The Use of Skolem Sequences to Generate Perfect One-Factorizations

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Abstract

We present several new non-isomorphic one-factorizations of K_{36} and K_{40} which were found through hill-climbing and testing Skolem sequences. We also give a brief comparison of the effectiveness of hill-climbing versus exhaustive search for perfect one-factorizations of K_{2n} for small values of $2n$.

1. Introduction

A *one-factor* in a complete graph K_{2n} is a set of edges in which every vertex appears exactly once. A *one-factorization* of K_{2n} is a partition of the edge-set of K_{2n} into $2n-1$ one-factors. A *perfect* one-factorization (P1F) is a one-factorization in which every pair of distinct one-factors forms a Hamiltonian cycle of K_{2n}. P1Fs of K_{2n} are known to exist when $2n-1$ or n is prime, and for $2n \in \{16, 28, 36, 40, 50, 126, 170, 244, 344, 730, 1332, 1370, 1850, 2198, 3126, 6860, 12168, 16808, 29792\}$. It has been conjectured that a perfect one-factorization of K_{2n} exists for all $n \geq 2$ [2]. The known results strongly suggest that P1Fs are difficult to construct.

For more details about P1Fs the reader is referred to [13, 7, 2].

Most of the known P1Fs are constructed through starters or even starters. A *starter* in $\mathbb{Z}_{2n+1}$ is a set $S = \{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_n, y_n\}\}$ such that:

(1) $x_1, y_1, \ldots, x_n, y_n$, are all the non-zero elements in $\mathbb{Z}_{2n+1}$

(2) $\pm(x_1 - y_1), \ldots, \pm(x_n - y_n)$, are all the non-zero elements in $\mathbb{Z}_{2n+1}$.

Let $S^* = S \cup \{0, \infty\}$ and define $\infty + z = z + \infty = \infty$ for all $z \in \mathbb{Z}_{2n+1}$. Then it is easy to see that $F = \{S^* + z : z \in \mathbb{Z}_{2n+1}\}$ is a one-factorization in K_{2n+2}.

For example, the starter $\{\{14,15\}$, $\{5,7\}$, $\{19,22\}$, $\{28,32\}$, $\{25,30\}$, $\{11,17\}$, $\{6,13\}$, $\{18,26\}$, $\{29,3\}$, $\{34,9\}$, $\{20,31\}$, $\{33,10\}$, $\{23,1\}$,

$\{2,16\}$, $\{12,27\}$, $\{8,24\}$, $\{4,21\}\}$ generates a one-factorization in K_{36}. Moreover, as demonstrated by Seah and Stinson [8], it generates a perfect one-factorization.

An *even starter* in $\mathbb{Z}_{2n}$ is a set $E = \{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_{n-1}, y_{n-1}\}\}$ such that:

(1) every non-zero element of $\mathbb{Z}_{2n}$ except one, denoted m, occurs as an element in E,

(2) every non-zero element of $\mathbb{Z}_{2n}$ except n occurs as a difference of some pair of E.

Let $E^* = E \cup \{\{0, \infty_1\}, \{m, \infty_2\}\}$, and define $z + \infty_i = \infty_i + z = \infty_i$, for all $z \in \mathbb{Z}_{2n}, i = 1, 2$. Let $Q^* = \{\{z, z+n\} : z \in \mathbb{Z}_{2n}\} \cup \{\{\infty_1, \infty_2\}\}$. It is easy to see that $F = \{E^* + z : z \in \mathbb{Z}_{2n}\} \cup \{Q^*\}$ is a one-factorization of K_{2n+2} (moreover, the structure of F is such that it is known as a *rotational* one-factorization [13]).

