Some Classes of Extended Mendelsohn Triple Systems and Numbers of Common Blocks

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Abstract

An extended Mendelsohn triple system of the order v with a idempotent element (EMTS(v, a)) is a collection of cyclically ordered triples of the type $[x, y, z]$, $[x, x, y]$ or $[x, x, x]$ chosen from a v-set, such that every ordered pair (not necessarily distinct) belongs to only one triple and there are a triples of the type $\{x, x, x\}$. If such a design with parameters v and a exist, then they will have $b_{v,a}$ blocks, where $b_{v,a} = (v^2 + 2a)/3$. A necessary and sufficient condition for the existence of EMTS($v, 0$) and EMTS($v, 1$) are $v \equiv 0 \pmod 3$ and $v \not\equiv 0 \pmod 3$, respectively. In this paper, we have constructed two EMTS(v,0)'s such that the number of common triples is in the set $\{0, 1, 2, \dots, b_{v,0} - 3, b_{v,0}\}$, for $v \equiv 0 \pmod 3$. Secondly, we have constructed two EMTS(v,1)'s such that the number of common triples is in the set $\{0, 1, 2, \dots, b_{v,1} - 2, b_{v,1}\}$, for $v \not\equiv 0 \pmod 3$.

1 Introduction

A Mendelsohn triple system of the order v, MTS(v), is a pair (V, B), where V is a v-set and B is a collection of cyclically ordered triples of distinct elements of V, such that every ordered pair of distinct elements of V is contained in only one member of B. This concept was introduced in [9] by N.S. Mendelsohn, who proved that a MTS(v) exists if and only if $v \equiv 0$ or $1 \pmod 3$, $v \neq 6$. In [1] F.E. Bennett introduced the concept of a system, similar to an MTS, in which a triple may have repeated elements.

An extended Mendelsohn triple system of the order v is a pair (V, B), where V is a v-set and B is a collection of cyclically ordered triples of elements of V, where each triple may have repeated elements and will be called a block, such that every ordered pair of elements of V, not necessarily distinct, is contained in only one block of B. It has been well established that an extended Mendelsohn triple system is co-extensive with the variety of quasigroup satisfying the identity $x(yx) = y$. (It is called a semi-symmetric quasigroup). There are three types of blocks: $[x, x, x]$, $[x, x, y]$, $[x, y, z]$ and we call them directed triangle, lollipop and idempotent, respectively. Observe that $[x, x, x]$ contains only the pair (x, x); $[x, x, y] = [x, y, x] = [y, x, x]$ contains the pairs (x, x), (x, y), (y, x); and $[x, y, z] = [y, z, x] = [z, x, y]$ contains (x, y), (y, z), (z, x). An extended Mendelsohn triple system of order v which has a idempotents will be denoted by EMTS(v, a). We define $\mathcal{EMTS}(v, a)$ as the class of all extended Mendelsohn triple systems on v element and having a idempotents. If (V, B) is an extended Mendelsohn triple system with parameters v and a, we say B is an EMTS(v, a) and write $B \in \mathcal{EMTS}(v, a)$. We say $\mathcal{EMTS}(v, a)$ exists if there exists an extended Mendelsohn triple system with parameters v and a. For $B \in \mathcal{EMTS}(v, a)$, $|B| = b_{v,a} = (v^2 + 2a)/3$.

In [1] it was shown that the necessary and sufficient conditions for the existence of an EMTS(v, a), with $0 \le a \le v$, are:

(i) if $v \equiv 0 \pmod 3$, then $a \equiv 0 \pmod 3$;

(ii) if $v \not\equiv 0 \pmod 3$, then $a \equiv 1 \pmod 3$;

(iii) if $v = 6$, then $a \le 3$.

Recently, some papers investigated the possible number of common blocks with two generalized triple systems with the same parameters, based on the same v-set. G. Lo Faro [7] considered this problem for extended triple systems without idempotent; W. -C. Huang [5, 6] for extended triple systems and C. M. Fu, Y. H. Gwo and F. C. Wu [2] for semi-symmetric latin squares.

In this paper, we have considered the intersection problems for the class $\mathcal{EMTS}$ $(v, 0)$ and $\mathcal{EMTS}(v, 1)$. Let $J[v, a]$ be the set of non-negative integers k such that there is a pair of EMTS(v, a) with k common blocks, let $I[v, a] = \{0, 1, 2, \ldots, b_{v,a} - k_a, b_{v,a}\}$, where $k_a = 2$ if $a = 1$ and $k_a = 3$ if $a = 0$.

Main Theorem $J[v, 0] = I[v, 0]$, for $v \equiv 0 (\text{mod } 3)$, and $J[v, 1] = I[v, 1]$, for $v \not\equiv 0 (\text{mod } 3)$.

