On normal quotients of transitive graphs

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Abstract

The notion of normal quotient of a vertex-transitive graph was introduced in [5]. It was shown there that many graph properties are inherited by normal quotients. The definition of a normal quotient was given in [5] in group-theoretical terms. In this note we give a combinatorial approximation to this notion which extends the original definition. We show that many of the properties that were inherited by group-theoretical normal quotients are also inherited by combinatorial ones.

1 Introduction

Graph-theoretical approach to the investigation of permutation groups goes back to the seminal papers of D.G.Higman [3] and C.C.Sims [6]. It exploits the connection between the properties of a permutation group and its 2-orbits: the orbits of the diagonal action of the group on the Cartesian square of the underlying set. The best known example of using this connection is the graph-theoretical criterion of primitivity of a transitive permutation group which asserts that a permutation group is primitive if and only if all non-diagonal 2-orbits of the group are connected (as graphs).

In this paper we consider two different weakenings of primitivity: "group-theoretical" and "combinatorial" ones. Group-theoretical weakening was introduced by C.Praeger in [5]. A transitive permutation group is called *quasiprimitive* if each normal subgroup of the group is transitive. A finite graph is called *quasiprimitive* if it is a 2-orbit of some quasiprimitive group. The combinatorial analog of a quasiprimitive graph, introduced in this paper, is based on the notion of the normal quotient of a graph which we define in a purely combinatorial way. Our main result shows that many of properties of a graph are inherited by its normal quotient. The definition of a normal quotient is given via a homogeneous coherent configuration.

It turns out that combinatorial quasiprimitivity is stronger than group-theoretical one but weaker than usual primitivity. Nevertheless, a combinatorial quasiprimitivity has an important advantage comparing the group-theoretical one: if r is combinatorially quasiprimitive graph then the automorphism group of r is quasiprimitive. This gives a partial answer on the Question 7.1(a) formulated in [5].

The paper is organized as follows. Next section contains preliminaries and formulation of main results. In Section 3 we prove some results about homogeneous coherent configurations that we need in the rest of the paper. We think that a part of these results are of independent interest. The last Section contains the proof of the main result formulated in Section 2.

2 Preliminaries

In this note a graph with a node set Ω means an arbitrary binary relation r on Ω. If $r, s \subseteq \Omega^2$, then rs will denote the usual product of relations. If $\omega \in \Omega$ and $r \subseteq \Omega^2$, then $\omega^r := \{\beta \in \Omega \,|\, (\omega, \beta) \in r\}$. If $H \subset 2^{\Omega\times\Omega}$, then we write ω^H for $\cup_{h\in H}\omega^h$. As usual $r, r \subseteq \Omega^2$ is said to be *regular* if $|\omega^r|$ is constant for all $\omega \in \Omega$. In this case we shall write $|r|$ for $|\omega^r|$. A permutation $g \in \mathsf{Sym}(\Omega)$ will be always treated as a binary relation on Ω.

A graph (Ω, r) is called *vertex-transitive* if its automorphism group $\mathsf{Aut}(r)$ is transitive on Ω. Following [5] we shall say that a graph (Ω, r) is *G-transitive*, $G \leq \mathsf{Aut}(r)$ if G acts transitively on Ω. If G acts transitively on r, then r is said to be $(G, 1)$-arc transitive. If $N \trianglelefteq G$, then the orbits of N form a G-invariant partition Ω/N of Ω. The quotient graph $r^{\Omega/N} := \{(\omega_1^N, \omega_2^N) \,|\, (\omega_1, \omega_2) \in r\}$ is called *a normal quotient* of (Ω, r) [5]. It inherits many of the properties of the original graph. If each normal subgroup of G is transitive, then G and r are called *quasiprimitive* [5]. However, it might happen that a $(G, 1)$-arc transitive quasiprimitive graph still has a "good" quotient. Let us say that a partition $\mathcal{P}$ of Ω is *r-normal* if the equivalence relation e corresponding to $\mathcal{P}$ satisfies the equality $er = re$. If r is a G-transitive graph, then the partition induced by a normal subgroup of G is always normal (Section 3). The converse is not true, in general. For example, the symmetric group S_{2m} acting on the set of all m-element subsets of $\{1, 2, ..., 2m\}$ is quasiprimitive, but there exists a partition which is normal with respect to each of S_{2m}-transitive graphs.