In 1957, Th. Skolem [11], when studying Steiner triple systems, considered the possibility of distributing the numbers $1, 2, \ldots, 2n$ in n pairs (a_r, b_r) such that $b_r - a_r = r$ for $r = 1, 2, \ldots, n$. For example, for $n = 4$, the pairs (1,2), (5,7), (3,6), and (4,8) form such a partition of the numbers $1, 2, \ldots, 8$. Later, this partition was written as a sequence, for which the previous partition would be written as (1,1,3,4,2,3,2,4), which is now known as a Skolem sequence of order 4.

Formally, a *Skolem sequence of order n* is a sequence $S = (S_1, S_2, \ldots, S_{2n})$ of $2n$ integers that satisfy the following conditions:

(1) for every $k \in \{1, 2, \ldots, n\}$ there exist exactly two elements S_i, S_j such that $S_i = S_j = k$,

(2) if $S_i = S_j = k$ and $i < j$, then $j - i = k$.

An *extended Skolem sequence* of order n is a sequence $ES = (S_1, S_2, \ldots, S_{2n+1})$ of $2n+1$ integers that satisfy conditions (1), (2), and:

(3) there is exactly one $i \in \{1, \ldots, 2n+1\}$ for which $S_i = 0$.

$S_i = 0$ is also known as the hook $(*)$ of the sequence, and if $S_{2n} = 0$, then the sequence is called *hooked Skolem sequence.* It has been shown that the necessary conditions for the existence of (hooked) (extended) Skolem sequences are sufficient.

Theorem 1 *[Skolem [11]] A Skolem sequence of order n exists if and only if $n \equiv 0$ or 1 (mod 4).*

[O'Keefe [5]] A hooked Skolem sequence of order n exists if and only if $n \equiv 2$ or 3 (mod 4).

[Abrham & Kotzig [1]] An extended Skolem sequence of order n exists for all n.

A Skolem sequence $(S_1, S_2, \ldots, S_{2n})$ of order n can be used to construct a starter in $\mathbb{Z}_{2n+1}$, and hence a one-factorization in K_{2n+2}. In particular, we obtain the starter set $S = \{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_n, y_n\}\}$ where $S_{x_i} = S_{y_i} = i$ for each $i = 1, 2, \ldots, n$.

For example, the Skolem sequence of order 8; (1,1,3,7,8,3,2,6,2,5,7,4,8,6, 5,4) gives rise to the starter $S = \{x_i, y_i\}$, $i = 1, \ldots, 8$, $\{\{1,2\}, \{7,9\}, \{3,6\}, \{12,16\}, \{10,15\}, \{8,14\}, \{4,11\}, \{5,13\}\}$, which also induces a perfect one-factorization of K_{18}. Further, the triples $\{0, i, y_i+n\}$ (or $\{0, x_i+n, y_i+n\}$), $i = 1, \ldots, 8$, give the base blocks, $\{0,1,10\}, \{0,2,17\}, \{0,3,14\}, \{0,4,24\}, \{0,5,23\}, \{0,6,22\}, \{0,7,19\}, \{0,8,21\}$ (mod 49) of a cyclic STS(49).

Similarly, any extended Skolem sequence $(S_1, S_2, \ldots, S_{2n+1})$ of order n can be used to construct an even starter in $\mathbb{Z}_{2n+2}$, and hence a one-factorization in K_{2n+4}. Specifically, the even starter is obtained from the set $E = \{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_n, y_n\}\}$ where $S_{x_i} = S_{y_i} = i$ for each $i = 1, 2, \ldots, n$.

For more details about (extended) Skolem sequences the reader is referred to [10].

In the following sections we present several P1Fs of K_{36} and K_{40} which were induced from Skolem sequences and extended Skolem sequences. Moreover, we make a statistical comparison of the effectiveness of hill-climbing versus exhaustive search to find P1Fs. We also consider the likelihood of finding a P1F of K_{2n} by means of (even) starters which are induced from (extended) Skolem sequences versus (even) starters which are not induced from (extended) Skolem sequences.

