# Constrained classes closed under unions and ne.c. structures

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**Abstract**

We investigate those classes $\mathcal{K}$ of relational structures closed under unions that are defined by excluding a fixed class of finite structures. We characterize such classes and show they contain an infinite family of pairwise non-embeddable members. $\mathcal{N}$e.c. structures are defined by certain extension conditions. We construct countable universal structures in $\mathcal{K}$ satisfying only finitely many of the $n$e.c. extension conditions.

## 1 Introduction

In this article we investigate classes of combinatorial structures defined by excluding certain finite structures (or *constrained classes*) which satisfy the additional condition that they are closed under unions. We characterize these classes via graph-theoretic properties of their constraints (see Proposition 16), and present various constructions possible in such classes (see Sections 4 and 5). $\mathcal{N}$-existentially closed or $n$e.c. structures are structures satisfying certain extension conditions (see Definition 24). $\mathcal{N}$e.c. graphs are precisely those graphs satisfying a certain adjacency property which has been actively studied by various authors (see [1], [3], [7]). In Section 6 we show that in constrained classes $\mathcal{K}$ closed under unions, for each $n \geq 1$ there are countable structures in $\mathcal{K}$ that are $n$e.c. but not $r$e.c. for some $r > n$, which embed all countable structures in $\mathcal{K}$. We emphasize that the advantage of our method is that all of the results stated above apply simultaneously in many combinatorial classes such as the classes of graphs, $K_n$-free graphs, and $k$-uniform hypergraphs, for $k \geq 3$.

## 2 Relational Structures

In this section we introduce the terminology needed to discuss relational structures. The reader who desires further information on relational structures is directed to [8].

**Definition 1** *1. A* ***signature*** *is a set of symbols $L = \{R_i : i \in I\}$ along with a function $ar : L \to \mathbb{N} - \{0\}$.*

*2. An* ***L-structure*** *$A$ is a pair consisting of a nonempty set $dom(A)$, the* ***domain*** *of $A$, and an operation $R \mapsto R^A$ defined on all $R \in L$ so that if $ar(R) = n$ then $R^A \subseteq dom(A)^n$. $R^A$ is the* ***interpretation*** *of $R$ in $A$.*

*3. The* ***order*** *of an L-structure $A$ is the cardinality of its domain.*

**Remark 2** *1.* We abuse notation and let $A$ stand for both a structure and its domain. Furthermore, we suppress mention of the operation $R \mapsto R^A$.

*2.* Given a structure $A$, $\bar{a} \in A$ is a finite (ordered) tuple of length $n \geq 1$. The length of $\bar{a}$ is written $|\bar{a}|$. Given a finite subset $S$ of $A$, let $\bar{a}$ be an enumeration of $S$ (without repetitions). We will abuse notation and identify $S$ with $\bar{a}$.

**Example 3** Let $L = \{E\}$, with $ar(E) = 2$.
A (simple) graph $G$ is an $L$-structure with $E^G$ irreflexive and symmetric. An order $P$ is an $L$ -structure with $E^P$ reflexive, anti-symmetric, and transitive.

**Example 4** Let $L = \{R\}$, $ar(R) = k$, for some $k \geq 3$.
A $k$-uniform hypergraph $H$ is an $L$-structure with $R^H$ interpreted as all permutations of a set of $k$-element subsets of $H$.

**Definition 5** *Let $A$, $B$ be L-structures for a fixed signature $L$.*

*1. A* ***homomorphism*** *$f$ from $A$ to $B$ is a map satisfying*

$$\bar{a} \in R^A \text{ implies } f(\bar{a}) \in R^B. \tag{1}$$

*2. $f$ is an* ***embedding*** *if $f$ is injective and the "implies" in (1) is replaced by an "if and only if". $f$ is an* ***isomorphism*** *if it is a surjective embedding. We write that $A$ and $B$ are isomorphic as $A \cong B$.*

*3. $S$ is a* ***substructure*** *of a structure $A$ if $S \subseteq A$ and the inclusion map $S \hookrightarrow A$ is an embedding. The substructure relation is written $S \leq A$; we write $S < A$ if $S \leq A$ and $S \neq A$.*

*4. Let $S \subseteq A$. Then the* ***induced substructure*** *of $A$ on $S$, in symbols $A \upharpoonright S$, is the structure with domain $S$ and relations $R^{A \upharpoonright S} = R^A \cap (S^n)$, for $R \in L$ with $ar(R) = n$.*

If $L$ is a signature, and $\mathcal{K}$ is a class of $L$-structures, we always assume that $\mathcal{K}$ is closed under isomorphism. The reason for this restriction is that we are interested only in the isomorphism types of structures.