We shall say that a $(G, 1)$-arc transitive graph r is *(G, n)-primitive* if there is no non-trivial G-invariant r-normal equivalence relation. Since the partition induced by an intransitive action of a normal subgroup of G is always an r-normal partition, each (G, n)-primitive graph is G-quasiprimitive. The most important property of (G, n)-primitive graphs is formulated below. Its proof is so easy that we omit it here.

Proposition 2.1 *Let (Ω, r) be a (G, n)-primitive graph. Then (Ω, r) is an (M, n)-primitive graph for each $M, G \leq M \leq \mathsf{Aut}(r)$.*

Thus if r is a G-quasiprimitive graph which is also (G, n)-primitive then r is M-quasiprimitive for each overgroup $M, G \leq M \leq \mathsf{Aut}(r)$. It is worth to mention that in general a G-quasiprimitive graph may be not a M-quasiprimitive for some $G \leq M \leq \mathsf{Aut}(r)$ [5].

It turns out that the r-normal partitions are also good enough for taking quotients of r. To formulate the corresponding claim we need to remind some definitions.

Let (Ω, r) be a graph and let $e \subseteq \Omega^2$ be an equivalence relation on Ω. The *quotient graph* $r^{\Omega/e}$ is a graph with the vertex set $\Omega/e := \{\omega^e \,|\, \omega \in \Omega\}$ and the edge set $\{(\alpha^e, \beta^e) \,|\, (\alpha, \beta) \in r\}$. We shall say that r is a *multicover* of $r^{\Omega/e}$ if for each edge (α^e, β^e) of $r^{\Omega/e}$ the intersections $\omega^r \cap \beta^e$, $\eta^{r^t} \cap \alpha^e$ are non-empty for each $\omega \in \alpha^e, \eta \in \beta^e$. A graph r is said to be a *cover* of its quotient $r^{\Omega/e}$ if $|\omega^r \cap \beta^e| = |\eta^{r^t} \cap \alpha^e| = 1$ for each $\omega \in \alpha^e, \eta \in \beta^e$. If r is symmetric, then these definitions coincides with the definitions of cover and multicover given in [5].

The result below is an analogue of Theorem 4.1 [5].

Theorem 2.2 *Let (Ω, r) be a $(G, 1)$-transitive connected graph and e be a G-invariant non-trivial r-normal equivalence relation on Ω. Denote by $\widehat{G} \leq \mathsf{Sym}(\Omega/e)$ the permutation group induced by the action of G on the equivalence classes of e; denote by $\overline{\alpha} \in \Omega/e$ the equivalence class of $\alpha \in \Omega$. Then the quotient graph $\overline{r} := r^{\Omega/e}$ is $(\widehat{G}, 1)$-arc transitive graph of valency k/l where k is the valency of r and $l = |\omega^r \cap \omega^e|$, and r is a multicover of $r^{\Omega/e}$. Moreover if $r^{\Omega/e}$ is not a directed full cycle on Ω/e then*

(i) *If r is G-locally primitive, then r is a cover of $r^{\Omega/e}$ and $(G_\alpha)^{\alpha^r}$ is transitively embedded into $(\widehat{G}_{\overline{\alpha}})^{\overline{\alpha}^{\overline{r}}}$ and $r^{\Omega/e}$ is $\widehat{G}$-locally primitive.*

(ii) *If r is (G, k)-transitive, $2 \leq k$, then $r^{\Omega/e}$ is $(\widehat{G}, k)$-transitive and r is a cover of $r^{\Omega/e}$, i.e., $l = 1$.*

Given a $(G,1)$-arc transitive graph (Ω, r), one can associate a *homogeneous coherent configuration* $2-\mathrm{orb}(G;\Omega)$ by taking the set of all *2-orbits* (*orbitals*) of $(G;\Omega)$. Since G acts arc transitively on r, $r \in 2-\mathrm{orb}(G;\Omega)$.