2. Methodology and Results

We implemented a hill-climbing heuristic, as described in [3], to search for Skolem sequences of order 17. For each sequence which was constructed, we then generated a one-factorization for K_{36} and then determined whether the one-factorization was perfect. Below we present eight Skolem sequences which give rise to P1Fs for K_{36}:

(17,12,8,15,10,6,9,14,11,3,8,6,3,12,10,9,16,17,15,11,13,14,2,7,2,4,5,1,1,4,7,5,16,13)
(15,3,9,14,3,16,12,17,4,1,1,9,4,13,2,15,2,14,12,10,8,16,11,7,17,6,13,5,8,10,7,6,5,11)
(10,14,9,13,17,8,6,16,12,5,10,9,6,8,5,14,13,15,7,11,12,17,2,16,2,7,1,1,3,4,11,3,15,4)
(16,4,7,17,13,4,1,1,6,7,9,14,10,15,6,11,16,13,5,9,17,12,10,5,8,14,11,3,15,2,3,2,8,12)
(11,8,1,1,14,2,6,2,15,8,3,11,6,3,16,10,17,12,14,13,7,5,9,15,4,10,5,7,4,12,16,9,13,17)
(4,13,5,7,4,8,12,5,15,3,7,17,3,8,13,10,16,9,12,14,11,1,1,15,6,10,9,2,17,2,6,11,16,14)
(7,12,15,17,11,3,13,7,3,5,8,10,14,12,5,11,16,15,8,13,17,10,9,1,1,4,14,6,2,4,2,9,16,6)
(11,3,13,17,3,8,2,5,2,14,7,11,5,8,12,13,16,7,15,10,17,9,6,14,1,1,12,4,6,10,9,4,16,15)

These eight P1Fs for K_{36} were tested by computer and found to be mutually non-isomorphic. Additionally, they are each non-isomorphic to the P1F published by Seah and Stinson [8].

We also implemented a hill-climbing heuristic to search for extended Skolem sequences of orders 16 and 18; such sequences naturally give rise to rotational one-factorizations in K_{36} and K_{40}, respectively. The following two extended Skolem sequences were found to produce P1Fs for K_{36}:

(14,9,1,1,4,16,10,2,4,2,9,13,7,12,14,8,10,15,11,7,6,16,5,8,13,12,6,5,3,11,0,3,15)
(13,8,4,14,3,9,4,3,15,8,12,7,16,13,9,10,11,14,7,2,6,2,12,15,5,10,6,11,16,5,1,1,0)

When tested, these two P1Fs were found to be non-isomorphic to each other, as well as non-isomorphic to the P1F for K_{36} which was published by Kobayashi et al [4].

We found three extended Skolem sequences of order 18 which produced P1Fs for K_{40}:

(4,15,10,17,4,2,16,2,3,18,8,3,10,12,9,11,15,14,8,13,17,7,16,9,6,12,11,18,7,5,6,14,13,0,5,1,1)
(3,1,1,3,8,9,2,18,2,7,10,17,8,14,9,15,7,6,16,13,10,5,11,6,12,18,5,14,17,0,15,4,13,11,16,4,12)
(17,12,18,0,10,7,2,16,2,6,9,15,7,12,10,6,11,17,13,9,18,14,8,16,1,1,15,11,4,5,8,13,4,3,5,14,3)

The three corresponding P1Fs are mutually non-isomorphic and are also non-isomorphic to that of Seah and Stinson [9].

Regarding the isomorphism tests, it should be noted that this was done by comparing the in-degree sequences of the trains of the P1Fs (see [13] for details on trains). All of the in-degree sequences produced by the P1Fs tested were distinct.

The results in this section improve on the known results [2, 4, 8, 9] and are summarized in the following theorem.