**Definition 6** *Let $L$ be a signature.*

1. *Let $\mathcal{K}(L)$ be the class of all $L$-structures.*
2. *Given $\mathcal{K} \subseteq \mathcal{K}(L)$, $A$ is a **$\mathcal{K}$-structure** if $A \in \mathcal{K}$.*
3. *Given $\mathcal{K} \subseteq \mathcal{K}(L)$, let $\mathcal{K}_{fin}$ be the class of finite structures in $\mathcal{K}$.*
4. *Given a cardinal $\lambda > 0$, let $\mathcal{K}_\lambda$ be the class of $\mathcal{K}$-structures of order $\lambda$.*

**Definition 7** *Let $A, B \in \mathcal{K}(L)$.*

1. *$A \hookrightarrow B$ if there is some embedding from $A$ to $B$.*
2. *$A \sim B$ if $A \hookrightarrow B$ and $B \hookrightarrow A$.*

**Remark 8** The relation $\hookrightarrow \subseteq \mathcal{K}(L) \times \mathcal{K}(L)$ is a pre-order; namely, it is a reflexive and transitive binary relation. $\sim$ is an equivalence relation; the $\sim$-equivalence classes of $\mathcal{K}(L)_{fin}$ are isomorphism classes of finite $\mathcal{K}(L)$-structures. We will identify $\mathcal{K}(L)_{fin}$ with the isomorphism classes of $\mathcal{K}(L)_{fin}$. With this identification, $\mathcal{K}(L)_{fin}$ equipped with $\hookrightarrow$ becomes an order.

We will be mainly interested in classes of structures defined by excluding certain finite $L$-structures, the so-called *constrained classes*. The main property of a constrained class $\mathcal{K}$ is that the induced substructures of $A \in \mathcal{K}$ are again in $\mathcal{K}$. Further, we will assume that $L$ is finite. The key effect of the latter assumption is that for each $n \in \omega - \{0\}$, $\mathcal{K}_n$ is finite (in fact, $|\mathcal{K}_n| \leq 2^{mn^\alpha}$, where $m = |L|$, and $\alpha$ is the maximum arity of a symbol of $L$ as the reader can check).

**Definition 9** *Let $\mathcal{C} \subseteq \mathcal{K}(L)_{fin}$, and let $\mathcal{K} \subseteq \mathcal{K}(L)$. Define*

$$\mathcal{K}(\neg\mathcal{C}) = \{B \in \mathcal{K} : C \not\hookrightarrow B, \text{ for each } C \in \mathcal{C}\}.$$

*$\mathcal{K}(\neg\mathcal{C})$ is called a **constrained class with constraints $\mathcal{C}$**.*

**Remark 10** The name "constrained class" originates from [6]. Given a class $\mathcal{K}(\neg\mathcal{C})$ we can always assume that $\mathcal{C}$ is an *antichain*: that is, the members of $\mathcal{C}$ are pairwise non-embeddable. If $\mathcal{C}$ is an antichain then each $M \in \mathcal{C}$ is *minimal*: that is, if $N < M$, then $N \in \mathcal{K}(\neg\mathcal{C})$.

**Example 11** The class of graphs has as its contraints the looped vertex and the directed edge. The class of oriented graphs has the looped vertex and the undirected edge as its constraints. The class of tournaments has as its constraints those of the class of oriented graphs as well as an additional constraint consisting of two isolated vertices. The class of bipartite graphs has as its constraints those of the class of graphs and the odd cycles. The class of orders has constraints as pictured in Figure 1.

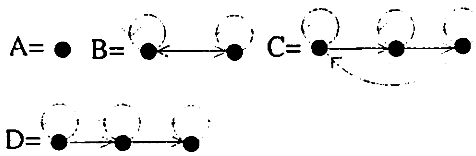


Figure 1: Order constraints.