In the next section we study the situation, when r is a basic relation of a homogeneous coherent configuration (Ω, R).

3 Homogeneous coherent configurations.

Let (Ω, R) be a homogeneous coherent configuration [3]. We write $\mathsf{Rel}(R)$ for the set of all binary relations on Ω which may be represented as a union of basic relations of (Ω, R). For each $S \subseteq R$, we write $|S|$ for the sum $\sum_{s\in S}|s|$. For arbitrary ring F, we denote by $F\Omega$ the F-module of all F-valued functions on Ω. We write a function $f \in F^{\Omega}$ as a formal sum $\sum_\omega f(\omega)\omega$. If $\Delta \subseteq \Omega$, then $\Delta^+ = \sum_{\delta\in\Delta}\delta \in F\Omega$ denotes the characteristic function of Δ. For each $g \in \mathsf{Rel}(R)$, we denote by $A(g)$ its adjacency matrix. For each pair A, B of $\Omega \times \Omega$-matrices, we write $\langle A, B\rangle$ for the following expression $\frac{1}{|\Omega|}\sum_{(\alpha,\beta)\in\Omega^2} A_{\alpha\beta}B_{\alpha\beta}$.

Proposition 3.1 *Let $e \in \mathsf{Rel}(R)$ be an equivalence relation. Then for each $r \in R$ there exist $l(r,e) \in \mathbb{N}$ such that*

(i) $A(r)A(e) = l(r,e)A(re), A(e)A(r) = l(r^t,e)A(er)$;

(ii) $\forall_{\omega',\omega\in\Omega}\ (\omega'^r \cap \omega^e \neq \emptyset \Rightarrow |\omega'^r \cap \omega^e| = l(r,e))$;

(iii) $l(r,e) \mid |r|$;

(iv) $|r||e| = l(r,e)|re| = l(r^t,e)|er|$;

(v)
$$|ere||re \cap er| l(r,e) l(r^t,e) = |e|^2|r|^2;$$

(vi)
$$|r^{\Omega/e}| = \frac{|e||r|^2}{l(r,e)l(r^t,e)|re\cap er|}.$$

Proof.

(i) It follows from Proposition 3, page 51,[8] that $A(r)A(e) = mA(re)$ for a suitable $m \in \mathbb{R}$. Since all structure constants of the Bose-Mesner algebra of R are non-negative integers, $m \in \mathbb{N}$. By setting $l(r,e) := m$ we obtain $A(r)A(e) = l(r,e)A(re)$. Now $A(e)A(r) = l(r^t,e)A(er)$ is a direct consequence of the equality $A(r^t)A(e) = l(r^t,e)A(r^te)$.

(ii) Let ω', ω be arbitrary points that satisfy $\omega'^r \cap \omega^e \neq \emptyset$. Then $|\omega'^r \cap \omega^e|$ is equal to the coeffcient of $E_{\omega',\omega}$ in the product $A(r)A(e^t) = A(r)A(e)$, i.e.,

$|\omega'^{r} \cap \omega^{e}| = l(r,e)$, as desired (here $E_{\alpha,\beta}$ is the matrix unit corresponding to the pair α, β).

(iii) Let $\Delta \subseteq \Omega$ intersect each e-class by one element, *i.e.*, Δ is a transversal of the partition Ω/e. Fix a point $\omega \in \Omega$ and consider the set $\Delta' = \{\delta \in \Delta \,|\, \delta^{e} \cap \omega^{r} \neq \emptyset\}$. Then $\omega^{r} = \cup_{\delta\in\Delta'}(\omega^{r} \cap \delta^{e})$, implying $|r| = |\omega^{r}| = |\Delta'| l(r,e)$.

Part (iv) follows if we apply $|\,|$ to the both sides of (i).