Theorem 2 *Let $NP(2n)$ denote the number of pairwise non-isomorphic P1Fs of K_{2n}. Then $NP(36) \geq 12$ and $NP(40) \geq 4$.*

3. Analysis

In Table 1, we enumerate the number of Skolem sequences generated from our hill-climbing heuristic and the number of corresponding P1Fs. Likewise, we present the total number of distinct Skolem sequences of each order and the number of corresponding P1Fs. These data allow us to compare the exact probability of finding a P1F from a starter generated from a Skolem sequence, as well as an estimated probability from the hill-climbing data.

Moreover, a least squares analysis of this data suggests that $S(n)$, the number of Skolem sequences that would have to be tested before finding a P1F of K_n, is $S(n) \approx 10^{0.3252n-4.0094}$, based on the hill-climbing data.

Graph	Hill-Climbing No. of SS	No. of P1F	Estimated Probability	Exhaustive Enumeration No. of SS	No. of P1F	Exact Probability
K_{18}	1 825 832	11 981	$0.6562 \cdot 10^{-2}$	504	6	$0.1905 \cdot 10^{-1}$
K_{20}	1 388 635	10 329	$0.7438 \cdot 10^{-2}$	2 656	12	$0.4518 \cdot 10^{-2}$
K_{26}	182 456 680	7 873	$0.4315 \cdot 10^{-4}$	455 936	22	$0.4825 \cdot 10^{-4}$
K_{28}	141 560 480	1 003	$0.7085 \cdot 10^{-5}$	3 040 560	18	$0.5920 \cdot 10^{-5}$
K_{34}	701 709 022	56	$0.7981 \cdot 10^{-7}$	1 400 156 768	122	$0.8713 \cdot 10^{-7}$

Table 1: Probabilities for finding P1Fs from Skolem Sequences

Graph	Total No. of Starters	Total No. of P1F	No. of non-SS starters	No. of P1F from non-SS starters	Exact Prob. of P1F from non-SS starter	Exact Prob. of P1F from SS starter
K_{18}	3 857	17	3 353	11	$0.3281 \cdot 10^{-2}$	$0.1905 \cdot 10^{-1}$
K_{20}	25 905	65	23 249	53	$0.2280 \cdot 10^{-2}$	$0.4518 \cdot 10^{-2}$
K_{26}	13 376 125	460	12 920 189	438	$0.3390 \cdot 10^{-4}$	$0.4825 \cdot 10^{-4}$
K_{28}	128 102 625	900	125 062 065	882	$0.7052 \cdot 10^{-5}$	$0.5920 \cdot 10^{-5}$

Table 2: Probabilities based on non-SS-induced and SS-induced starters

Based on the exhaustive data, the approximation is $S(n) \approx 10^{0.3287n-4.1083}$. For K_{52}, our data suggests that about $10^{12.90}$ or $10^{12.98}$ Skolem sequences will have to be tested before finding a P1F, based on the hill-climbing and exhaustive data, respectively.

In Table 2, we show the exact probability of finding a P1F from starters that are not induced by Skolem sequences as well as the exact probability for starters that are induced by Skolem sequences. From a comparison of these data, it seems to be more efficient to search for a P1F by using starters induced by Skolem sequences than by using general starters.

Similar to Table 1, in Table 3 we present probability information for finding P1Fs from even starters that are generated from extended Skolem sequences. A least squares analysis of this data suggests that $E(n)$, the number of extended Skolem sequences that would have to be tested before finding a P1F of K_n, is $E(n) \approx 10^{0.2940n-3.3707}$, based on the hill-climbing data. Based on the exhaustive data, the approximation is $E(n) \approx 10^{0.3201n-3.9783}$.

For K_{40}, this suggests that we would need to test approximately $10^{8.39}$ or $10^{8.83}$ extended Skolem sequences, based on the hill-climbing or exhaustive data, respectively. For K_{52}, our data suggests that about $10^{11.92}$ or $10^{12.66}$, respectively, extended Skolem sequences will have to be tested before finding a P1F.

