SHORT EVEN CYCLES IN CAGES WITH ODD GIRTH

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ABSTRACT. A $(k;g)$-cage is a smallest k-regular graph with girth g. Harary and Kovacs[2] conjectured that for all $k \geq 3$ and odd $g \geq 5$, there exists a $(k;g)$-cage which contains a cycle of length $g+1$. Among other results, we prove the conjecture for all $k \geq 3$ and $g \in \{5,7\}$.

1. INTRODUCTION

We consider only simple graphs. The *odd girth* (*even girth*) of G is the length of a shortest odd (even) cycle in G. If there is no odd (even) cycle in G then the odd (even) girth of G is taken as ∞. Let $g = g(G)$ denote the smaller of the odd and even girths, and let $h = h(G)$ denote the larger. Then g is the called *girth* of G, and (g,h) is called the *girth pair* of G. A k-regular graph with girth g is called a $(k;g)$-graph; the minimum number of vertices of a $(k;g)$-graph is denoted by $f(k;g)$. A $(k;g)$-graph with $f(k;g)$ vertices is called a *(k;g)-cage.* Equivalently, a $(k;g)$-cage is a smallest k-regular graph with girth at least g (see [1]). A k-regular graph with girth pair (g,h) is called a $(k;g,h)$-graph; the minimum number of vertices of a $(k;g,h)$-graph is denoted by $f(k;g,h)$.

The girth pairs were introduced by Harary and Kovacs[2] in 1983. Several interesting questions concerning girth pairs remain open. For example, it is clear that $f(k;g,h) \geq f(k;g)$, and this inequality may be strict; the unique $(k;4)$-cage is $K_{k,k}$, which does not contain a 5-cycle, thus $f(k;4,5) > f(k;4)$. Related to this is the following conjecture by Harary and Kovacs[2].

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Conjecture. *For all $k \geq 3$, and odd $g \geq 5$, $f(k;g,g+1) = f(k;g)$.*

In other words, they conjected that for all $k \geq 3$ and odd $g \geq 5$, there exists a $(k;g)$-cage with even girth $g+1$. We prove this conjecture for all $k \geq 3$ and $g \in \{5,7\}$. We do so by obtaining, for all $k \geq 3$ and odd $g \geq 5$, an upper bound on the smallest possible even girth of a $(k;g)$-cage. Given a cycle C and two vertices x, y on it, we use $C[x,y]$ to denote the portion of C from x to y in the clockwise direction. We use $[m]$ to denote the set $\{1,\ldots,m\}$.

2. Main results

Lemma 1. *Let G be a bipartite graph with bipartion X, Y, where $|X| = |Y| = m$. If G contains more than $m^2 - m$ edges, then G contains a perfect matching.*

Proof. By Konig-Egervary's Theorem, it suffices to show that a minimum vertex cover of G contains at least m vertices. Since $\Delta(G) \leq m$, each vertex can cover at most m edges. In order to cover more than $m^2 - m = m(m-1)$ edges, at least m vertices are needed. Hence a minimum vertex cover contains at least m vertices. □

The following lemma is interesting in its own right.

Lemma 2. *Let G be a $(k;g)$-cage, where $k \geq 3$ and $g \geq 5$. Then every edge of G is contained in at least $k-1$ cycles of length at most $g+1$.*

Proof. Suppose there exists an edge uv which belongs to fewer than $k-1$ cycles of length at most $g+1$ in G. Suppose $N(u) - v = \{u_1,\ldots,u_{k-1}\}$ and $N(v) - u = \{v_1,\ldots,v_{k-1}\}$. Since G has girth $g \geq 5$, $N(u) \cap N(v) = \emptyset$. Now construct a bipartite graph H with bipartition $X = \{x_1,\ldots,x_{k-1}\}$ and $Y = \{y_1,\ldots,y_{k-1}\}$ such that $x_i y_j \in E(H)$ if and only if path $u_i u v v_j$ does not belong to a cycle of length at most $g+1$ in G. Since uv belongs to fewer than $k-1$ cycles of length at most $g+1$ in G, H has more than $(k-1)^2 - (k-1)$ edges. By Lemma 1 H contains a perfect matching. Without loss of generality, suppose $\{x_1y_1,\ldots,x_{k-1}y_{k-1}\}$ is a perfect matching in H. By our definition of H, for each $i \in [k-1]$, $u_i u v v_i$ does not belong to a cycle of length at most $g+1$ in G; in other words every cycle in G containing $u_i u v v_i$ has length at least $g+2$. Now let G'' be the graph obtained from G by deleting vertices u, v, and adding edges $u_i v_i$ for $i \in [k-1]$. It's easy to verify that G'' is a k-regular graph with girth at least g, contradicting the minimality of G. □

Given a path P, we use $l(P)$ to denote the number of edges in it.

