All c-Bhaskar Rao Designs With Block Size 3 and c ≥ -1 Exist

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Abstract: It is shown that the necessary conditions are sufficient for the existence of c-BRD(v, 3, λ) for all c ≥ -1. This was previously known for c = 0 and for c = 1.

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1. Introduction

A Bhaskar Rao design (BRD) is the incidence matrix of a BIBD(v, b, r, k, λ) when the one's are assigned a plus or minus sign in such a way that the rows are orthogonal under the standard inner product. We consider designs with an assignment of plus/minus signs which yield a constant inner product c, but c is not necessarily zero. Such matrices were introduced by Dey and Midha [1976], who referred to them as GBM's or Generalized Balanced Matrices, but, to be more consistent with present terminology, we choose to call them c-BRD's. In Hurd and Sarvate [1999] it was shown that the necessary conditions were sufficient for the existence of 1-BRD(v, 3, λ). Here we first solve the case for $c = -1$. The results are then extended to all c ≥ -1.

Originally 0-BRD's were introduced in [1] and [2]. Such matrices and their generalizations have been studied by numerous authors, e.g., see [3], [4], [5], [9], [11], [12], [13] and the references therein.

As usual, we do not distinguish between the incidence matrix of a BIBD and the BIBD. The incidence matrix of a BIBD(v, k, λ) with no minus signs is a λ-BRD. Recall that all BIBD's satisfy (1) vr = bk and (2) λ(v - 1) = r(k - 1).

2. New Necessary Conditions.

In [8] it was shown that:

Lemma 1: *(A) For every c-BRD(3, 3, λ), $c \equiv \lambda$ (mod 4); this extends the well-known condition that $c \equiv \lambda$ (mod 2) for every c-BRD.*

(B) For every 1-BRD(v, 3, 3), $v \equiv 1$ (mod 4).

(C) For k odd, every c-BRD satisfies $b(k - 1) + cv(v - 1) \equiv 0$ (mod 8).

The proof of (C) established that, if s_i is the i^{th} column sum of the c-BRD, then

$$\sum_{i=1}^{b} s_i^2 = vr + cv(v - 1).$$

But this implies a *new condition*, namely, that

Theorem 2: *(A) For every c-BRD, $vr + cv(v - 1) \geq 0$.*

(B) For $k = 3$, $c = -1$, $2b \equiv v(v - 1)$ (mod 8).

We now apply these ideas to the case $c = -1$ seeking a condition analogous to (B). Suppose $\lambda = 3$. As $k = 3$, and as $\lambda(v - 1) = r(k - 1)$, we have $r = 3(v - 1)/2$. But from $vr = bk$, we see $b = vr/k = v(v - 1)/2$. From Lemma 1(C), and as $c = -1$, we have, for some t, $8t = 2b - v(v - 1) = 0$. But this is no restriction at all. Thus, for all odd v, a (-1)-BRD(v, 3, 3) should exist! This is in striking contrast to the $c = 1$ case.

We next explore the main result of this section, a rather important hidden connection between the parameters c and λ.

Theorem 3. Suppose $v(v - 1) \not\equiv 0$ (mod 12). Then, for any c-BRD(v, 3, λ):

(A) If $c = 2s$ and $\lambda = 2t$, then $s \equiv t$ (mod 2).

(B) Suppose $x = 1$ or 5. If $c = 2s + 1 > 0$ and $\lambda = 6t + x$, then $s \equiv t$ (mod 2). If $c = -2s - 1 < 0$ and $\lambda = 6t + x$, then $s \not\equiv t$ (mod 2).

(C) Suppose $\lambda = 6t + 3$. If $c = 2s + 1 > 0$, then $s \not\equiv t$ (mod 2). If $c = -2s - 1 < 0$, then $t \equiv s$ (mod 2). In particular, if $c = -1$, i.e., $s = 0$, then t must be even.