## 2.1 Unions of structures

In this subsection, we develop machinery to work with unions of structures.

**Definition 12** *Let $A$, $B$ be $L$-structures, so that $A$ and $B$ **agree**, that is,*

$$A \restriction A \cap B = B \restriction A \cap B$$

*if $A \cap B \neq \emptyset$. The **union of $A$ and $B$**, $A \cup B$, is the $L$-structure with domain $A \cup B$, and with relations $R^{A\cup B} = R^A \cup R^B$, for $R \in L$. We also call $A \cup B$ the **free amalgam** of $A$ and $B$ over $A \cap B$.*

**Remark 13** *1.* Let $A$,$B$,$C$ be $L$-structures so that $C \leq A$ and $C \leq B$. By taking isomorphic copies we can assume that $A \cap B = C$. In this way, it makes sense to discuss the *free amalgam of $A$ and $B$ over $C$*.

*2.* Let $\mathcal{K} \subseteq \mathcal{K}(L)$. If $B, C \in \mathcal{K}$ agree, it does not necessarily follow that $B \cup C \in \mathcal{K}$. For example, let $\mathcal{K}$ be the class of orders, and let $B$ and $C$ be two-element chains that agree over the least element of $B$ and the greatest element of $C$. Then $B \cup C$ fails to be transitive.

*3.* Unions of several structures are defined in a similar way as in Definition 12. We therefore omit the details.

We will make frequent use of the following observation about unions that we call "freeness".

**Freeness:** If $A$ and $B$ agree over $A \cap B \neq \emptyset$, then in $G(A \cup B)$, there is no edge between a vertex of $A - A \cap B$ and a vertex of $B - A \cap B$.

A crucial tool we use is the notion of the graph of a structure.

**Definition 14** *Let $A$ be an $L$-structure. Define the **graph** of $A$, denoted by $G(A)$, to be the graph with vertices $A$, and edges $\{(x, y) : x, y \in A$ so that $x \neq y$ and there exists $R \in L$ and $\bar{a} \subseteq A$ so that $x, y \in \bar{a}$ and $\bar{a} \in R^A\}$.*

**Example 15** *1.* If $A$ itself is a graph, then $G(A) = A$. If $A$ is a directed graph, then $G(A)$ results by forming the symmetric closure of the directed edges of $A$.

*2.* If $A$ is an order, then $G(A)$ is the *comparability graph* of $A$: $xE^{G(A)}y$ iff $x < y$ or $y < x$.

# 3 A characterization of constrained classes closed under unions

Constrained classes closed under unions are intimately linked to complete graphs via the following theorem which characterizes such classes.

**Proposition 16** *Let $\mathcal{K}$ be a constrained $L$-class, with $\mathcal{K} = \mathcal{K}(\neg\{M_i : i \in I\})$ so that the $M_i$ are minimal (see Remark 10). Then the following are equivalent.*

1. *$\mathcal{K}$ is closed under unions.*

2. *For each $i \in I$, $G(M_i)$ is complete.*

Constrained classes of graphs closed under unions include the class of all graphs, where the minimal constraints have graphs $K_1$ and $K_2$ (see Example 11), and for each $n \geq 3$, the classes of $K_n$-free graphs, where the minimal constraints are the minimal constraints of graphs and $K_n$. Constrained classes of directed graphs closed under unions include the class of all oriented graphs: the minimal constraints have graphs $K_1$ and $K_2$ (see Example 11). A complete oriented graph is precisely a tournament. Hence, the $2^{\aleph_0}$ many Henson classes of digraphs, defined by excluding a countable set of pairwise non-embeddable tournaments (see [9]) are all closed under unions.

The class of orders is not closed under unions, since in Example 11 above, there is a minimal constraint of order 3 whose graph is the path with two edges, $P_2$. Similarly, the class of tournaments is not closed under unions, since in Example 11, $\overline{K_2}$ is the graph of one of the minimal constraints.

Proof of Proposition 16. ($1 \Rightarrow 2$)

We first show that for each $i \in I$, $G(M_i)$ is connected.