Lemma 3. *Let G be a simple graph with odd girth g and even girth h. Suppose u, v are two vertices in G and P_1, P_2 are two different u, v-paths whose lengths have the same parity, then $l(P_1) + l(P_2) \geq \min(2g, h)$.*

Proof. Let F be the subgraph formed by the set of edges belonging to exactly one of P_1 and P_2. Since P_1, P_2 are different and $l(P_1), l(P_2)$ have the same parity, F is nonempty and contains an even number of edges. Furthermore, each vertex in F has degree 2 or 4. Hence F is an edge-disjoint union of cycles. If F contains an even cycle, then it contains at least h edges; if F does not contain an even cycle, then it must consist of an even number of odd cycles, in which case it contains at least $2g$ edges. Hence $l(P_1) + l(P_2) \geq e(F) \geq \min\{2g, h\}$. □

Lemma 4. *Let G be a $(k; g)$-cage, where $k \geq 3, g \geq 5$ and g is odd. If G contains a cycle of length g which shares at most one edge with every other cycle of length g in G, then there exists a $(k; g)$-cage with even girth $g + 1$.*

Proof. We may assume that G has no cycle of length $g+1$, since otherwise we are done. Let C_1 be a cycle of length g in G that shares at most one edge with every other cycle of length g. Suppose the vertices on C_1 are labeled with $v_1, \ldots, v_g$ in the clockwise direction. Consider edge v_1v_2, by Lemma 2 there exists another cycle C_2 of length g containing it. By our assumption, $E(C_1) \cap E(C_2) = v_1v_2$. Now, let G'' be the graph obtained from G by replacing edges v_1v_2, v_3v_4 with edges v_1v_3, v_2v_4. We show that G'' is a $(k; g)$-cage containing a cycle of length $g + 1$. Clearly, G'' is k-regular. To show that G'' has girth at least g, we consider an arbitrary cycle C in G''. If C avoids edges v_1v_3, v_2v_4, then it is also a cycle in G and hence has length at least g. If C uses both v_1v_3 and v_2v_4, then it consists of v_1v_3, v_2v_4 and two paths P_1, P_2 in G (and in G''), where either P_1 is a v_1, v_2-path and P_2 is a v_3, v_4-path or P_1 is a v_1, v_4-path and P_2 is a v_2, v_3-path. In the former case, P_1, P_2 each has length at least $g - 1$, since $P_1 \cup v_1v_2$ and $P_2 \cup v_3v_4$ are both cycles in G, and hence C has length at least $2(g-1) + 2 > g$. In the latter case, $C \cup \{v_1v_2, v_3v_4\} - \{v_1v_3, v_2v_4\}$ is a cycle in G having the same length as C, hence C has length at least g. It remains to consider the case where C uses exactly one of v_1v_3, v_2v_4. Without loss of generality, suppose C uses v_1v_3 but not v_2v_4. Then C consists of edge v_1v_3 and a v_1, v_3-path P, where P is a path in both G and G''. It suffices to show that P has length at least $g - 1$. Since P is a path in G'' and hence avoids v_1v_2, v_3v_4, $P \neq C_1[v_1, v_3], P \neq C_1[v_3, v_1]$. If P has length less than $g - 2$, then $P \cup C_1[v_1, v_3]$ would contain a cycle of length less than g in G; if P has length $g - 2$, then $P \cup C_1[v_1, v_3]$ either contains a cycle of length less than g in G or is itself a cycle of length g in G, which is different from C_1 and shares at least two common edges with C_1, namely v_1v_2 and v_2v_3, contradicting our assumption. Hence P has length at least $g-1$. This proves that G'' has girth at least g. On the other hand, $C_1 \cup \{v_1v_3, v_2v_4\} - \{v_1v_2, v_3v_4\}$ is a cycle of length g in G''. Since G'' also has the same number of vertices as G, G'' is a $(k;g)$-cage. Now $(C_2 - v_1v_2) \cup v_2v_3 \cup v_1v_3$ is a cycle of length $g+1$ in G''. □

Theorem 1. *For all $k \geq 3$ and odd $g \geq 5$, there exists a $(k;g)$-cage with even girth at most $g + \lfloor \frac{g}{2} \rfloor - \epsilon$, where $\epsilon = 1$ if $g \equiv 1 \pmod 4$, and $\epsilon = 2$ if $g \equiv 3 \pmod 4$.*

Proof. First notice that $g + \lfloor \frac{g}{2} \rfloor - \epsilon$, where $\epsilon = 1$ if $g \equiv 1 \pmod 4$, and $\epsilon = 2$ if $g \equiv 3 \pmod 4$, is the largest even integer strictly less than $g + \lfloor \frac{g}{2} \rfloor$. Hence it suffices to show that there exists a $(k;g)$-cage with even girth less than $g + \lfloor \frac{g}{2} \rfloor$.