Proof: As $k = 3$, we get $6b = \lambda v(v - 1)$. Since by hypothesis, $v(v - 1)$ is not 0 mod 12, it follows that $v(v - 1) = 2n$ for some necessarily odd n. Now from

Lemma 1(C),

$$2b + cv(v - 1) \equiv 0 \pmod 8$$
$$\Rightarrow \lambda v(v-1)/3 + cv(v-1) \equiv 0 \pmod 8$$
$$\Rightarrow 2n\lambda + 6nc \equiv 0 \pmod 8$$
$$\Rightarrow \lambda + 3c \equiv 0 \pmod 4, \text{ as n is odd.}$$

If $c = 2s$ and $\lambda = 2t$, the last congruence reduces to $t + s \equiv 0 \pmod 2$. This proves (A).

Now suppose $c = -2s - 1 < 0$ and $\lambda = 6t + x$. Then

$$\lambda + 3c \equiv 6t + x - 6s - 3 \equiv 2(t - s - 1) \equiv 0 \pmod 4$$
$$\Rightarrow t - s - 1 \equiv 0 \pmod 2.$$

But this means exactly one of s and t must be odd and the other even. If $c = 2s + 1 > 0$, the conditions reduce to $t + s \equiv 0 \pmod 2$. Thus, $s \equiv t \pmod 2$. This proves (B).

We prove half of part (C), the other half being similar. Suppose $\lambda = 6t + 3$ and $c = -2s - 1$. From $\lambda + 3c \equiv 0 \pmod 4$ we get

$$6t + 3 + 3(-2s - 1) \equiv 0 \pmod 4$$
$$\Rightarrow 6(t - s) \equiv 0 \pmod 4$$
$$\Rightarrow 2(t - s) \equiv 0 \pmod 4$$
$$\Rightarrow t - s \equiv 0 \pmod 2$$

■

Corollary. Suppose $v(v-1) \not\equiv 0 \pmod{12}$. Then for any c-BRD(v, 3, λ), $c \equiv \lambda \pmod 4$.

Lemma 4: [8] *Suppose there exists a BIBD(v, k′, λ) and a c-BRD(k′, k, μ). Then there exists a cλ-BRD(v, k, λμ).*

■

Table 1 below is used explicitely throughout the rest of the paper. Parts A-D are taken from [10] and the rest from [12]. The examples in Tables 2 and 3 will be referred to later.

Necessary and Sufficient Conditions for
λ-fold Triple Systems and 0-BRD's

	λ	v
A.	0 mod 6	all $v \neq 2$
B.	1, 5 mod 6	$v \equiv 1, 3 \mod 6$
C.	2, 4 mod 6	$v \equiv 0, 1 \mod 3$
D.	3 mod 6	all odd v

E. 0-BRD(v, 3, 2) exist if and only if $v(v - 1) \equiv 0 \pmod{12}$.

F. 0-BRD(v, 3, 4) exist if and only if $v(v - 1) \equiv 0 \pmod{3}$.

G. 0-BRD(v, 3, 6) exist if and only if $v(v - 1) \equiv 0 \pmod{4}$.

H. 0-BRD(v, 3, 2t) exists if and only if $2tv(v - 1) \equiv 0 \pmod{24}$.

Table 1

2-BRD(3, 3, 6)					
1	1	1	1	1	1
1	1	1	1	-1	-1
1	1	1	-1	-1	1

Table 2

1-BRD(5, 3, 3)									
1	1	0	0	-1	1	0	1	1	0
-1	1	1	0	0	0	1	0	1	1
0	-1	1	1	0	1	0	1	0	1
0	0	-1	1	1	1	1	0	1	0
1	0	0	-1	1	0	1	1	0	1

Table 3

Section 3. The Cases c = 2 and c = -1.

