to see from the previous proposition that if C is a circuit of M and A is a subset of M^* then

$$C \cap A = \emptyset \Leftrightarrow C \cap \mathrm{cl}_{M^*}(A) = \emptyset.$$

Therefore if A and B are subsets of M^* then

$$\mathrm{cl}_{M^*}(A) = \mathrm{cl}_{M^*}(B) \Rightarrow R_A = R_B. \tag{3}$$

2.4 Proposition

Let F^* be an r-special basis of M^*. Let T be an r-transversal of M^*. If there exists a basis of $\mathrm{cl}_{M^*}(T)$ contained in F^*, then the following holds:

(i) T is an r-transversal of $M[\mathcal{A}_{F^*}]$.

(ii) $\mathrm{cl}_{M^*}(T) \setminus \mathcal{L}(M^*) = \mathrm{cl}_{M[\mathcal{A}_{F^*}]}(T)$.

Proof

(i) Let B^* be a basis of $(M^*)^{(r)}$ and let

$$F^* \cup B_2^* \cup \cdots \cup B_r^*$$

be an r-factorization of B^* in M^*. Let U be a basis of $\mathrm{cl}_{M^*}(T)$ contained in F^*. It is easy to see that $|R_U| = |U|$. Then, using (3), we have $R_T = R_U$. Therefore

$$\mathrm{rk}^*(T) = |U| = |R_T|.$$

By definition of $M[\mathcal{A}_{F^*}]$ we have

$$\mathrm{rk}_{M[\mathcal{A}_{F^*}]}(T) \leq \mathrm{rk}^*(T) = |R_T|.$$

Since the independents of M^* are also independent in $M[\mathcal{A}_{F^*}]$ we have

$$\mathrm{rk}^*(T) \leq \mathrm{rk}_{M[\mathcal{A}_{F^*}]}(T).$$

Therefore

$$\mathrm{rk}^*(T) = \mathrm{rk}_{M[\mathcal{A}_{F^*}]}(T) = |R_T|.$$

Again, since the independent sets of M^* are also independent in $M[\mathcal{A}_{F^*}]$ we get that T is independent in $M[\mathcal{A}_{F^*}]^{(r)}$ and

$$r\,\mathrm{rk}_{M[\mathcal{A}_{F^*}]}(T) = r\,\mathrm{rk}^*(T) = |T|.$$

Thus T is an r-transversal of $M[\mathcal{A}_{F^*}]$.

(ii) Let $x \in \mathrm{cl}_{M^*}(T) \setminus \mathcal{L}(M^*)$. Then, if $x \in F^*$ it is obvious that $R_x \neq \emptyset$. If $x \notin F^*$, assume, to get a contradiction, $R_x = \emptyset$. Let C^* be the fundamental circuit of x in F^*. Then if $y_j \in C^* \cap F^*$ we have $C^* \cap H_j = \{y_j\}$. Contradiction. Therefore $R_x \neq \emptyset$. On the other hand we have using (3)

$$\begin{aligned} x \in \mathrm{cl}_{M^*}(T) &\Rightarrow \mathrm{cl}_{M^*}(T \cup x) = \mathrm{cl}_{M^*}(T) \\ &\Rightarrow R_{T \cup x} = R_T \\ &\Rightarrow R_x \subseteq R_T \end{aligned}$$

Since we have already seen that $|R_T| = \mathrm{rk}_{M[\mathcal{A}_{F^*}]}(T)$ we conclude from the former implications that $x \in \mathrm{cl}_{M[\mathcal{A}_{F^*}]}(T)$. Then we have $\mathrm{cl}_{M^*}(T) \setminus \mathcal{L}(M^*) \subseteq \mathrm{cl}_{M[\mathcal{A}_{F^*}]}(T)$. The reverse inclusion can be obtained by using Theorem 2.1.

■

3 The covering number of the elements of M^*

The main results we are going to present in this section are the following.