(v) It follows from Proposition 3, page 51,[8] that $A(e)A(r)A(e) = \lambda A(ere)$ for a suitable $\lambda \in \mathbb{N}$. Hence $\lambda|ere| = |e||r||e|$.
Since $A(r)$ appears in $A(ere)$ with coefficient 1,

$$\lambda = \frac{1}{|r|}\langle A(e)A(r)A(e), A(r)\rangle = \frac{1}{|r|}\langle A(e)A(r), A(r)A(e)\rangle.$$

By part (i),

$$\langle A(r)A(e), A(e)A(r)\rangle = l(r,e)l(r^{t},e)\langle A(re), A(er)\rangle = l(r,e)l(r^{t},e)|re \cap er|,$$

implying

$$\lambda = \frac{l(r,e)l(r^{t},e)|re \cap er|}{|r|}.$$

Now part (v) is a direct consequence of this equality.

(vi) According to [9] $|r^{\Omega/e}| = \frac{|ere|}{|e|}$. Now part (v) of our claim implies the desired result. □

Remark 1. More general version of part (i) was proved in [1] for generalized table algebras.

Remark 2. In general $l(r,e) \neq l(r^{t},e)$.

Proposition 3.2 *Let* $e \in \mathrm{Rel}(R)$ *be an equivalence relation Then for each* $r \in \mathrm{R}$ *the following are equivalent:*

(i) $re = er$;

(ii) $A(r)A(e) = A(e)A(r)$;

(iii) Ω/e *is an equitable partition with respect to* r *and* r^{t}.

(iv) r *is a multicover of* $r^{\Omega/e}$.

Proof. (i) $\Rightarrow$ (ii). By Proposition 3.1 $A(r)A(e) = l(r,e)A(re), A(e)A(r) = l(r^{t},e)A(er)$ where

$$l(r,e) = \frac{|r||e|}{|re|}, l(r^{t},e) = \frac{|r||e|}{|er|}.$$

Since $re = er$, $l(r,e) = l(r^t,e)$, and, consequently, $A(r)A(e) = A(e)A(r)$.

(ii)$\Rightarrow$ (iii) Since $A(e)$ commutes with $A(r)$, $\mathsf{Im}(A(e))$ is an $A(r)$-invariant subspace. But $\mathsf{Im}(A(e)) = \mathsf{Span}\langle\Pi^+\rangle_{\Pi\in\Omega/e}$. Thus $\mathsf{Span}\langle\Pi^+\rangle_{\Pi\in\Omega/e}$ is an $A(r)$-invariant subspace, which is one of the equivalent definitions of equitable partition. Applying ^{t} to the both parts of (ii) we obtain $A(e)A(r^t) = A(r^t)A(e)$. Hence Ω/e is also an r^t-equitable partition.

(iii)$\Rightarrow$ (i) Let $(\alpha,\gamma) \in er$. Then there exists $\beta \in \alpha^e$ such that $(\beta,\gamma) \in r$. So $\beta^r \cap \gamma^e \neq \emptyset$. Since Ω/e is r-equitable and $(\alpha,\beta) \in e$, $\alpha^r \cap \gamma^e \neq \emptyset$. Hence there exists $\delta \in \Omega$ such that $(\alpha,\delta) \in r$ and $(\delta,\gamma) \in e$ which implies $(\alpha,\gamma) \in re$. Thus $er \subseteq re$.

Since Ω/e is r^t-equitable, $er^t \subseteq r^te$. Applying ^{t} to the both sides of the inclusion we obtain $re \subseteq er$, and, therefore $er = re$.

The equivalence (i)$\Leftrightarrow$(iv) follows directly from the definition of a multicover. □

As a direct consequence we obtain the following

Proposition 3.3 *Let $e \in \mathsf{Rel}(R)$ be an equivalence relation. Then e is R-normal if and only if Ω/e is an R-equitable partition of Ω.*

In what follows we call an equivalence relation $e \in \mathsf{Rel}(R)$ *r-normal*, $r \in R$ if $re = er$. There are two trivial r-normal equivalence relations: id_Ω and Ω^2. We shall say that r is *(R,n)-primitive* if id_Ω, Ω^2 are the only r-normal equivalence relations. If an equivalence relation $e \in \mathsf{Rel}(R)$ is r-normal for every $r \in R$, then e is called *R-normal* (see [9]). The homogeneous coherent configuration (Ω, R) will be called *n-primitive*, if id_Ω, Ω^2 are the only normal equivalence relations in $\mathsf{Rel}(R)$.