Theorem 5. *2-BRD(v, 3, 6) exist for all $v \geq 3$.*
Proof: From Table 2, we have a 2-BRD(3, 3, 6). We construct a 2-BRD(4, 3, 6) by juxtaposition of 0-BRD(4, 3, 2) and two copies of 2-BRD(4, 3, 2). These exist by Table 1. A 2-BRD(5, 3, 6) is formed from two copies of 1-BRD(5, 3, 3), from Table 3. Juxtapose a 0-BRD(6, 3, 4) and a 2-BRD(6, 3, 2) to make a 2-BRD(6, 3, 6). When v = 8, in Table 4 we have a signing for 2-BRD(8, 3, 6). Thus, we have constructed a 2-BRD(v, 3, 6) for v = 3, 4, 5, 6, and 8. The theorem now follows by Lemma 4 above and the Hanani's Lemma 5.3 **[7, p.289]** which states for every integer $v \geq 3$, $v \in B(K_3, 1)$ holds, where $K_3 = \{3, 4, 5, 6, 8\}$.

■

Theorem 5 stands in contrast to the case for 0-BRD(v, 3, 6) many of which do not exist (Table 1).

Theorem 6: *A (-1)-BRD(v, 3, 3) exists for all odd v.*

Proof: From **[10, p.49]**, let $(Q, \circ)$ be an idempotent commutative semigroup of order v, for v odd. Then

$$\{\{a, b, a \circ b\} \mid a < b \in Q\}$$

forms a BIBD(v, 3, 3), (or actually a 3-BRD(v, 3, 3)).We sign $a \circ b$ with -1 in the incidence matrix. This forms a (-1)-BRD(v, 3, 3) since all pairs (a, b) with $a < b$ occur only once as the first two entries in a block.

■

Theorem 7. *If $\lambda \equiv 3 \pmod 6$, then the necessary conditions are sufficient for (-1)-BRD(v, 3, λ) to exist.*
Proof: First suppose $v(v - 1) \equiv 0 \pmod{12}$. Then certainly $v(v - 1) \equiv 0 \pmod 4$. Hence a 0-BRD(v, 3, 6) exists (Table 1). But a (-1)-BRD(v, 3, 3) exists by Theorem 6. By juxtaposition of the t-copies of the first matrix with the second, we get a (-1)-BRD(v, 3, 6t + 3). Now suppose $v(v - 1) \not\equiv 0 \pmod{12}$. By Theorem 3(C), t must be even. Hence, $\lambda = 12y + 3$ for some y. But a 0-BRD(v, 3, 12) exists. Hence, we can juxtapose y-copies of 0-BRD(v, 3, 12) with a (-1)-

BRD(v, 3, 3) to make a (-1)-BRD(v, 3, 6t + 3).

■

Theorem 8. *If $\lambda \equiv 1, 5 \pmod 6$, then the necessary conditions are sufficient for (-1)-BRD(v, 3, λ) to exist.*
Proof: Suppose $\lambda \equiv 5 \pmod 6$. Now by Theorem 3(B), since s is even, t must be odd. So $\lambda = 6t + 5 = 6(2y + 1) + 5 = 12y + 11$ for some y. By Table 1, v is ncessarily 1 or 3 mod 6. But in either case, a 0-BRD(v, 3, 4) exists. We construct a (-1)-BRD(v, 3, 12y + 11) by juxtaposing y copies of 0-BRD(v, 3, 12), two copies of 0-BRD(v, 3, 4), and one copy of (-1)-BRD(v, 3, 3). When $\lambda \equiv 1 \pmod 6$, the conditions are the same and the construction is the same except we use only one copy of 0-BRD(v, 3, 4).

■

Theorems 7 and 8 establish:

Theorem 9. *The necessary conditions are sufficient for the existence of (-1)-BRD(v, 3, λ).*

4. The Cases c = 3 and c = 5.

We now construct the family 3-BRD(v, 3, 6t + 3). First suppose that $v(v - 1) \equiv 0 \pmod{12}$. Then by Table 1, we may combine t-copies of 0-BRD(v, 3, 6) with one 3-BRD(v, 3, 3). Now suppose $v(v - 1) \not\equiv 0 \pmod{12}$. As s = 1, by Theorem 3(C), t is even. Hence $6t + 3 = 12y + 3$ for some y. Now we take one copy of 3-BRD(v, 3, 3) and y-copies of 0-BRD(v, 3, 12), and, by adjoining them, get a 3-BRD(v, 3, 6t + 3).