3.5 Theorem

Let M be a matroid on S. We have

$$D_2 \setminus \mathcal{L}(M) = \{x \in S \,:\, s_x(M^*) = 2\}.$$

3.6 Theorem

Let x be an element of S which is not coloop of M. If C is a circuit of M and $x \in C$ then

$$s_x(M^*) \leq |C|.$$

3.7 Theorem

Let F^* be a special basis of M^*. Then for all $x \in S \setminus \mathcal{L}(M^*)$ we have

$$s_x(M^*) = s_x(M[\mathcal{A}_{F^*}]).$$

We start be proving some auxiliary results.

3.8 Proposition

Let M be a matroid on S. Then we have

$$\rho_2^* - \rho_1^* = \rho_2 - \rho_1.$$

Proof

Let B^* be a special basis of M^* and $B = S \setminus B^*$ be the corresponding basis of M. We say that $X \subseteq B$ can be replaced if there exists $Y \subseteq B^*$ such that $(B \setminus X) \cup Y$ is a basis of M. Since $(B \setminus X) \cup Y$ is a basis of M we conclude that if X can be replaced then it is an independent of M^*. Moreover if X is a maximal (by inclusion) subset of B independent in M^* and C^* is a basis of M^* containing X, we have $C^* \cap B = X$. On the other hand, fixing $Y = (S \setminus C^*) \cap B^*$ we have

$$(B \setminus X) \cup Y = S \setminus C^*.$$

Therefore X can be replaced. Thus

$$\rho_1(M^*|B) = \max\{|X| : X \subseteq B \text{ and } X \text{ can be replaced }\}.$$

Since B^* is special $\rho_2^* - \rho_1^* = \rho_1(M^*|B)$. So

$$\begin{aligned}
\rho_2^* - \rho_1^* &\leq \max\{|X| : X \subseteq B \text{ and } X \text{ can be replaced }\} \\
&\leq \max\{|Y| : Y \text{ is independent and contained in } S \setminus B\} \\
&\leq \rho_2 - \rho_1
\end{aligned}$$

Now from $(M^*)^* = M$ we have

$$\rho_2^* - \rho_1^* = \rho_2 - \rho_1.$$

■

3.9 Proposition

Let B a basis of $M^{(2)}$. Let $B_1 \cup B_2$ be a 2-factorization of B in M. Let $Z \subseteq B_1$ be such that $(B_2 \cup Z) \cup (B_1 \setminus Z)$ is a 2-factorization of B in M. Then the following is true:

1. *$S \setminus Z$ is a basis of $(M^*)^{(2)}$;*
2. *$S \setminus Z = (S \setminus B_1) \cup (B_1 \setminus Z)$;*
3. *$D_2(M^*) \cap (B_1 \setminus Z) = D_2(M^*) \cap (S \setminus (B_2 \cup Z))$, is a 2-factorization of $S \setminus Z$ in M^*.*

Proof

1. Since $S \setminus (B_2 \cup Z)$ is a basis of M^* then

$$B_1 \setminus Z \subseteq (S \setminus (B_2 \cup Z))$$

is an independent set of M^*. Therefore $S \setminus Z = (S \setminus B_1) \cup (B_1 \setminus Z)$ is independent in $(M^*)^{(2)}$ (remind that $S \setminus B_1$ is a basis of M^*). Thus

$$\begin{aligned}
\mathrm{rk}_2^*(S \setminus Z) &= |S \setminus Z| \\
&= |S| - |Z| \\
&= |S \setminus B_1| + |B_1 \cup B_2| - |B_2 \cup Z| \\
&= \rho_1^* + \rho_2 - \rho_1 \\
&= \rho_1^* + \rho_2^* - \rho_1^* \\
&= \rho_2^*
\end{aligned}$$

The fifth equality follows from Proposition 3.8

2. Since $S \setminus B_1$ is a basis of M^* and $S \setminus Z = (S \setminus B_1) \dot{\cup} (B_1 \setminus Z)$ we conclude that $(S \setminus B_1) \dot{\cup} (B_1 \setminus Z)$ is a 2-factorization of $(M^*)^{(2)}$.

