All Cycles are Edge-Magic

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Abstract

We prove that all cycles are edge-magic, thus solving a problem presented by [2]. In [3] it was shown that all cycles of odd length are edge-magic. We give explicit constructions that show that all cycles of even length are edge-magic. Our constructions differ for the case of cycles of length $n \equiv 0(\text{mod}4)$ and $n \equiv 2(\text{mod}4)$.

1 Introduction

Let $G = (V, E)$ be a graph, where $V = \{v_1, v_2, ..., v_n\}$ is the set of vertices, and $E = \{e_1, e_2, ..., e_m\}$ is the set of edges of the graph.

Our goal is to label the vertices and edges of G with distinct labels, where $S = \{1, 2, 3, ..., n + m\}$ is the set of possible labels.

Definition 1 *Let L be a labeling of the vertices and edges of G using the set of labels S. Denote by $\ell(v)$ the label given by L to vertex v, and by $\ell(e)$ the label given to edge e. Let $L(V)$ be the set of n labels given by L to the vertices, and $L(E)$ the set of m labels given to the edges of G.*

Definition 2 *Let L be some labeling of G. The* valence *of edge $e = (u, v) \in E$ under the labeling L is $val(e) = \ell(u) + \ell(v) + \ell(e)$.*

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Definition 3 *A graph G is called* edge-magic *if there exists a labeling L, for which all edge valencies are equal. In this case denote the value of all edge valencies by val(L). The labeling L will be called a* magic labeling, *and val(L) will be called the* valence of *L*.

In fact if L is a magic labeling, then it is enough to state the labeling of the vertices under L. Then there is exactly one possible way to label the edges which will result in equal edge valencies $val(L)$. In order to see this, note that if L is a magic labeling then:

$$\forall e = (u, v) \in E, \quad \ell(u) + \ell(v) = val(L) - \ell(e). \tag{1}$$

Therefore the sums of labels of adjacent vertices and the value of $val(L)$, determine the labels of all edges. Denote by $Sum(L(V))$ the set of sums of labels of adjacent vertices. Using Eq 1, we get that:

$$Sum(L(V)) = \{val(L) - \ell(e)\}_{e \in E} \tag{2}$$

Thus, in order to prove that a labeling L is magic, we will have to show only that $Sum(L(V))$ is as claimed. Note also that if L is a magic labeling then the valence of L should satisfy the following equality:

$$val(L) = \frac{\sum_{i=1}^{n} \ell(v_i) \cdot d(v_i) + \sum_{i=1}^{m} \ell(e_i)}{m} \tag{3}$$

where $d(v)$ is the degree of vertex v. This expression should of course be an integer.

Previous Work: The notion of an edge-magic graph was introduced in [1] and [2]. In [2] it was proved that all caterpillars are edge magic and it was conjectured that all trees are edge-magic. The authors noted that the cycles C_n, $3 \leq n \leq 7$, are edge-magic, and asked whether all cycles C_n are edge-magic. In [3] it was proved that all odd cycles are edge-magic. Some examples of even edge-magic cycles were introduced, as well.

We solve this problem completely for cycles and prove:

Theorem 1 *All cycles C_n, $n \geq 3$, are edge-magic.*

2 Proof of Theorem 1

In case $G = C_n$ is the cycle on $n \geq 3$ vertices, the label set is $S = \{1, 2, 3, ..., 2n\}$, and Eq. 3 simplifies to the following equation:

$$val(L) = \frac{2\sum_{i=1}^{n} \ell(v_i) + \sum_{i=1}^{n} \ell(e_i)}{n} \tag{4}$$

We prove Theorem 1 by presenting an explicit magic labeling for each cycle C_n. The labeling differs for an even and odd cycle, and for the even case - it differs for $n \equiv 0(\text{mod}4)$ and $n \equiv 2(\text{mod}4)$.