Fix $i \in I$. If $G(M_i)$ is not connected, then $G(M_i) = A \uplus B$. Then $M_i = (M_i \restriction A) \uplus (M_i \restriction B)$. However, $M_i$ is minimal so $M_i \restriction A$ and $M_i \restriction B$ are in $\mathcal{K}$. This is a contradiction as $\mathcal{K}$ is closed under disjoint unions.

Fix $i \in I$ so that $G(M) = G(M_i)$ is not complete. Let $x, y \in M$ so that $x$ is not adjacent to $y$ in $G(M_i)$ with $x$ and $y$ distinct.

Define $A = M \restriction M - \{x, y\}$, $B = M \restriction M - \{y\}$, and $C = M \restriction M - \{x\}$.

Then $B$ and $C$ agree: as $B \cap C = A$, it is enough to check that for each $R \in L$, $R^{B|A} = R^{C|A}$. But this is immediate as $B, C \leq M$.

But then $M = B \cup C$. Since $x$ is not adjacent to $y$ in $G(M)$ this is in turn equivalent to $\bar{a} \in R^{B \cup C}$.

Finally, as $G(M)$ is connected, $A \neq \emptyset$: $x$ and $y$ are not adjacent and so there must be some path connecting $x$ to $y$. Hence, we may realize $M$ as a free amalgam of proper substructures $A$, $B$, and $C$, all of which are in $\mathcal{K}$ by the minimality of $M$. This is a contradiction, as $\mathcal{K}$ is closed under union.

($2 \Rightarrow 1$)

If $\mathcal{K}$ is not closed under unions, then there are $B, C \in \mathcal{K}$ so that some minimal constraint $M_i$ of $\mathcal{K}$ embeds in $B \cup C$.

Since $B, C \in \mathcal{K}$, $M_i \cap B$ and $M_i \cap C$ are nonempty. Define $M_B = M_i \restriction M \cap B$ and $M_C = M_i \restriction M \cap C$; if $A = B \cap C \neq \emptyset$ let $M_A = M_i \restriction M \cap A$ (in this case note that $M_B$ and $M_C$ agree with $M_A$). We claim that $M_B \cup M_C = M_i$. We use the fact that $M_i \leq B \cup C$ and that for $R \in L$, $R^{B\cup C} = R^B \cup R^C$. For $R \in L$, $\bar{a} \in R^{M_i}$ iff $\bar{a}$ is from $M_i \cap B$ and $\bar{a} \in R^{M_i}$ or $\bar{a}$ is from $M_i \cap C$ and $\bar{a} \in R^{M_i}$. In turn, this is equivalent to $\bar{a} \in R^{M_B}$ or $\bar{a} \in R^{M_C}$, which itself is equivalent to $\bar{a} \in R^{M_B \cup M_C}$.

Hence, we can realize $M_i$ as the union of $M_B$ and $M_C$. But then in $G(M_i)$ there is no edge from some element of $M_B - M_A$ to any element of $M_C - M_A$, contradicting that $G(M_i)$ is complete. □

# 4 Antichains in classes closed under unions

**Definition 17** *Let $\mathcal{K}$ be a constrained class of $L$-structures.*

1. *$\mathcal{K}$ **has edges** if there is some $A \in \mathcal{K}$ so that $G(A)$ has edges.*
2. *If $\mathcal{K}$ has edges, suppose $A \in \mathcal{K}$ is such that $G(A)$ has edges. Assume that $\{a_1, \ldots, a_n\}$ is a multiset of elements from $A$ so that $\{a_1, \ldots, a_n\}$ contains at least two distinct elements and there is some $R \in L$ so that $\bar{a} = (a_1, \ldots, a_n) \in R^A$. Then $A \restriction \{a_1, \ldots, a_n\} \in \mathcal{K}$ is called an **edge**. The **length** of $A \restriction \{a_1, \ldots, a_n\}$ is the number of distinct elements in $\{a_1, \ldots, a_n\}$.*
3. *A class $\mathcal{C}$ of $\mathcal{K}$-structures is an **antichain** if distinct members of $\mathcal{C}$ are pairwise non-embeddable.*

The class of chordless cycles form an antichain in the class of graphs. In our next proposition, we generalize this fact to classes closed under unions that have edges.