Let A and B be two sets of integers and k a positive integer. We define $A + B = \{a + b \mid a \in A, b \in B\}$, $k + A = \{k\} + A$, and $kA = \{k \cdot a \mid a \in A\}$.

For convenience, we denote the k-triple $\langle v_1, v_2, \ldots, v_k\rangle$ by $\{[v_1, v_1, v_2], [v_2, v_2, v_3], \ldots, [v_{k-1}, v_{k-1}, v_k], [v_k, v_k, v_k]\}$ where $v_i \neq v_j$ for all $i \neq j$. And $\langle v_1, v_2, \ldots, v_k, v_1\rangle = \{[v_1, v_1, v_2], [v_2, v_2, v_3], \ldots, [v_{k-1}, v_{k-1}, v_k], [v_k, v_k, v_1]\}$. From this notation, we have the following lemma which makes it easy to construct another system such that it has small different blocks in the original system.

Lemma 1.1 *Let (V, B) be an EMTS(v, a) and $\langle v_1, v_2, \ldots, v_k\rangle \subset B$. Then $b_{v,a} - \{k-1\} \in J[v, a]$, if $v_1 = v_k$; and $b_{v,a} - i \in J[v, a]$, $i = 2, 3, \ldots, k$, if $v_1 \neq v_k$.*

Proof. For $v_1 = v_k$, we can replace the blocks $\langle v_1, v_2, \ldots, v_k\rangle$ by $\langle v_k, v_{k-1}, \ldots, v_1\rangle$. Thus, $b_{v,a} - \{k-1\} \in J[v, a]$. For $v_1 \neq v_k$, we can replace the blocks $\langle v_{k-i}, v_{k-i+1}, \ldots, v_k\rangle$ by $\langle v_k, v_{k-1}, \ldots, v_{k-i}\rangle$, for $i = 1, 2, \ldots, k-1$. Thus, $b_{v,a} - j \in J[v, a]$, for $j = 2, 3, \ldots, k$.

2 Auxiliary constructions of EMTS

As usual, K_v is the complete graph on v vertices. An r-cycle is an elementary cycle of length r and is denoted by the sequence of its vertices $(x_1, x_2, \ldots, x_r)$. In [3], if v is even then K_v can be decomposed into $v-1$ 1-factors and if v is odd then K_v can be decomposed into $(v-1)/2$ edge-disjoint spanning cycles. In each case, we can construct directed triangles as follows:

Method 1. Let F be a 1-factor of K_v on V and a be any vertex not in V. $\mathcal{T}(F, a) = \{[a, x, y], [a, y, x] \mid \{x, y\} \in F\}$.

Mathod 2. Let $C = (c_1, c_2, \ldots, c_v)$ be a spanning cycle of K_v on V and a and b be any two different vertices not in V. $\mathcal{T}(C, a, b) = \{[a, c_1, c_2], [a, c_2, c_3], \ldots, [a, c_v, c_1], [b, c_v, c_{v-1}], [b, c_{v-1}, c_{v-2}], \ldots, [b, c_1, c_v]\}$.

In order to count the number of common blocks of the two extended Mendelsohn triple systems, we need some special embedding constructions. Let (V_1, B_1) be an METS(v, a), where $V_1 = \{a_1, a_2, \ldots, a_v\}$.

(1) v to $2v$, v even

Let $\mathcal{F} = \{F_i \mid i = 1, 2, \ldots, v-1\}$ be a 1-factorization of K_v on $V_2 = \{x_1, x_2, \ldots, x_v\}$. Let $V = V_1 \cup V_2$ and $B = B_1 \cup T \cup L$, where $T = \cup\{\mathcal{T}(F_i, a_i) \mid i = 1, 2, \ldots, v-1\}$ and $L = \{a_v x x \mid x \in V_2\}$. Then (V, B) is an EMTS$(2v, a)$.

(2) v to $2v$, v odd

Let $\mathcal{C} = \{C_i \mid i = 1, 2, \ldots, (v-1)/2\}$ be the edge-disjoint spanning cycles of K_v on $V_2 = \{x_1, x_2, \ldots, x_v\}$. Let $V = V_1 \cup V_2$ and $B = B_1 \cup T \cup L$, where $T = \cup\{\mathcal{T}(C_i, a_{2i-1}, a_{2i}) \mid i = 1, 2, \ldots, (v-1)/2\}$ and $L = \{a_v xx \mid x \in V_2\}$. Then (V, B) is an EMTS$(2v, a)$.