3. Let T be the maximal 2-transversal of $(M^*)^{(2)}$ contained in $S \setminus Z$. Then

$$T = (D_2(M^*) \cap (B_1 \setminus Z)) \cup (D_2(M^*) \cap (S \setminus B_1))$$

is a 2-factorization of T. Therefore $\mathrm{cl}_{M^*}(D_2(M^*) \cap (B_1 \setminus Z)) = D_2(M^*)$. Since $S \setminus (B_2 \cup Z)$ is a basis of M^* and $D_2(M^*) \cap (B_1 \setminus Z)$ and $D_2(M^*) \cap (S \setminus (B_2 \cup Z))$ are bases of $D_2(M^*)$, all elements of

$S\backslash(B_2 \cup Z)$ that do not belong to $B_1 \backslash Z$ are not elements of $D_2(M^*)$. Thus

$$D_2(M^*) \cap (B_1 \setminus Z) = D_2(M^*) \cap (S \setminus (B_2 \cup Z)).$$

■

Proof of Theorem 3.5

Let $x \in D_2 \setminus \mathcal{L}(M)$. Assume, to get a contradiction, that $s_x(M^*) > 2$. Then there exists a basis B of $(M^*)^{(2)}$ such that $x \notin B$. Let $B = B_1 \cup B_2$ a 2-factorization of B in M^*. Let $Z \subseteq B_1$ be such that $(B_2 \cup Z) \cup (B_1 \setminus Z)$ is a 2-factorization of B. Since we have assumed that $x \notin B$ we have

$$x \notin D_2 \cap (B_1 \setminus Z).$$

Using the part 3) of Proposition 3.9 we have

$$x \notin D_2 \cap (S \setminus (B_2 \cup Z)).$$

Then $x \notin S \setminus (B_2 \cup Z)$ and, consequently, $x \in B_2 \cup Z$. Contradiction. So we conclude that $s_x(M^*) \leq 2$. Since $x \notin \mathcal{L}(M)$ we have $s_x(M^*) = 2$. Therefore

$$D_2 \setminus \mathcal{L}(M) \subseteq \{x \in S \ : \ s_x(M^*) = 2\}.$$

To show that $\{x \in S \ : \ s_x(M^*) = 2\} \subseteq D_2 \setminus \mathcal{L}(M)$ we are going to prove that if $x \notin D_2$ then $s_x(M^*) \geq 3$.

Claim

Let B be a basis of $M^{(2)}$. If $x \notin D_2$ there exists a factorization in M

$$B = B_1 \cup B_2$$

satisfying $x \in B_1$ and $x \notin B_2$.

Proof

Let x' be an element that does not belong to S and $\mathcal{M}_{\{x\}}$ the principal modular cut defined by the flat $\{x\}$. Let

$$N = M +_{\{x\}} x'.$$

Assume that does not exist any 2-factorization of B in the conditions of the claim. Then it is easy to see that $B \cup x'$ is dependent in $N^{(2)}$. So B is a basis of $N^{(2)}$. We can now see that $x' \in D_2(N) = D_2(M) \cup \{x'\}$. Then $x \in D_2(M)$. Contradiction. ■

Assume that $x \notin D_2$ then, using the claim, if B is a basis of $M^{(2)}$ there exists a 2-factorization of B, $B = B_1 \cup B_2$, such that $x \notin B_2$. Let $Z \subseteq B_1$ be such that $x \in Z$ and $B_2 \cup Z$ is a basis of M. By Proposition 3.9 $S \setminus Z$ is a basis of $(M^*)^{(2)}$. Since $x \notin S \setminus Z$, we have $s_x(M^*) \geq 3$. ■

Proof of Theorem 3.6

Let C be a circuit of M and let $x \in C$. Let p be the cardinality of C. Assume, to get a contradiction, that $s_x(M^*) > p$. Then there exists a basis B^* of $(M^*)^{(p)}$ such that $x \notin B^*$. Let

$$B_1^* \cup \cdots \cup B_p^*$$

be a p-factorization of B^*. Therefore there exists an integer j, $1 \leq j \leq p$ such that $B_j^* \cap C = \emptyset$. Using Proposition 2.3 we have that $B_j^* \cup x$ is independent. Then B^* is not a basis of $(M^*)^{(p)}$. Contradiction. ■

Proof of Theorem 3.7

Let $s_x(M^*) = r$ and let B^* be a basis of $(M^*)^{(r)}$ such that

$$F^* \cup B_2^* \cdots \cup B_{r-1}^* \cup B_r^*$$

is an r-factorization of B^*.

