A Note on the Toughness of Certain Graphs

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ABSTRACT. The toughness $t(G)$ of a noncomplete graph G is defined as

$$t(G) = \min\{|S|/\omega(G-S) \mid S \subset V(G), \omega(G-S) \geq 2\},$$

where $\omega(G-S)$ is the number of components of $G-S$. We also define $t(K_n) = +\infty$ for every n.

In this article, we discuss the toughness of the endline graph of a graph and the middle graph of a graph.

1 Introduction

In this article, all graphs are finite, undirected, without loops or multiple edges. The toughness of a graph is an invariant first introduced by Chvátal [1]. He observed some relationships between this parameter and the existence of hamiltonian cycles or k-factors. The toughness is an interesting invariant in graph theory.

Let G be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges, respectively.

We denote the order of G by $|G|$ and the number of connected components of G by $\omega(G)$. If S is a subset of G with $\omega(G-S) \geq 2$, we call it a cutset of G. If $S \subset V(G)$, $\langle S \rangle$ is the subgraph of G induced by S. We write $G-S$ for $\langle V(G)-S \rangle$. Terms not defined here can be found in [2].

A graph G is t-tough if the implication

$$\omega(G-S) > 1 \rightarrow |S| \geq t \cdot \omega(G-S)$$

holds for any $S \subset V(G)$.

A complete graph is t-tough for any real number t. If G is not complete, there exists the largest t such that G is t-tough. This number is denoted by

$t(G)$ and is called the toughness of G. We define $t(K_n) = +\infty$ for every n. If G is not complete, $t(G) = \min\{|S|/\omega(G-S) \mid S \subset V(G), \omega(G-S) \geq 2\}$.

In this article, we study the toughness of the endline graph of a graph and the middle graph of a graph.

2 Results

We first give the definition of the endline graph of a graph. Let G be a graph and $V(G) = \{v_1, v_2, \ldots, v_n\}$. We add to G n new vertices and n edges $\{u_i, v_i\}$ $(i = 1, 2, \ldots, n\}$, where u_i are different from any vertex of G and from each other. Then we obtain a new graph G^+ with $2n$ vertices, called the endline graph of G.

Theorem 1. *Let G be a graph with at least two vertices, then*

$$t(G^+) = \begin{cases} t(G)/(1+t(G)) & \text{if } 0 \leq t(G) \leq 1 \\ 1/2 & \text{if } 1 < t(G). \end{cases}$$

In order to prove Theorem 1, we need the following two lemmas.

Lemma 1. *Let G be noncomplete and S be a cutset of G minimizing $|S|/\omega(G-S)$, further let U be a cutset of G and let us set $|U| = u$, $\omega(G-U) = m$ and $t(G) = t$. Then $t/(1+t) \leq u/(u+m)$.*

Proof: Let us set $|S| = s$ and $\omega(G-S) = k$. From the minimality of S, we easily check that $s/(s+k) \leq u/(u+m)$, which implies $t/(1+t) \leq u/(u+m)$ since $s/k = t$. □

Lemma 2. *Let G be noncomplete, then $t(G^+) \leq t(G)/(1+t(G))$.*

Proof: Let S be a cutset such that $t(G) = |S|/\omega(G-S)$. Then S would be a cutset of G^+. Hence from the definition of $t(G^+)$, we have $t(G^+) \leq s/(s+k)$, where $|S| = s$ and $\omega(G-S) = k$. This implies the result. □

Proof of Theorem 1: If G is not connected, there is nothing to show. Next let G be a complete graph K_n and S be a cutset of the endline graph K_n^+. Then since K_n has not a cutset, $\omega(G-S) = s+1$, where $|S| = s$. Hence

$$t(K_n^+) = \min\{|S|/\omega(G-S)\} = \min\{s/(s+1) \mid s \geq 1\} = 1/2.$$

Therefore we may assume G is connected and noncomplete. Let U be a cutset of G^+ such that

$$t(G^+) = \min\{|U|/\omega(G^+ - U)\}.$$

We here distinguish two cases.

Case 1. U is a cutset of G.