**Proposition 18** *Let $\mathcal{K}$ be a constrained class closed under unions that has edges. Then $\mathcal{K}_{fin}$ contains a countably infinite antichain.*

PROOF. Fix $A$ an edge, and let $a, b$ be distinct elements of $A$. Define a set of finite $L$-structures $\{C_n(A) : n \geq 4 \text{ and } n \text{ even}\}$ as follows. Fix an even integer $n \geq 4$. List $n$ copies of $A$ as $A_1, \ldots, A_n$. By taking isomorphic copies we may assume that for $1 \leq i < j \leq n$, $A_i \cap A_j \neq \emptyset$ if and only if $j = i + 1$, and for $1 \leq i \leq n-1$,

$$A_i \cap A_{i+1} = \begin{cases} \{a\} & \text{if } i \text{ is odd,} \\ \{b\} & \text{else.} \end{cases}$$

Define $P_2(A)$ to be the free amalgam of $A_1$ and $A_2$ over $A_1 \restriction \{a\}$. Assume $P_n(A)$ has been defined for $n \geq 2$, so that the domain of $P_n(A)$ is $A_1 \cup \cdots \cup A_n$. Define $P_{n+1}(A)$ to be the free amalgam of $P_n(A)$ and $A_{n+1}$ over $P_n(A) \restriction A_n \cap A_{n+1}$. Then $P_n(A) \in \mathcal{K}_{fin}$, for each $n \geq 1$.

Let $B = P_2(A)$, and assume the copies of $A$ forming $B$ are listed as $A_1, A_2$. Let $C = P_{n-2}(A)$, and assume the copies of $A$ forming $C$ are listed as $B_1, \ldots, B_{n-2}$. By taking isomorphic copies assume that

$$B \cap C = (A_1 \cap B_1) \cup (A_2 \cap B_{n-2}),$$

where $A_1 \cap B_1 = \{b\}$, $A_2 \cap B_{n-2} = \{b\}$. Define $C_n(A)$ to be the free amalgam of $B$ and $C$ over $B \cap C$. Then $C_n(A) \in \mathcal{K}_{fin}$ (note that if $\mathcal{K}$ is the class of graphs with $A = K_2$, $C_n(A) = C_n$, the chordless $n$-cycle).

**Claim:** $\{C_n(A) : n \geq 4$ and $n$ even$\}$ is an antichain.

To see this fix $m$ and $n$ even positive integers $\geq 4$, so that $m < n$. It is enough to show that $C_m(A)$ does not embed in $C_n(A)$.

Assume otherwise. List the copies of $A$ in $C_m(A)$ as $A_1, \ldots, A_m$, and the copies of $A$ in $C_n(A)$ as $B_1, \ldots, B_n$. $A_1$ must embed as some $B_i$ by freeness: without loss of generality, we may assume $A_1$ embeds as $B_1$. Proceeding inductively, assume $A_i$ embeds as $B_i$ for each $1 \leq i \leq j$. Again by freeness, $A_{j+1}$ must embed as $B_{j-1}$ or $B_{j+1}$. Hence, $A_{j+1}$ must embed as $B_{j+1}$.

Now, fix $x \in A_m - A_1$; in particular, $x$ is adjacent (in $G(C_m(A))$ to some element of $A_1$. But then the image of $x$ in $C_n(A)$, as an element of $B_m$, is not adjacent (in $G(C_n(A))$ to any element of $B_1$, by freeness and choice of $m$ and $n$. This is a contradiction. □

# 5 Further constructions in classes closed under union

If $\mathcal{K}$ is closed under unions another consequence is that we may "delete edges" from $\mathcal{K}$-structures and remain in $\mathcal{K}$.

**Definition 19** *Let $A \in \mathcal{K}(L)$, with $|A| \geq 2$. Let $x, y \in A$. Define $A_{-xy}$ to be the $L$-structure with domain $A$ and for $R \in L$,*

$$R^{A_{-xy}} = \{\bar{a} : \bar{a} \in R^A \text{ and } \{x, y\} \not\subseteq \bar{a}\}.$$

**Lemma 20** *Let $\mathcal{K} \subseteq \mathcal{K}(L)$ be a constrained class closed under unions, and let $A \in \mathcal{K}$. Then for all $x, y \in A$, $A_{-xy} \in \mathcal{K}$.*

Proof. Let $B = A \restriction A - \{y\}$, $C = A \restriction A - \{x\}$. Then $B$ and $C$ agree over $A \restriction A - \{x, y\}$. Further, $B \cup C = A_{-xy}$, as $B \cup C$ contains all of the relations of $A$ except those involving $x, y$. □

A further consequence of being closed under unions is that, under certain restrictions on $\mathcal{K}$, the class of graphs of $\mathcal{K}$-structures contains all triangle free-graphs.