Since connected components of a basic graph $r \in R$ always form an r,r^t-equitable partition of Ω, each (R,n)-primitive basic graph is always connected. In order to formulate next proposition we remind that an m-arc of graph $r \subseteq \Omega^2$ is an $(m+1)$-tuple $(\omega_0,\omega_1,...,\omega_m)$ of points of Ω such that $(\omega_i,\omega_{i+1}) \in r$ for each $i = 0,...,m-1$ and $\omega_{i-1} \neq \omega_{i+1}, i = 0,...,m-2$ (the latter condition always holds of r is a non-symmetric basic graph of a homogeneous coherent configuration).

Proposition 3.4 *Let $e \in \mathsf{Rel}(R)$ be a r-normal equivalence relation. Then for each m-arc $(\Delta_0,...,\Delta_m)$, $\Delta_i \in \Omega/e$ of $r^{\Omega/e}$ there exists an m-arc $\omega_0,...,\omega_m$ of r such that $\Delta_i = \omega_i^e, i = 0,...,m$.*

Proof. Induction on m. If $m = 1$ then our claim follows directly from the definition. Assume now that $m > 1$. By induction hypothesis there exists an $(m-1)$-arc $\omega_0,...,\omega_{m-1}$ of r such that $\omega_i^e = \Delta_i, 0 \leq i \leq m-1$. By definition of $r^{\Omega/e}$ there exist $\beta_1 \in \Delta_{m-1}, \beta_2 \in \Delta_m$ such that $(\beta_1,\beta_2) \in r$. Since Ω/e

is an equitable partition of Ω, there exists $\omega_m \in \Delta_m$ with $(\omega_{m-1}, \omega_m) \in r$. Since $\Delta_{i-1} \neq \Delta_{i+1}$, $\omega_{i-1} \neq \omega_{i+1}$ implying that $(\omega_0, ..., \omega_m)$ is an m-arc of r. □

Proposition 3.5 *Let $e \in \mathrm{Rel}(R)$ be an equivalence relation. Then*

(i) $(\omega^r)^e \subseteq (\omega^e)^{r^{\Omega/e}}$;

(ii) *If e is r-normal, then* $(\omega^r)^e = (\omega^e)^{r^{\Omega/e}}$ *and* $|r^{\Omega/e}| = \frac{|r|}{|l(r,e)|}$.

Proof. (i) If $\Delta \in (\omega^r)^e$, then $\Delta = \alpha^e$ for some $\alpha \in \omega^r$. Therefore $(\omega, \alpha) \in r$ implying $(\omega^e, \alpha^e) \in r^{\Omega/e}$. Hence $\Delta = \alpha^e \in (\omega^e)^{r^{\Omega/e}}$, as desired.

(ii) Take an arbitrary $\Delta \in (\omega^e)^{r^{\Omega/e}}$. Then $\Delta = \alpha^e$ for some $\alpha \in \Omega$ and $(\omega^e, \alpha^e) \in r^{\Omega/e}$. Therefore there exist $\omega' \in \omega^e, \alpha' \in \alpha^e$ such that $(\omega', \alpha') \in r$. Since e is r-normal, there exists $\alpha'' \in \alpha'^e = \alpha^e$ such that $(\omega, \alpha'') \in r$. Therefore $\Delta = \alpha'^e = \alpha''^e \in (\omega^r)^e$.

The equality $|r^{\Omega/e}| = |r|/l(r,e)$ follows immediately from part (vi) of Proposition 3.1. □

4 Proof of the main result.

We recall some standard definitions from permutation group theory. Let $G \leq \mathrm{Sym}(\Omega)$ be a transitive permutation group. The orbits of G in its natural action on Ω^2 will be called *2-orbits* [7]. The set of 2-orbits of the permutation group G will be denoted by $2-\mathrm{orb}(G;\Omega)$. It is well known fact that $2-\mathrm{orb}(G;\Omega)$ is a homogeneous coherent configuration on Ω.