(3) v to $2v+3$, v even

Let $\mathcal{C} = \{C_i \mid i = 1, 2, \ldots, v/2+1\}$ be the edge-disjoint spanning cycles of K_{v+3} on $V_2 = \{x_1, x_2, \ldots, x_{v+3}\}$. Let $V = V_1 \cup V_2$ and $B = B_1 \cup T \cup L$, where $T = \cup\{\mathcal{T}(C_i, a_{2i-1}, a_{2i}) \mid i = 1, 2, \ldots, v/2\}$ and $L = \langle x_{i_1}, x_{i_2}, \ldots, x_{i_{v+3}}, x_{i_1}\rangle$ for the last spanning cycle $C_{v/2+1} = (x_{i_1}, x_{i_2}, \ldots, x_{i_{v+3}})$. Then (V, B) is an EMTS$(2v+3, a)$.

Let $\mathcal{F} = \{F_i \mid i = 1, 2, \ldots, 2v-1\}$ be a 1-factorization of K_{2v} on $N = \{1, 2, \ldots, 2v\}$. If $F_a, F_b \in \mathcal{F}$, the notation $F_a \cdot F_b$ [7] will denote the following set of blocks: $\langle 1, x_{i_2}, x_{i_3}, \ldots, x_{i_r}, 1\rangle \cup \langle x_{j_1}, x_{j_2}, x_{j_3}, \ldots, x_{j_s}, x_{j_1}\rangle \cup \ldots \cup \langle x_{p_1}, x_{p_2}, x_{p_3}, \ldots, x_{p_t}, x_{p_1}\rangle \cup \langle x_{q_1}, x_{q_2}, x_{q_3}, \ldots, x_{q_m}, x_{q_1}\rangle$ where $x_{j_1} = min(N \setminus \{1, x_{i_2}, x_{i_3}, \ldots, x_{i_r}\})$, $\ldots$, $x_{q_1} = min(N \setminus \{1, x_{i_2}, x_{i_3}, \ldots, x_{i_r}, x_{j_1}, x_{j_2}, x_{j_3}, \ldots, x_{j_s}, \ldots, x_{t_1}, x_{t_2}, x_{t_3}, \ldots, x_{t_m}\})$; $F_a = \{1x_{i_2}, x_{i_3}x_{i_4}, \ldots, x_{i_{r-1}}x_{i_r}, x_{j_1}x_{j_2}, x_{j_3}x_{j_4}, \ldots, x_{j_{s-1}}x_{j_s}, \ldots, x_{p_1}x_{p_2}, x_{p_3}x_{p_4}, \ldots, x_{p_{t-1}}x_{p_t}, x_{q_1}x_{q_2}, x_{q_3}x_{q_4}, \ldots, x_{q_{m-1}}x_{q_m}\}$ and $F_b = \{x_{i_2}x_{i_3}, x_{i_4}x_{i_5}, \ldots, x_{i_r}1, x_{j_2}x_{j_3}, x_{j_4}x_{j_5}, \ldots, x_{j_s}x_{j_1}, \ldots, x_{p_2}x_{p_3}, x_{p_4}x_{p_5}, \ldots, x_{p_t}x_{p_1}, x_{q_2}x_{q_3}, x_{q_4}x_{q_5}, \ldots, x_{q_m}x_{q_1}\}$.

(4) v to $2v+3$, v odd

Let $\mathcal{F} = \{F_i \mid i = 1, 2, \ldots, v+2\}$ be a 1-factorization of K_{v+3} on $V_2 = \{x_1, x_2, \ldots, x_{v+3}\}$. Let $V = V_1 \cup V_2$ and $B = B_1 \cup T \cup L$, where $T = \cup\{\mathcal{T}(F_i, a_i) \mid i = 1, 2, \ldots, v\}$ and $L = F_{v+1} \cdot F_{v+2}$. Then (V, B) is an EMTS$(2v+3, a)$.

3 For the Class of EMTS$(v, 0)$

The spectrum of EMTS$(v, 0)$ is $v \equiv 0 \pmod 3$. Since the smallest possible mutually balanced subsets of an EMTS$(v, 0)$ are $\langle x, y, z, x\rangle$ (which can be changed to $\langle x, z, y, x\rangle$), it follows that $J[v, 0] \subseteq I[v, 0]$.

Lemma 3.1 *If* $J[v, 0] = I[v, 0]$ *and* v *is an integer* ≥ 9 *then* $J[2v, 0] = I[2v, 0]$.

Proof. By using constructions 1 and 2, we can embed an EMTS(v,0) in an EMTS($2v$,0). By replacing an EMTS(v,0) and interchanging any two vertices of V corresponding to different 1-factors or spanning cycles to form different directed triangles or lollipops, we obtain $J[2v,0] \supseteq J[v,0] + \{0, v, 2v, \ldots, (v-2)v, v^2\}$. If $v \geq 9$ and $J[v,0] = I[v,0]$ then $J[2v,0] \supseteq I[2v,0]$. Therefore $J[2v,0] = I[2v,0]$.