**Definition 21** *Let $\mathcal{K} \subseteq \mathcal{K}(L)$. Define*

$$G(\mathcal{K}) = \{G(A) : A \in \mathcal{K}\}.$$

**Definition 22** *Let $\mathcal{K} \subseteq \mathcal{K}(L)$.*

*1. $A \in \mathcal{K}$ is a 2-edge if $A$ is a two-element $\mathcal{K}$-structure with graph $K_2$.*

2. *$\mathcal{K}$ has a 2-edge if there is some 2-edge $A \in \mathcal{K}$.*

Let $\mathcal{C}$ be the class of all triangle-free graphs.

**Lemma 23** *Let $\mathcal{K}$ be a constrained class closed under unions that has a 2-edge $A$ and assume that $|\mathcal{K}_1| = 1$. Then $\mathcal{C}_{fin} \subseteq G(\mathcal{K})$.*

PROOF. We proceed by induction: the induction hypothesis is that $\mathcal{C}_n \subseteq G(\mathcal{K})$, for $n \geq 1$.

$\mathcal{C}_1 = \{K_1\}$ can be realized as the graph of a one-element substructure of $A$. $\mathcal{C}_2 = \{\overline{K_2}, K_2\}$. $\overline{K_2}$ can be realized as the disjoint union of a structure in $\mathcal{K}$ realizing $K_1$ with itself; $K_2$ can be realized by $A$ itself.

Let $B \in \mathcal{C}_{n+1}$. Then $B$ is a 1-element extension of a $\mathcal{C}_n$-structure $B'$, which, by induction, is realized as the graph of a $\mathcal{K}$-structure $C''$. Let $a \in B - B'$. $B$ is determined by the vertices $X$ that $a$ is adjacent to in $B'$. Note that if $X \neq \emptyset$, then $X$ is independent: an edge in $X$ will result in a triangle in $B$. Let $|X| = m \geq 1$. Let $A = \{x, y\}$. Using closure under union in $\mathcal{K}$ we can form a $\mathcal{K}$-structure $S_X$, with domain $X \cup \{x\}$, so that $G(S_X)$ is the rooted tree with $m$-leaves as in Figure 2.

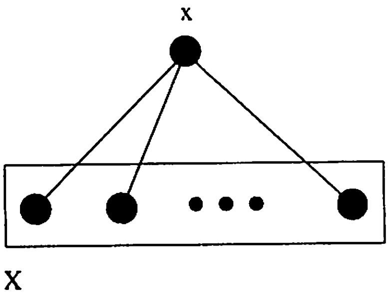


Figure 2: The structure $S_X$.

Let $A_1, \ldots, A_n$ be $n$ copies of $A$. Let $S_X(1) = A_1$. Assume $S_X(k) \in \mathcal{K}$ has been defined for some $1 \leq k < n$. Assume, by taking isomorphic copies, that $G(S_X(k))$ is a rooted tree with root $x$ and that

$$S_X(k) \cap A_{k+1} = \{x\}$$

Define

$$S_X(k+1) = S_X(k) \cup A_{k+1}.$$

Define

$$S_X = S_X(n).$$

As there is only one isomorphism type of one-element structures in $\mathcal{K}$, $S_X$ and $C''$ agree; hence, we can form the free amalgam of $S_X$ and $C''$ over $S_X \upharpoonright X = C'' \upharpoonright X$. Then

$$G(S_X \cup C'') = B.$$

□

## 6 Ne.c. structures

**Definition 24** *Let $\mathcal{K}$ be a constrained class of L-structures. Fix $n \geq 1$.*

1. *Let $A, B, C \in \mathcal{K}$ with $A \leq B$ and $A \leq C$. $C$ **extends** $A$ **in** $B$ if there is an embedding $f$ of $C$ into $B$ so that $f$ is the identity on $A$.*