In Table 4, we compare the probability of finding a P1F based on even starters which are not induced by extended Skolem sequences versus those

Graph	Hill-Climbing: No. of ESS	Hill-Climbing: No. of P1F	Hill-Climbing: Estimated Probability	Exhaustive Enumeration: No. of ESS	Exhaustive Enumeration: No. of P1F	Exhaustive Enumeration: Exact Probability
K_{18}	813 153	24 318	$0.2991 \cdot 10^{-1}$	636	20	$0.3145 \cdot 10^{-1}$
K_{20}	20 576 354	16 893	$0.8210 \cdot 10^{-3}$	3 556	4	$0.1125 \cdot 10^{-2}$
K_{22}	16 303 799	12 899	$0.7912 \cdot 10^{-3}$	19 488	24	$0.1232 \cdot 10^{-2}$
K_{24}	25 057 057	14 329	$0.5719 \cdot 10^{-3}$	95 872	26	$0.2712 \cdot 10^{-3}$
K_{26}	65 243 860	1 540	$0.2360 \cdot 10^{-4}$	594 320	18	$0.3029 \cdot 10^{-4}$
K_{28}	55 331 468	1 126	$0.2035 \cdot 10^{-4}$	4 459 888	64	$0.1435 \cdot 10^{-4}$
K_{30}	52 103 606	143	$0.2745 \cdot 10^{-5}$	32 131 648	94	$0.2925 \cdot 10^{-5}$
K_{32}	131 180 634	71	$0.5412 \cdot 10^{-6}$	227 072 544	92	$0.4052 \cdot 10^{-6}$
K_{34}	2 330 980	1	$0.4290 \cdot 10^{-6}$	1 875 064 880	240	$0.1280 \cdot 10^{-6}$

Table 3: Probabilities for finding P1Fs from Extended Skolem Sequences

Graph	Total No. of Starters	Total No. of P1F	No. of non-ESS starters	No. of P1F from non-ESS starters	Exact Prob. of P1F from non-ESS starter	Exact Prob of P1F fron ESS starte
K_{18}	5 760	80	5 124	60	$0.1171 \cdot 10^{-1}$	$0.3145 \cdot 10^{-}$
K_{20}	42 816	120	39 260	116	$0.2955 \cdot 10^{-2}$	$0.1125 \cdot 10^{-}$
K_{22}	320 512	272	301 024	248	$0.8239 \cdot 10^{-3}$	$0.1232 \cdot 10^{-}$
K_{24}	2 366 080	440	2 270 208	414	$0.1824 \cdot 10^{-3}$	$0.2712 \cdot 10^{-}$
K_{26}	20 857 088	576	20 262 768	558	$0.2754 \cdot 10^{-4}$	$0.3029 \cdot 10^{-}$
K_{28}	216 731 392	2 016	212 271 504	1 952	$0.9196 \cdot 10^{-5}$	$0.1435 \cdot 10^{-}$

Table 4: Probabilities based on non-ESS-induced and ESS-induced even starters

which are. Again, we find that there tends to be a higher probability when using even starters that are induced by extended Skolem sequences.

4. Conclusions and Questions

It is clear that we were successful in generating numerous P1Fs from (extended) Skolem sequences. The data presented also suggests that it is generally better to use Skolem starters than using non-Skolem starters. Additionally, Theorem 2, Table 1, and Table 3 update information published in Theorem VI.4.45, Table IV.43.20, and Table IV.43.23, respectively, of the Handbook of Combinatorial Designs [2, 10].

Regarding questions which arise, it is natural to ask if the properties and direct (or recursive) constructions of Skolem sequences can be applied to generate further P1Fs. This is not clear to us yet.

An extended Skolem sequence of order n with $S_{n+1} = 0$ (known also as a Rosa sequence) can be used to construct a cyclic STS$(6n+3)$ [6]. There are no known examples of Rosa sequences that induce P1Fs, and so we ask whether there exist any Rosa sequences which generate P1Fs.

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