Let G be a $(k;g)$-cage. We may assume that G has even girth $h(G) \geq g + \lfloor \frac{g}{2} \rfloor$; otherwise we are done. We are going to modify G to obtain a $(k;g)$-cage with even girth less than $g + \lfloor \frac{g}{2} \rfloor$. By our assumption, G contains no cycles of length $g+1$. Let C_1 be a cycle of length g. Among all other cycles of length g in G, we choose one that shares maximum number of edges with C_1, call it C_2. Suppose C_1, C_2 share t edges, by Lemma 4 we may assume that $t \geq 2$. It is easy to see that those common edges must be consecutive on both C_1 and C_2, otherwise we would get a cycle of length less than g. Suppose the vertices on C_1 are labeled with $v_1, \ldots, v_g$ in the clockwise direction such that the common segment that C_1 and C_2 share is $C_1[v_1, v_{t+1}]$. Since $(C_1 - C_2) \cup (C_2 - C_1)$ is an even cycle of length $2g - 2t \geq h(G)$, we have $2t \leq 2g - h(G) \leq g - 2$ and hence $t \leq \lfloor \frac{g}{2} \rfloor - 1$.

Let m be the even one of $\lfloor \frac{g}{2} \rfloor$ and $\lceil \frac{g}{2} \rceil$. Let G'' be the graph obtained from G by deleting edges $v_1v_2, v_{m+1}v_{m+2}$ and adding edges v_1v_{m+1}, v_2v_{m+2}. Clearly, G'' is k-regular. We show that G'' has girth at least g. Let C be a cycle in G''. If C uses neither or both of the new edges v_1v_{m+1}, v_2v_{m+2}, then we argue as in Lemma 4. It hence remains to consider the case where C contains exactly one of the new edges. Without loss of generality, suppose C contains v_1v_{m+1} but not v_2v_{m+2}. Then C consists of edge v_1v_{m+1} and a v_1, v_{m+1}-path P, where P is a path in both G and G''. It suffices to show that P has length at least $g-1$. Let $P_1 = C_1[v_1, v_{m+1}]$ and $P_2 = C_1[v_{m+1}, v_1]$; one of them has length $\lfloor \frac{g}{2} \rfloor$ and the other has length $\lceil \frac{g}{2} \rceil$. Since P does not use edges $v_1v_2, v_{m+1}v_{m+2}$, $P \neq P_1, P \neq P_2$. Depending the parity of the length of P, we can apply Lemma 3 to either

P, P_1 or P, P_2 to get either $l(P)+l(P_1) \geq \min\{2g, h(G)\}$ or $l(P)+l(P_2) \geq \min\{2g, h(G)\}$. In either case, we have $l(P) \geq (g + \lfloor \frac{g}{2} \rfloor) - \lceil \frac{g}{2} \rceil = g - 1$. This completes the proof that G'' has girth at least g. On the other hand, $C_1 \cup \{v_1 v_{m+1}, v_2 v_{m+2}\} - \{v_1 v_2, v_{m+1} v_{m+2}\}$ is a cycle of length g in G''. Since G'' also has the same number of vertices as G, G'' is a $(k;g)$-cage. Now $C = (C_2 - C_1) \cup C_1[v_{t+1}, v_{m+1}] \cup v_1 v_{m+1}$ (note that $t+1 < m+1$) is a cycle in G'' with even length $g + (m+1) - 2t$. Since $m = \lfloor \frac{g}{2} \rfloor$ or $\lceil \frac{g}{2} \rceil$ and $t \geq 2$, $g + (m+1) - 2t \leq g + (\lfloor \frac{g}{2} \rfloor + 2) - 2t \leq g + \lfloor \frac{g}{2} \rfloor - 2$. Hence G'' has even girth less than $g + \lfloor \frac{g}{2} \rfloor$. □

Corollary 1. *For all* $k \geq 3$, $f(k;5,6) = f(k;5)$, $f(k;7,8) = f(k;7)$.

Proof. By Theorem 1, for all $k \geq 3$, there exists a $(k;5)$-cage with even girth (at most) 6, and a $(k;7)$-cage with even girth (at most) 8. □

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