To make our notation more transparent we fix an arbitrary G-invariant equivalence relation $e \subseteq \Omega^2$ and set $\overline{\omega} := \omega^e, \overline{\Omega} := \Omega/e$. For each $f \subset \Omega^2$ we set $\overline{f} := \{(\overline{\alpha}, \overline{\beta}) \mid (\alpha, \beta) \in f\}$. If $g \in \mathrm{Sym}(\Omega)$ is a permutation that leaves Ω/e invariant, then $\widehat{g} \in \mathrm{Sym}(\overline{\Omega})$ is a permutation which acts on $\overline{\Omega}$ by the formula: $\Delta^{\widehat{g}} := \Delta^g, \Delta \in \overline{\Omega}$. It follows directly from the definition that $\overline{\omega^g} = \overline{\omega}^{\widehat{g}}$.

Fix an arbitrary point $\omega \in \Omega$. For each $f \in \mathrm{Rel}(2-\mathrm{orb}(G;\Omega))$ we set $G_\omega(f) := \{g \in G \mid (\omega, \omega^g) \in f\}$. For each 2-orbit s of G, the set $G_\omega(s)$ is a double coset of G_ω. Moreover, the correspondence $s \leftrightarrow G_\omega(s), s \in 2-\mathrm{orb}(G;\Omega)$ is a bijection between the 2-orbits of G and the G_ω-double cosets of G. This bijection is well-known in the theory of permutation groups, see, for example, [7].

A direct check shows that for each $r, s \in \mathrm{Rel}(2-\mathrm{orb}(G;\Omega))$ the following properties hold:

$$G_\omega(r \cup s) = G_\omega(r) \cup G_\omega(s) \quad (1)$$

$$G_\omega(rs) = G_\omega(r) G_\omega(s) \quad (2)$$

The latter equality shows that $e \in \mathsf{Rel}(2\text{-}\mathrm{orb}(G;\Omega))$ is an equivalence relation if and only if $G_\omega(e)$ is an overgroup of G_ω. As a direct consequence of the definition of being r-normal we obtain the following

Theorem 4.1 *Let e be a G-invariant equivalence relation and r be an arbitrary 2-orbit of G. Denote $H := G_\omega(e), G_\omega g G_\omega := G_\omega(r)$. Then*

(i) *e is r-normal if and only if $G_\omega g H = H g G_\omega$;*

(ii) *e is $2\text{-}\mathrm{orb}(G;\Omega)$-normal if and only if the equality $KxG_\omega = G_\omega x K$ holds for each $x \in G$.*

Let $N \leq \mathsf{Aut}(r)$ be an arbitrary subgroup normalized by G, i.e., $[N, G] \leq N$. Then the equivalence relation $e_N = \{(\alpha,\beta) \mid \alpha \in \beta^N\}$ is G-invariant. We claim that $re_N = e_N r$. Indeed, if $(\alpha,\gamma) \in re_N$ then $(\alpha,\beta) \in r, (\beta,\gamma) \in e_N$ for a suitable $\beta \in \Omega$. Therefore $\gamma = \beta^n$ for some $n \in N$. Since $n \in \mathsf{Aut}(r)$, $(\alpha^n, \beta^n) = (\alpha^n, \gamma) \in r$ which implies $(\alpha,\gamma) \in e_N r$. Thus $re_N \subseteq e_N r$. Analogously, $e_N r \subseteq re_N$. Combining altogether we obtain

$$re_N = e_N r, \tag{3}$$

as claimed. If $N \trianglelefteq G$, then we have a stronger result

Corollary 4.2 *Let $N \trianglelefteq G$. Then e_N is a $2\text{-}\mathrm{orb}(G;\Omega)$-normal equivalence relation of G.*

Proof is a direct consequence of (3).