In similar fashion we construct the family 5-BRD(v, 3, 6t + 3). Again, suppose that $v(v - 1) \equiv 0 \pmod{12}$. Using Table 1, we combine t-copies of 0-BRD(v, 3, 6), one 3-BRD(v, 3, 3), and one 2-BRD(v, 3, 2). Now suppose $v(v - 1) \not\equiv 0 \pmod{12}$. We must note that s = 2, and so t = 2y + 1 for some y; thus, $6t + 3 = 12y + 9$. Take one copy of 3-BRD(v, 3, 3), one copy of 2-BRD(v, 3, 6), and y-copies of 0-BRD-(v, 3, 12). Together this gives 5-BRD(v, 3, 6t + 3). We have proved:

Theorem 10. *The necessary conditions are sufficient for the existence of 5-BRD(v, 3, 6t + 3) and 3-BRD(v, 3, 6t + 3).*

■

BRD(v, 3, λ).

Proof: We now construct a 3-BRD for λ = 6t + 1 and 6t + 5. It is again necessary to consider two cases. First suppose that v(v - 1) ≡ 0 (mod 12). Use one (-1)-BRD(v, 3, 3), one 3-BRD(v, 3, 3), one 1-BRD(v, 3, 1) and (t - 1)-copies of 0-(BRD(v, 3, 6). These exist by Table 1. Together we have a 3-BRD(v, 3, 6t + 1). For 6t + 5, use one copy of 0-BRD(v, 3, 2), one 3-BRD(v, 3, 3), and a 0-BRD(v, 3, 6t). Now suppose v(v - 1) ≢ 0 (mod 12). For this case, we note s = 1. By Theorem 3, t is odd. So 6t + 1 = 12y + 7 for some y. We use one 0-BRD(v, 3, 4), one 3-BRD(v, 3, 3), and y-copies of 0-BRD(v, 3, 12). For λ = 6t + 5, we add one more 0-BRD(v, 3, 4).

■

Theorem 12. *The necessary conditions are sufficient for the existence of 5-BRD(v, 3, λ).*

Proof: Following Theorem 10, we only need to construct the remaining 5-BRD's for λ = 6t + 1, 6t + 5. First suppose that v(v - 1) ≡ 0 (mod 12). Juxtapose one copy of 0-BRD(v, 3, 6t), one copy of 3-BRD(v, 3, 3), and a 2-BRD(v, 3, 2). This constructs a 5-BRD(v, 3, 6t + 5). Now juxtapose one copy of 0-BRD(v, 3, 2), one 0-BRD(v, 3, 6(t-1)), and one 5-BRD(v, 3, 5). This builds a 5-BRD(v, 3, 6t + 1). When v(v - 1) ≢ 0 (mod 12), by Theorem 3, t is even since s is even. Thus 6t + 1 = 12y + 1 for some y. For 12y + 1, we juxtapose (y-1)-copies of 0-BRD(v, 3, 12), two copies of 0-BRD(v, 3, 4), one copy of 1-BRD(v, 3, 1), and a 4-BRD(v, 3, 4). When λ = 6t + 5 = 12y + 5, we use y-copies of 0-BRD(v, 3, 12) and one copy of 5-BRD(v, 3, 5).

■

Section 4. Main Result

This section is devoted to proving the following theorem.

Theorem 13. *The necessary conditions are sufficient for the existence of all c-BRD(v,3,λ) where c ≥ -1.*

Proof: The cases c = -1 (Theorem 9) and c = 0 [10] are done. we only need to consider positive c. We divide the argument into two cases.