2. *$A \in \mathcal{K}$ is ne.c. if*

   *(a) $A$ embeds each isomorphism type of one element structure in $\mathcal{K}$.*

   *(b) If $A \leq B$ and $A \leq C$ with $|A| = n$, $|C| = n + 1$, then $C$ extends $A$ in $B$.*

3. *$A \in \mathcal{K}$ is **strongly ne.c.** if $A$ is ne.c. but there is some $r > n$ so that $A$ is not re.c..*

Ne.c. structures have been primarily studied in the class of finite graphs, where it is known that almost all finite graphs are ne.c. for a fixed $n$ (this follows from the 0-1 law for graphs proved in [2]). Caccetta et al [3] refer to such graphs as having property $P(n)$: for any two sets $A$ and $B$ with $A \cap B = \emptyset$ and $|A \cup B| = n$ there is a vertex $u \notin A \cup B$ that is joined to every vertex of $A$ and not joined to any vertex of $B$. Very few constructive examples of ne.c. graphs are known, especially for $n \geq 3$, the exception being large Paley graphs (see Theorem 5.1 of [1]).

**Definition 25** *$A \in \mathcal{K}$ is **universal** if $A$ embeds each countable structure in $\mathcal{K}$.*

**Remark 26** For $\mathcal{K}$ a constrained class of $L$-structures closed under unions, there is a unique (up to isomorphism) countable $A \in \mathcal{K}$ that is ne.c. for all $n \geq 1$. $A$ is universal and is sometimes called the *Fraïssé limit of* $\mathcal{K}$ (see [4] for more on Fraïssé limits) or an *existentially closed* structure in $\mathcal{K}$. The Fraïssé limit of the class of graphs is known as the infinite *random graph* (see [5] for a survey of results on the random graph).

In our next theorem we show that in constrained classes $\mathcal{K}$ closed under union, there exist ne.c. structures that are universal but not isomorphic to the Fraïssé limit of $\mathcal{K}$.

**Theorem 27** *Let $\mathcal{K} \subseteq \mathcal{K}(L)$ be a constrained class closed under unions that has edges. Then for each $n \geq 1$, there is a strongly ne.c. universal structure in $\mathcal{K}_{\aleph_0}$.*

Proof of Theorem 27. By hypothesis, we can choose an edge $A \in \mathcal{K}$ (see Definition 17).

Fix $a \in A$.

Define

$$1_a = A \restriction a \in \mathcal{K}.$$

Inductively define

$$(n+1)_a = (n)_a \uplus 1_a.$$

As $\mathcal{K}$ is closed under unions, $(n)_a \in \mathcal{K}$, for all $n \geq 1$.

Note that $(n)_a$ is an $n$-element structure so that

$$G((n)_a) = \overline{K_n}.$$

For each $n \geq 1$ we define $M_n$ as the union of a chain of finite $\mathcal{K}$-structures $M_n^k$, $k \geq 0$ (and hence, $M_n \in \mathcal{K}_{\aleph_0}$ as $\mathcal{K}$ is constrained).

Let

$$M_n^0 = (n+1)_a.$$

Assume that for $k \geq 0$, $M_n^k \in \mathcal{K}_{fin}$ with $M_n^0$ a substructure of $M_n^k$.

Define $M_n^{k+1}$ as follows. List $S \leq M_n^k$ so that $M_n^0 \cap S \neq \emptyset$ and $|S| \leq n$, as $S_1, \ldots S_r$; list $S \leq M_n^k$ so that $M_n^0 \cap S = \emptyset$ and $|S| \leq k+1$, as $S_{r+1}, \ldots, S_u$.

For each $1 \leq p \leq r$, list the isomorphism types of extensions of $S_p$ to a $\mathcal{K}_{fin}$-structure of order $\leq n+1$ as $T_1, \ldots, T_j$ (there are only finitely many as $L$ is finite). For each $r+1 \leq q \leq u$, list the isomorphism types of extensions of $S_q$ to a $\mathcal{K}_{fin}$-structure of order $\leq k+2$ as $T_1', \ldots, T_l'$.