Not every r-normal equivalence relation is induced by a subgroup $N \leq \mathsf{Aut}(r)$ normalized by G. Nevertheless in some cases we can show that an equivalence relation is induced by some G-normalized subgroup of $\mathsf{Aut}(r)$. In order to formulate the result we recall the definition of *the thin radical* of a coherent configuration [9].

Let $R \subset 2^{\Omega\times\Omega}$ be a homogeneous coherent configuration. A relation $s \in R$ is called *thin* (see [9]) if $|s| = 1$, i.e., s is a permutation on Ω. The set of all thin elements of R form a semiregular subgroup of $\mathsf{Sym}(\Omega)$ which is called the thin radical of R and is denoted as $\mathbf{O}_\theta(R)$[9]. Following [9] we set $\mathbf{C}_{\mathbf{O}_\theta(R)}(r) := \{s \in \mathbf{O}_\theta(R) \mid sr = rs\}$. It is easy to see that $\mathbf{C}_{\mathbf{O}_\theta(R)}(r)$ is a semiregular subgroup of $\mathsf{Sym}(\Omega)$. Therefore $e_{\mathbf{C}_{\mathbf{O}_\theta(R)}(r)}$ is an equivalence relation on Ω. It follows direclty from the definition that $e_{\mathbf{C}_{\mathbf{O}_\theta(R)}(r)}$ is an r-normal equivalence relation.

Proposition 4.3 *Let r be a $(G,1)$-transitive graph and $R := 2\text{-}\mathrm{orb}(G;\Omega)$. Then*

(i) $\mathbf{C}_{\mathsf{Aut}(r)}(G) = \mathbf{C}_{\mathbf{O}_\theta(R)}(r)$;

(ii) $e_{\mathrm{Co}_\theta(R)}(r)$ *is an r-normal equivalence relation induced by the subgroup* $\mathrm{C}_{\mathsf{Aut}(r)}(G)$*;*

(iii) *For each* $\omega \in \Omega$, $\mathrm{C}_{\mathsf{Aut}(r)}(G) \cong F/G_\omega$, *where* $F = \{h \in \mathrm{N}_G(G_\omega) \mid g^h \in G_\omega g G_\omega\}$.

Proof. (i) Let $s \in \mathrm{C}_{\mathsf{Aut}(r)}(G)$. Then s is G-invariant binary relation, and, therefore, $s \in \mathsf{Rel}(R)$. Since s is of valency one, $s \in R$. Therefore $s \in \mathrm{O}_\theta(R)$. Together with $s \in \mathsf{Aut}(r)$ we obtain $sr = rs$ which, in turn, implies $s \in \mathrm{C}_{\mathrm{O}_\theta(R)}(r)$.

Vice versa, if $s \in \mathrm{C}_{\mathrm{O}_\theta(R)}(r)$ then $sr = rs$ or, equivalently, $s \in \mathsf{Aut}(r)$. Since s is a G-invariant relation, s centralizes G, *i.e.*, $s \in \mathrm{C}_{\mathsf{Aut}(r)}(G)$, as desired.

Part (ii) is a direct consequence of (i) and (3).

(iii) Fix an arbitrary $\omega \in \Omega$. It is well-known that $s \in \mathrm{O}_\theta(R) \Leftrightarrow G_\omega(s) \subseteq \mathrm{N}_G(G_\omega)$. So the mapping $s \mapsto G_\omega(s), s \in \mathrm{O}_\theta(R)$ is an isomorphism between $\mathrm{O}_\theta(R)$ and $\mathrm{N}_G(G_\omega)/G_\omega$. Now the claim follows directly from the definition of $\mathrm{C}_{\mathrm{O}_\theta(R)}(r)$. □

We shall say that the permutation group $(G;\Omega)$ is *n-primitive* if its 2-orbit configuration is n-primitive.

Proof of Theorem 2.2.

It is evident that $\overline{r}$ is $(\widehat{G},1)$-transitive graph. Since e is r-normal, the valency of $\overline{r}$ is equal $k/l(r,e)$ (Propositon 3.5 (ii)) where $l(r,e) = |\omega^r \cap \omega^e|, \omega \in \Omega$. Thus r is a multicover of $\overline{r}$. The graph $\overline{r}$ is a directed cycle if and only if $|\overline{r}| = 1$, or, equivalently, $l(r,e) = |r|$.