We are going to prove that

$$s_x(M[\mathcal{A}_{F^*}]) \geq r.$$

If $r = 1$ the inequality follows from the definitions. Let $r > 1$. Using Proposition 3.2 of [3] we know that there exists a maximal $(r-1)$-transversal T such that

$$x \in \mathrm{cl}_{M^*}(T) \setminus T.$$

Let P be the maximal transversal contained in $F^* \cup B_2^* \cdots \cup B_{r-1}^*$. Since

$$\mathrm{cl}_{M^*}(T) = \mathrm{cl}_{M^*}(P) = \mathrm{cl}_{M^*}(P \cap F^*)$$

we have, using Proposition 2.4, T is an r-transversal of $M[\mathcal{A}_{F^*}]$ and $x \in \mathrm{cl}_{M[\mathcal{A}_{F^*}]}(T) \setminus T$. Using again Proposition 3.2 of [3] we have

$$s_x(M[\mathcal{A}_{F^*}]) \geq r.$$

Next we prove that $s_x(M[\mathcal{A}_{F^*}]) \leq r$.

Since $s_x(M^*) = r$, $B^* \setminus x$ is a basis of $(M^*)^{(r)} \setminus x = (M^* \setminus x)^{(r)}$. We will show that $B^* \setminus x$ is a basis of $M[\mathcal{A}_{F^*}] \setminus x$. It is obviously an independent set if $M[\mathcal{A}_{F^*}]$ by Theorem 2.1 so, an independent set of $M[\mathcal{A}_{F^*}] \setminus x$.

Let T be the inclusion maximum r-transversal of $M^* \setminus x$ contained in $B^* \setminus x$. Then T is an r-transversal of M^* contained in B^* and

$$\mathrm{cl}_{M^* \setminus x}(T) = \mathrm{cl}_{M^*}(T) = \mathrm{cl}_{M^*}(T \cap F^*). \tag{4}$$

Then by (2) we have $S \setminus B^* \subseteq \mathrm{cl}_{M^*}(T)$. Bearing in mind (4) and Proposition 2.4 we conclude that T is a transversal of $M[\mathcal{A}_{F^*}]$ and

$$S \setminus B^* \subseteq \mathrm{cl}_{M[\mathcal{A}_{F^*}]}(T)$$

Therefore the independent set of $(M[\mathcal{A}_{F^*}])^{(r)} \setminus x$ cannot be extended to a larger independent set of $M[\mathcal{A}_{F^*}]^{(r)} \setminus x$. This means that $B^* \setminus x$ is a basis of $M[\mathcal{A}_{F^*}]^{(r)} \setminus x$ and x is a coloop of $M[\mathcal{A}_{F^*}]^{(r)}$. Then

$$s_x(M[\mathcal{A}_{F^*}]) \leq r.$$

■

4 The rank partition of M^*

Let t ba a nonegative integer. Let $\lambda = (\lambda_1, \ldots, \lambda_t)$ and $\mu = (\mu_1, \ldots, \mu_t)$ be two partitions of t. We say that $\lambda \succeq \mu$ if

$$\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i, \quad k = 1, \ldots, t.$$

It is well known that $X \subseteq S$ is a *spanning set* of the matroid M on S if $\mathrm{cl}_M(X) = S$. Let M_1 and M_2 be matroids on S and $\mathcal{S}_1$ and $\mathcal{S}_2$ be the collections of spanning sets of M_1 and M_2 respectively. The intersection $M_1 \wedge M_2$ is the matroid on S whose family of spanning sets is the collection

$$\mathcal{S}_1 \wedge \mathcal{S}_2 = \{X_1 \cap X_2 \, : \, X_1 \in \mathcal{S}_1, \, X_2 \in \mathcal{S}_2\}.$$

It is known that

$$M_1 \wedge M_2 = (M_1^* \vee M_2^*)^* \tag{5}$$

The purpose of this section is to get information on the rank partition of M^* coming from the matroid structure of M without an explicit reference to the independents of M^*. The main results we are going to present are the following:

4.10 Theorem

Let F^* be a basis of M^*. Then

$$\rho(M[\mathcal{A}_{F^*}]) \succeq \rho(M^*). \tag{6}$$

If F^* is special then

$$\rho(M[\mathcal{A}_{F^*}]) = \rho(M^*). \tag{7}$$

4.11 Theorem

Let $k > 1$ be an integer. Then

$$\rho^*_k - \rho^*_{k-1} = \mathsf{rk}_{(\wedge^{k-1}M)\vee M}(S) - \mathsf{rk}(S),$$

where $\wedge^t M$ denotes the matroid intersection of t copies of M.