For $1 \leq \alpha \leq u$, form $M_{n,\alpha}^k$ as follows. If $1 \leq p \leq r$, freely amalgamate $T_1, \ldots, T_j$ and $M_n^k$ over $S_p$ to obtain $M_{n,p}^k \in \mathcal{K}_{fin}$. If $r+1 \leq q \leq u$, freely amalgamate $T_1', \ldots, T_l'$ and $M_n^k$ over $S_q$ to obtain $M_{n,q}^k \in \mathcal{K}_{fin}$.

Freely amalgamate $M_{n,1}^k, \ldots, M_{n,u}^k$ over $M_n^k$ to obtain $(M_n^{k+1})' \in \mathcal{K}_{fin}$.

Form $M_n^{k+1} \in \mathcal{K}_{fin}$ by taking the disjoint union of $(M_n^{k+1})'$ with all isomorphism types of $\mathcal{K}_{k+1}$-structures (there are only finitely many).

Define

$$M_n = \bigcup_{k \geq 0} M_n^k.$$

Then $M_n \in \mathcal{K}$ and by construction, $M_n$ is $n$e.c. (every one-element extension of a $\mathcal{K}_n$-structure that embeds in $M_n$ is realized in $M_n$). $M_n \restriction M_n - M_n^0$ is $k$e.c. for all $k \geq 1$ (each finite extension of a substructure of $M_n \restriction M_n - M_n^0$ is realized in $M_n \restriction M_n - M_n^0$); by Remark 26, $M_n \restriction M_n - M_n^0$ is universal, and hence, $M_n$ is universal.

We now form an extension $C \in \mathcal{K}_{fin}$ of $M_n^0$ that is not realized in $M_n$.

Let $A_1, \ldots, A_{n+1}$ be $n+1$ copies of $A$.

Let $C \in \mathcal{K}_{fin}$ be the free amalgam of $A_1, \ldots, A_{n+1}$ over $A \restriction A - \{a\}$.

It can be arranged that $C \geq M_n^0$. See Figure 3. Observe that each vertex of $C - M_n^0$ is adjacent with each vertex of $M_n^0$ in $G(C)$.

We show that $C$ is not realized in $M_n$ by showing that $C$ is not realized in $M_n^k$ for all $k \geq 0$. $C$ is not realized in $M_n^0$ as $|C| > |M_n^0|$. Assume $C$ is not realized in $M_n^k$, for a fixed $k \geq 0$.

(1) $C$ is not realized in $(M_n^{k+1})' - M_n^k$.

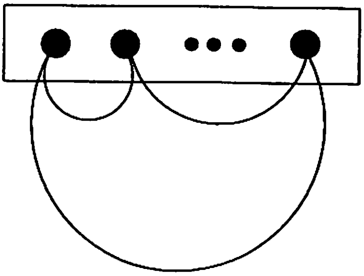


Figure 3: The extension $C$ of $M_n^0$.

To see this, we must check two cases.

**Case 1:** $C$ is realized as an extension $T$ of $S \leq M_n^k$ so that $M_n^0 \cap S \neq \emptyset$.

In this case $|S| \leq n$ and so by freeness each vertex of $T$ in $G(M_n)$ is adjacent to at most $n$ of the vertices of $M_n^0$, which is contrary to the choice of $C$.

**Case 2:** $C$ is realized as an extension $T$ of $S \leq M_n^k$ so that $M_n^0 \cap S = \emptyset$.

In this case, by freeness, each vertex of $T$ in $G(M_n)$ is adjacent to none of the vertices of $M_n^0$.

(2) $C$ is not realized in $M_n^{k+1} - (M_n^{k+1})'$.

(2) follows as no element of $M_n^{k+1} - (M_n^{k+1})'$ is adjacent (in the graph of $M_n^{k+1}$) to an element of $M_n^0$.

(1) and (2) together show that $C$ is not realized in $M_n$, so that $M_n$ is not $(|A| + n - 1)$e.c. (as $|C| = |A| - 1 + n + 1 = |A| + n$). □

The results of Theorem 27 are in stark contrast to some other known results on $n$e.c. structures for constrained classes not closed under union. In particular, it was shown in [10] that every 4e.c. order is $n$e.c. for all $n \geq 1$. It is easy to show that every 2e.c. linear order is dense and without endpoints, from which it follows that every countable 2e.c. linear order is $n$e.c., for all $n \geq 2$.

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