Let us set $|U| = u$ and $\omega(G-U) = k$. Then we have $\omega(G^+ - U) = k+u$. From the minimality of U, we have

$$t(G^+) = u/(k+u). \tag{1}$$

On the other hand, from Lemma 1,

$$t(G)/(1+t(G)) \leq u/(k+u). \tag{2}$$

Combining (1) with (2), we have $t(G)/(1+t(G)) \leq t(G^+)$.

By the way, from Lemma 2, $t(G^+) \leq t(G)/(1+t(G))$.

Hence, we obtain $t(G^+) = t(G)/(1+t(G))$.

Case 2. U is not a cutset of G.

Let us set $|U| = u$. Then $\omega(G^+ - U) = u+1$. Hence we have

$$t(G^+) = \min\{|U|/\omega(G^+ - U)\} = \min\{u/(u+1) \mid u \geq 1\} = 1/2.$$

Therefore, from case 1 and case 2, we obtain

$$t(G^+) = \min\{t(G)/(1+t(G)), 1/2\}.$$

This completes the proof. □

Let us denote the vertex-connectivity of a graph G by $\kappa(G)$ and the vertex-independence number of a graph G by $\beta(G)$ respectively. Then we have, $t(G) \geq \kappa(G)/\beta(G)$, which is proved by Chvátal [1]. Therefore we immediately have the following:

Corollary 1. *If* $|G| \geq 2$ *and* $\kappa(G) > \beta(G)$*, then* $t(G^+) = 1/2$.

Using the theorem 1, we can construct the family $\{G_n\}$ such that $t(G_n) = 1/(n+1)$ $(n = 1, 2, \ldots)$.

In fact, let us set that $G_1 = P_3$, $G_n = G_{n-1}$ $(n \geq 1)$, where P_n is a path with order n. Then, from Theorem 1, we obtain the following recurrence formula:

$$t_1 = 1/2, t_{n+1} = t_n/(1+t_n)(n \geq 1),$$

where $t_n = t(G_n)$.

Hence, we have $t(G_n) = 1/(n+1)$.

Finally we shall give a bound of the toughness of the middle graph of a graph. The middle graph $M(G)$ of a graph G is the graph obtained from G by inserting a new vertex into every edge of G and by joining by edges those pairs of these new vertices which lie on adjacent edges of G.

Let us denote the line graph of a graph G by $L(G)$. Then, from the definition of the endline graph and the middle graph of a graph G, we have, $L(G^+) = M(G)$, which is proved in [3].

Let us denote the edge-connectivity of graph G by $\lambda(G)$ and let G be a connected and noncomplete graph, then it is already known [1] that $\lambda(G)/2 \leq t(L(G))$. Hence we can obtain the following result.

Theorem 2. *Let G be a connected and a noncomplete graph, then*

$$1/2 \leq t(M(G)) \leq \lambda(G)/2.$$

Proof: Since $L(G^{+}) = M(G)$ and $\lambda(G^{+}) = 1$, the inequality on the left side is clear. Hence we may prove only the inequality on the right side. From now on we denote $\lambda(G)$ by λ.

As λ is the edge-connectivity of graph G, there exists an edge-set $F \subset E(G)$ such that $|F| = \lambda$ and $G - F$ is disconnected.

Now, let $F = \{e_1, e_2, \ldots, e_\lambda\}$ and let v_i be a new vertex inserting into an edge e_i of G. Then $S = \{v_1, v_2, \ldots, v_\lambda\}$ would be a cutset of $M(G)$. Let us set $\omega(M(G) - S) = k$, then we have

$$t(M(G)) \leq |S|/k \leq \lambda(G)/2.$$

This completes the proof. □

Corollary 2. *If a connected graph G has a bridge, $t(M(G)) = 1/2$.*

References

[1] V. Chvátal, Tough graphs and Hamiltonian circuits, *Discrete Math.* 5 (1973), 215–228.

[2] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Second edition, Wadsworth & Brooks / Cole, Advanced Books & Software Monterey, CA (1986).

[3] T. Hamada and I. Yoshimura, Traversability and connectivity of the middle graph of a graph, *Discrete Math.* 14 (1976), 247–255.