Proof of Theorem 4.10

To prove (6) just remark that, by Theorem 2.1 every independent of M^* is an independent of $M[\mathcal{A}_{F^*}]$. Then using the definition of rank partition we get (6).

To prove (7) consider B^* a basis of $(M^*)^m$ and let

$$F^* \cup B^*_2 \cup \cdots \cup B^*_m$$

be a factorization of B^*.

Let $k \in \{1, \ldots, m\}$. We prove that

$$F^* \cup \cdots \cup B^*_k$$

is a basis of $M[\mathcal{A}_{F^*}]^{(k)}$. Using Theorem 2.1 we conclude that $F^* \cup \cdots \cup B^*_k$ is independent in $M[\mathcal{A}_{F^*}]^{(k)}$. To prove that $F^* \cup \cdots \cup B^*_k$ is a basis of $M[\mathcal{A}_{F^*}]^{(k)}$ it suffices to show that $S \setminus (F^* \cup \cdots \cup B^*_k)$ is included in

$$\mathsf{cl}_{M[\mathcal{A}_{F^*}]}(F^* \cup \cdots \cup B^*_k).$$

Let P the maximal transversal contained in the basis of $M^{(k)}$,

$$F^* \cup B^*_1 \cup \cdots \cup B^*_k.$$

Using (2) we can conclude that

$$S \setminus (F^* \cup B^*_1 \cup \cdots \cup B^*_k) = \mathsf{cl}_{(M^*)^{(k)}}(T) \setminus T.$$

The Proposition 2.4 allows us to get from the former equality, that

$$S \setminus (F^* \cup B_1^* \cup \cdots \cup B_k^*) = \mathrm{cl}_{M[\mathcal{A}_{F^*}]}(T) \setminus T.$$

■

4.12 Corollary

Let M be a matroid and M^* the dual of M. Then

$$\rho(M^*) = \min_{F^* \text{ basis of } M^*} \rho(M[\mathcal{A}_{F^*}]),$$

where min *is considered for the majorization order in the set of partitions of $m - |\mathcal{L}(M^*|$.*

Remarks

1) The definition of dual matroid implies that the first term of $\rho(M^*)$ is $m - \mathrm{rk}\,(S)$. The second term of $\rho(M^*)$ is also known by Proposition 3.8.

2) Bearing in mind that $(M^*)^* = M$ we can easily see from Theorem 4.10 that every matroid M is a weak map image of a transversal matroid with the same rank partition.

Proof of Theorem 4.11

Using (5) and the fact that, [5, pg 72],

$$\mathrm{rk}^*(X) = |X| - \mathrm{rk}\,(S) + \mathrm{rk}\,(S \setminus X)$$

we get:

$$\begin{aligned}
\rho_k^* &= \min_{X \subseteq S}(\mathrm{rk}_{(M^*)^{(k-1)} \vee M^*}(X) + \mathrm{rk}^*(X) + |S \setminus X|) \\
&= \min_{X \subseteq S}(\mathrm{rk}_{(\wedge^{k-1} M)^*}(X) + \mathrm{rk}^*(X) + |S \setminus X|) \\
&= \min_{X \subseteq S}(\mathrm{rk}_{\wedge^{k-1} M}(S \setminus X) + \mathrm{rk}_M(S \setminus X) + |X|) - \mathrm{rk}_{\wedge^{k-1} M}(S) - \mathrm{rk}\,(S) + |S| \\
&= \mathrm{rk}_{(\wedge^{k-1} M) \vee M}(S) - \mathrm{rk}\,(S) + \rho_{k-1}^*.
\end{aligned}$$

■

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