Caes 1: Assume v(v − 1) ≡ 0 (mod 12). If λ = 2t + 1 is odd, then necessarily c = 2s + 1 for some s (Lemma 1(A)). Note λ = 2(t - s) + 2s + 1. To form a c-

$c = 2s + 1$ for some s (Lemma 1(A)). Note $\lambda = 2(t - s) + 2s + 1$. To form a c-BRD(v 3, λ), we need only juxtapose (t-s)-copies of a 0-BRD(v, 3, 2) and a (2s+1)-BRD(v, 3, 2s+1). The former exists from the Case 1 hypothesis and by Table 1(E), and the latter exists, from Table 1(B), since the necessarily odd v satisfies $v \equiv 1, 3 \pmod 6$ with the Case 1 hypothesis. If $\lambda = 2t$, we form a 2s-BRD(v, 3, 2t) by combining s-copies of 2-BRD(v, 3, 2) and (t-s)-copies of 0-BRD(v, 3, 2).

Case 2: Suppose $v(v - 1) \not\equiv 0 \pmod{12}$. For even λ and c, we refer the reader to Table 5. Each case there provides a 2s-BRD(v, 3, 2t) when the necessary conditions are satisfied. These conditions include $v(v - 1) \not\equiv 0 \pmod{12}$, Table 1, Theorem 3 and its corollary, s and t have the same parity, and necessary conditions for BIBD. Note that, in reading Table 5, *x means use x-copies. The array uses the Corollary to Theorem 3, namely that $c \equiv \lambda \pmod 4$. When λ is odd, see Table 6. The same comments apply here as for Table 5.

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	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12	C13	C14
v1	-1	1	1	1	1									
v2	1	-1	1	1	1	1	1	-1	-1	1				
v3						-1	1	1	-1	1	1	1	1	1
v4	1	1	-1	1	1						-1	1	1	1
v5						1	1	1	1	1				
v6											1	1	-1	1
v7														
v8														

	C15	C16	C17	C18	C19	C20	C21	C22	C23	C24	C25	C26	C27	C28
v1							-1	1	1	1	1			
v2												1	1	-1
v3	1													
v4	1	-1	1	1	1	1								
v5		1	1	1	-1	1	1	1	1	1	1			
v6	1						1	1	-1	1	1	-1	1	-1
v7		1	1	-1	1	1						1	1	1
v8														

	C29	C30	C31	C32	C33	C34	C35	C36	C37	C38	C39	C40	C41	C42
v1			1	-1	1	1	1					1		
v2	1	1							1					
v3			-1	1	1	1	1				-1		1	
v4								1	1			1		
v5								1		-1			1	
v6	1	1								1				1
v7	1	1	1	1	1	1	1				1			1
v8								1	1	1	1	1	1	1

	C43	C44	C45	C46	C47	C48	C49	C50	C51	C52	C53	C54	C55	C56
v1			-1	1					1		1			1
v2	1			1			1					1	1	
v3					1		1			1				1
v4		1						1		-1				
v5	1					1					-1			
v6		1			-1				1				-1	
v7			1			1		1				-1		
v8	1	1	1	-1	1	1	-1	-1	1	1	1	1	1	1

Table 4: 2-BRD(8, 3, 6)

λ	c = 12x	c = 12x + 4	c = 12x + 8
12y	12-BRD(v,3,12)*x 0-BRD(v,3,12)*(y-x)	2-BRD(v,3,6)*2 12-BRD(v,3,12)*x 0-BRD(v,3,12)*(y-x-1)	(12x+6)-BRD(v,3,12x+6) 0-BRD(v,3,12)*(y-x-1) 2-BRD(v,3,6)
12 y + 4	12-BRD(v,3,12)*x 0-BRD(v,3,12)*(y-x) 0-BRD(v,3,4)	(12x+4)-BRD(v,3,12x+4) 0-BRD(v,3,12)*(y-x)	(12x+4)-BRD(v,3,12x+4) 2-BRD(v,3,6)*2 0-BRD(v,3,12)*(y-x-1)
12y + 8	12-BRD(v,3,12)*x 0-BRD(v,3,12)*(y-x) 0-BRD(v,3,4)*2	(12x+4)-BRD(v,3,12x+4) 0-BRD(v,3,4) 0-BRD(v,3,12)*(y-x)	(12x+8)-BRD(v,3,12x+8) 0-BRD(v,3,12)*(y-x)