Assume now that $\overline{r}$ is not a directed cycle, *i.e.*, $l(r,e) < |r|$.

(i) Fix an arbitrary $\omega \in \Omega$. The equivalence relation $e \cap (\omega^r)^2$ is G_ω-invariant. Since G_ω acts primitively on ω^r, either $e \cap (\omega^r)^2 = (\omega^r)^2$ or $e \cap (\omega^r)^2 = id_{\omega^r}$. In the first case we have $l(r,e) = k$ contrary to the assumption. So we may assume that $e \cap (\omega^r)^2 = id_{\omega^r}$. In this case $l(r,e) = 1$ and, therefore, the valency of $\overline{r}$ is equal to k,*i.e.*, r is a covering of $\overline{r}$.

For each $g \in G_\omega$ and $\alpha \in \omega^r$, we have $\widehat{g} \in \widehat{G}_{\overline{\omega}}, \overline{\alpha} \in \overline{\omega}^{\overline{r}}$. Thus the map $(g,\alpha) \mapsto (\widehat{g},\overline{\alpha})$ is a morphism between the group actions $(G_\omega;\omega^r)$ and $(\widehat{G}_{\overline{\omega}};\overline{\omega}^{\overline{r}})$. If $\widehat{g}$ acts trivially on $\overline{\omega}^{\overline{r}}$ then $\alpha^{eg} = \alpha^e$ for every $\alpha^e \in \overline{\omega}^{\overline{r}} = \{\beta^e \mid \beta \in \omega^{er}\}$. In particular, $\alpha^{eg} = \alpha^e$ for every $\alpha \in \omega^r$. Since $g \in G_\omega$, $\omega^{rg} = \omega^r$ implying $(\alpha^e \cap \omega^r)^g = \alpha^e \cap \omega^r$ for all $\alpha \in \omega^r$. Now combining $\alpha \in \alpha^e \cap \omega^r$ with $|\alpha^e \cap \omega^r| \leq l(r,e) = 1$ we obtain $\alpha^e \cap \omega^r = \{\alpha\}$ which, in turn, implies that for each $\alpha \in \omega^r$, $\alpha^g = \alpha$. Hence g acts trivially on ω^r and the map $g^{\omega^r} \mapsto \widehat{g}^{\overline{\omega}^{\overline{r}}}$ is an embedding of $G_\omega^{\omega^r}$ into $\widehat{G}_{\overline{\omega}}^{\overline{\omega}^{\overline{r}}}$. The image of this embedding is transitive, because by Proposition 3.5 (ii) $\overline{\omega^r} = \overline{\omega}^{\overline{r}}$. Since

$\widehat{G_{\overline{\omega}}^{\overline{\omega}^r}}$ is an overgroup of $\widehat{G_\omega^{\omega^r}}$ and $\widehat{G_\omega^{\omega^r}}$ acts primitively on $\overline{\omega}^r$, the group $\widehat{G_{\overline{\omega}}^{\overline{\omega}^r}}$ also acts primitively.

(ii) Since r is (G,k)-transitive for $k \geq 2$, r is also G-locally primitive and we may apply the previous part of the Theorem. Let now $(\Delta_0, ..., \Delta_k)$ and $(\Delta_0', ..., \Delta_k')$ be two k-arcs of $\overline{r}$. By Proposition 3.4 there exist k-arcs of r $(\omega_0, ..., \omega_k), (\omega_0', ..., \omega_k')$ such that $\Delta_i = \overline{\omega_i}, \Delta_i' = \overline{\omega_i'}$. Now it is easy to see that

$$(\Delta_0', ..., \Delta_k') = (\Delta_0, ..., \Delta_k)^{\widehat{g}}$$

where $g \in G$ is an arbitrary permutation that moves $(\omega_0', ..., \omega_k')$ into $(\omega_0, ..., \omega_k)$. □

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