2.1 Odd Cycles

The following magic labeling for cycles of odd length n appeared in [3][1]. The labeling presented, is such that:

$$L(V) = \{1, 2, ..., n\}, \quad L(E) = \{n+1, n+2, ..., 2n\}.$$

By Eq. 4, the valence should be:

$$val(L) = \frac{5}{2}n + \frac{3}{2}$$

Therefore, by Eq. 2, any magic labeling L should satisfy:

$$Sum(L(V)) = \left\{\frac{n+3}{2}, \frac{n+5}{2}, \frac{n+7}{2}..., \frac{3n+1}{2}\right\}$$

We now present the magic labeling L. Start at any vertex and label the vertices in a consecutive cyclic way as follows:

$$1 \to \frac{n+3}{2} \to 2 \to \frac{n+5}{2} \to 3 \to \frac{n+7}{2} \to ... \to \frac{n-1}{2} \to n \to \frac{n+1}{2} \to 1.$$

This labeling can also be represented by the following simple recursion:

$$\begin{aligned} \ell(v_1) &= 1, \\ \ell(v_{i+1}) &= 1 + \left(\ell(v_i) + \tfrac{n-1}{2}\right) \bmod n, \quad 1 \leq i \leq n. \end{aligned} \tag{5}$$

It is easy to verify that $Sum(L(V))$ is as claimed. Thus, L is a magic labeling.

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[1]In fact, the dual labeling which will be presented shortly is given there.

Note that the dual labeling in which a vertex that was labeled i will now be labeled $i+n$, and an edge that was labeled i will be labeled by $i-n$, is also magic. In this case $L(V) = \{n+1, n+2, ..., 2n\}$ and $L(E) = \{1, 2, ..., n\}$.

Another interesting fact about the above labeling is that if we rotate all labels by one position, i.e. the edges will receive the labels of the vertices and vice versa, then the resulting labeling is *edge anti-magic*, that is all edge valencies are distinct. In fact in [2] it was proved that all graphs are edge anti-magic.

2.2 Even Cycles

First note that in this case there is no magic labeling L for which $L(V) = \{1, 2, ..., n\}$, because then by Eq. 4, the valence would be:

$$val(L) = \frac{5}{2}n + \frac{3}{2},$$

and this is not an integer when n is even.

Since this is the lowest valence possible for a cycle, we must increase the label of at least one vertex, so that the resulting valence is an integer. In general by Eq. 4, we must increase the sum $\sum_{i=1}^{n} L(v_i)$ by $\frac{n}{2} + kn$ for some $k \geq 0$. In this case the valence will be

$$val(L) = \frac{5}{2}n + k + 2.$$

We present an explicit labeling for even n in which $k = 0$, that is $val(L) = \frac{5}{2}n + 2$. The vertices will be labeled with the set of labels $L(V) = \{1, 2, 3, ..., n-1, \frac{3}{2}n\}$. Thus,

$$Sum(L(V)) = \left\{\frac{n}{2} + 2, ..., n, n+1, n+3, n+4, ..., \frac{3}{2}n, \frac{3}{2}n + 1, \frac{3}{2}n + 2\right\}$$

We show a different labeling for cycles of length $n \equiv 0(\text{mod}4)$ and $n \equiv 2(\text{mod}4)$.

A labeling for $n \equiv 0(\text{mod}4)$

Start at any vertex and label the vertices in a consecutive cyclic way as follows:

- $1 \to \frac{3}{2}n \to 2$.
- $2 \to \frac{n}{2} \to 3 \to \frac{n}{2}+1 \to 4 \to ... \to \frac{n}{4}-1 \to \frac{3}{4}n-3 \to \frac{n}{4} \to \frac{3}{4}n-2 \to \frac{n}{4}+1$
- $\frac{n}{4}+1 \to \frac{3}{4}n \to \frac{3}{4}n-1 \to \frac{3}{4}n+1$
- $\frac{3}{4}n+1 \to \frac{n}{4}+2 \to \frac{3}{4}n+2 \to \frac{n}{4}+3 \to ... \to n-3 \to \frac{n}{2}-2 \to n-2 \to \frac{n}{2}-1 \to n-1$
- $n-1 \to 1$.

A labeling for $n \equiv 2(\text{mod}4)$

Start at any vertex and label the vertices in a consecutive cyclic way as follows:

- $1 \to \frac{3}{2}n \to 2$.
- $2 \to \frac{n}{2} \to 3 \to \frac{n}{2}+1 \to 4 \to ... \to \frac{n+2}{4}-1 \to \frac{3n+2}{4}-3 \to \frac{n+2}{4} \to \frac{3n+2}{4}-2$
- $\frac{3n+2}{4}-2 \to \frac{3n+2}{4} \to \frac{3n+2}{4}-1 \to \frac{n+2}{4}+1$
- $\frac{n+2}{4}+1 \to \frac{3n+2}{4}+1 \to \frac{n+2}{4}+2 \to \frac{3n+2}{4}+2 \to ... \to \frac{n}{2}-2n-2 \to \frac{n}{2}-1 \to n-1$
- $n-1 \to 1$.