λ	c = 12x + 2	c = 12x + 6	c = 12x + 10
12y + 2	(12x+2)-BRD(v,3,12x+2) 0-BRD(v,3,12)*(y-x)	2-BRD(v,3,6)*2 (12x+2)-BRD(v,3,12x+2) 0-BRD(v,3,12)*(y-x-1)	(12x+10)-BRD(v,3,12x+10) 0-BRD(v,3,12)*(y-x-1) 0-BRD(v,3,4)
12y + 6	12-BRD(v,3,12)*x 2-BRD(v,3,6) 0-BRD(v,3,12)*(y-x)	(12x+6)-BRD(v,3,12x+6) 0-BRD(v,3,12)*(y-x)	(12x+6)-BRD(v,3,12x+6) 2-BRD(v,3,6)*2 0-BRD(v,3,12)*(y-x-1)
12y + 10	(12x+2)-BRD(v,3,12x+2) 0-BRD(v,3,4) 0-BRD(v,3,12)*(y-x)	(12x+6)-BRD(v,3,12x+6) 0-BRD(v,3,12)*(y-x) 0-BRD(v,3,4)	(12x+10)-BRD(v,3,12x+10) 0-BRD(v,3,12)*(y-x)

Table 5: 2s-BRD(v, 3, 2t)

λ	c = 12x + 1	c = 12x + 5	c = 12x + 9
12y + 1	(12x+1)-BRD(v,3,12x+1) 0-BRD(v,3,12)*(y-x)	(12x+5)-BRDv, 3, 12x+5) 0-BRD(v,3,4)*(3y-3x-1)	(12x+9)-BRD(v,3,12x+9) 0-BRD(v,3,4)*(3y-3x-2)
12 y + 5	(12x+1)-BRD(v,3,12x+1) 0-BRD(v,3,12)*(y-x) 0-BRD(v,3,4)	(12x+5)-BRD(v,3,12x+5) 0-BRD(v,3,12)*(y-x)	(12x+9)-BRD(v,3,12+9) 0-BRD(v,3,12)*(y-x-1) 0-BRD(v,3,4)*2
12y + 9	(12x+3)-BRD(v,3,12x+3) 0-BRD(v,3,12)*(y-x) (-1)-BRD(v,3,3)*2	(12x+6)-BRD(v,3,12x+6) 0-BRD(v,3,12)*(y-x) (-1)-BRD(v,3,3)	(12x+9)-BRD(v,3,12x+9) 0-BRD(v,3,12)*(y-x)

λ	c = 12x + 3	c = 12x + 7	c = 12x + 11
12y + 3	(12x+3)-BRD(v,3,12x+3) 0-BRD(v,3,12)*(y-x)	(12x+9)-BRD(v,3,12x+9) 0-BRD(v,3,12)*(y-x-1) (-1)-BRD(v,3,3)*2	(12x+9)-BRD(v,3,12x+9) 0-BRD(v,3,12)*(y-x-1) 2-BRD(v,3,6)
12y + 7	(12x+3)-BRD(v,3,12+3) 0-BRD(v,3,4) 0-BRD(v,3,12)*(y-x)	(12x+7)-BRD(v,3,12x+7) 0-BRD(v,3,12)*(y-x)	(12x+11)-BRD(v,3,12x+11) 0-BRD(v,3,12)*(y-x-1) 0-BRD(v,3,4)*2
12y + 11	(12x+3)-BRD(v,3,12x+3) 0-BRD(v,3,4)*2 0-BRD(v,3,12)*(y-x)	(12x+7)-BRD(v,3,12x+7) 0-BRD(v,3,12)*(y-x) 0-BRD(v,3,4)	(12x+11)-BRD(v,3,12x+11) 0-BRD(v,3,12)*(y-x)

Table 6: (2s+1)-BRD(v, 3, 2t+1)