It is easy to verify in both cases that $Sum(L(V))$ is as claimed, and therefore both labelings are magic. See for example Figure 1.

3 Extensions

There are many more magic labelings for cycles, and it seems that the number of solutions grows exponentially with n.

For example here is another magic labeling for a cycle of even length n, where $n \equiv 2(\text{mod}4)$. In this case $L(V) = \{1, 2, ..., \frac{n}{2}, \frac{n}{2}+2, ..., n, n+1\}$, and $Sum(L(V)) = \{2n+1, \frac{n}{2}+2, \frac{n}{2}+3, ..., \frac{3}{2}n\}$. Again, start at any vertex and label the vertices in a consecutive cyclic way as follows:

- $1 \to \frac{n}{2}+2 \to 2 \to \frac{n}{2}+3 \to 3 \to \frac{n}{2}+4 \to \dots \to \frac{n-2}{4} \to \frac{3n+2}{4} \to \frac{n+2}{4}$
- $\frac{n+2}{4} \to \frac{n+2}{4}+1.$
- $\frac{n+2}{4}+1 \to \frac{3n+2}{4}+1 \to \frac{n+2}{4}+2 \to \frac{3n+2}{4}+2 \to \dots \to \frac{n}{2}-1 \to n-1 \to \frac{n}{2} \to n$
- $n \to n+1 \to 1.$

As claimed above, it seems that the number of magic labelings of C_n grows exponentially with n. It would be interesting to find the number of magic labelings for C_n, and to characterize these solutions.

Some magic labelings for C_n can be transformed easily into a magic labeling of a path on n vertices. For example, if the label $2n$ appears on an edge, simply delete this edge and its label to get a magic labeling of the resulting path. If the label 1 appears on an edge, then we can delete this edge and its label, and decrease all other labels by 1. The resulting labeling is again a magic labeling of the resulting path. If both labels 1 and $2n$ appear on vertices, it is not clear whether it is possible to transform a magic labeling of C_n into a magic labeling of the path.

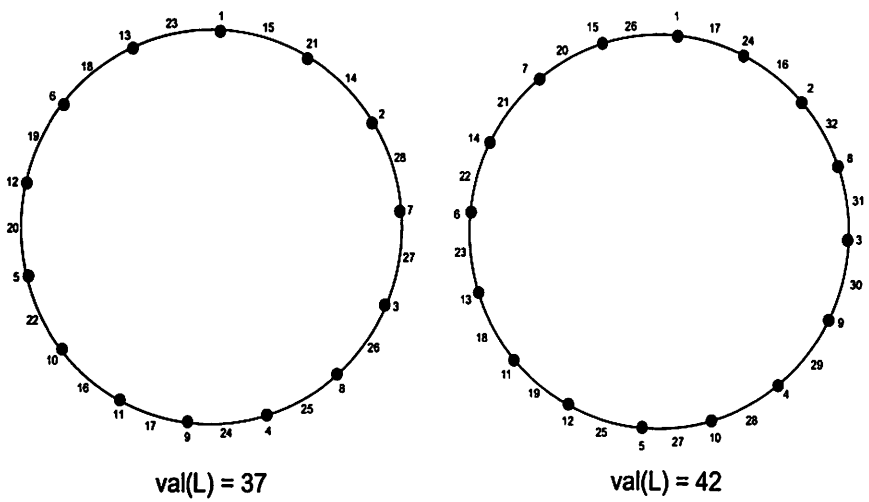

Figure 1: A magic labeling of C_{14} and C_{16}.

References

[1] N. Hartsfield and G. Ringel, *Pearls in Graph Theory,* Academic Press, INC, Boston, 1994 pp. 108–109.

[2] G. Ringel and A.S. Llado, *Another Tree Conjecture.* Bulletin of the ICA, 18, pp. 83–85, 1996.

[3] Y. Roditty and T. Bachar, *A Note on Edge-Magic Cycles.* Bulletin of ICA (to appear).