Lemma 3.2 *If $J[v,0] = I[v,0]$ and v is an integer ≥ 6 then $J[2v+3,0] = I[2v+3,0]$.*

Proof. By using constructions 3 and 4, we can embed an EMTS(v,0) in an EMTS($2v+3$,0). By replacing an EMTS(v,0) and interchanging any two vertices of V corresponding to different 1-factors or spanning cycles to form different directed triangles or lollipops, we obtain $J[2v+3,0] \supseteq J[v,0] + \{0, v+3, 2(v+3), \ldots, (v-2)(v+3), v(v+3)\}$. If $v \geq 9$ and $J[v,0] = I[v,0]$ then $J[2v+3,0] \supseteq I[2v+3,0]$. Therefore $J[2v+3,0] = I[2v+3,0]$.

Lemma 3.3 *$J[v,0] = I[v,0]$, for v =3, 6, 9, 12.*

Proof. There are precisely two EMTS(3,0): $\{[1,1,2], [2,2,3], [3,3,1]\}$ and $\{[1,1,3], [3,3,2], [2,2,1]\}$. So, we have $J[3,0] = \{0,3\} = I[3,0]$.

For $v = 6$, using a similiar argment to Lemma 3.1, we obtain $\{0,3,6,9,12\} \subseteq J[6,0]$. Let $T_1 = \{[1,1,2], [2,2,6], [3,3,1], [4,4,1], [5,5,1], [6,6,1], [2,3,5], [2,4,3], [2,5,4], [3,4,6], [3,6,5], [4,5,6]\}$ and $T_2 = \{[1,1,2], [3,3,1], [4,4,1], [6,6,2], [1,5,6], [1,6,5], [3,4,5], [3,6,4]\} \cup A$, where $A = \{[2,2,3], [2,4,5], [2,5,4], [5,5,3\}$. Now, N_1 comes from T_2 by removing the blocks A and replacing them with $\{[2,2,4], [2,3,5], [2,5,3], [5,5,4]\}$. Then, $|T_2 \cap N_1| = 8$. Using the isomorphic designs obtained from T_1 by permuting elements in Table 1, we have $J[6,0] = I[6,0]$.

Table 1

Intersection	Size	Intersection	Size
$T_1 \cap (123)(56)T_1$	1	$T_1 \cap (23)(45)T_1$	5
$T_1 \cap (12)(456)T_1$	2	$T_1 \cap (456)T_1$	7
$T_1 \cap (23)T_1$	4		

For $v = 9$, using a similiar argment to Lemma 3.2, we obtain $\{0,3,6,9,12,15,18,21,24,27\} \subseteq J[9,0]$. Let $T_1 = \{[1,1,2], [2,2,3], [3,3,1], [4,4,8], [8,8,5], [5,5,6], [6,6,7], [7,7,9], [9,9,4], [1,4,5], [1,5,4], [1,6,9], [1,9,6], [1,7,8], [1,8,7], [2,4,6], [2,6,4], [2,5,7], [2,7,5], [2,8,9], [2,9,8], [3,4,7], [3,7,4], [3,5,9], [3,9,5], [3,6,8], [3,8,6]\}$. $T_2 = \{[1,1,2], [2,2,3], [3,3,1],$

[4,4,6], [6,6,5], [5,5,4], [7,7,8], [8,8,9], [9,9,7], [1,4,7], [1,7,4], [1,5,9], [1,9,5], [1,6,8], [1,8,6], [2,4,9], [2,9,4], [2,5,8], [2,8,5], [2,6,7], [2,7,6], [3,4,8], [3,8,4], [3,5,7], [3,7,5], [3,6,9], [3,9,6]}. T_3 = {[1,1,2], [2,2,3], [3,3,4], [4,4,5], [5,5,1], [6,6,1], [7,7,2], [8,8,3], [9,9,4], [1,3,7], [1,4,8], [1,7,4], [1,8,9], [1,9,3], [2,4,6], [2,5,8], [2,6,9], [2,8,4], [2,9,5], [3,5,7], [3,6,5], [3,9,6], [4,7,6], [5,6,8], [5,9,7], [6,7,8], [7,9,8]}. T_4 = {[1,1,2], [2,2,3], [3,3,4], [4,4,1], [5,5,1], [6,6,2], [7,7,3], [8,8,4], [9,9,4], [1,3,6], [1,6,8], [1,7,9], [1,8,3], [1,9,7], [2,4,5], [2,5,7], [2,7,4], [2,8,9], [2,9,8], [3,5,9], [3,8,5], [3,9,6], [4,6,5], [4,7,6], [5,6,9], [5,8,7], [6,7,8]}. T_5 = {[1,1,2], [2,2,3], [3,3,4], [4,4,5], [5,5,6], [6,6,7], [7,7,1], [8,8,1], [9,9,1], [1,3,5], [1,4,6], [1,5,3], [1,6,4], [2,4,7], [2,5,8], [2,6,9], [2,7,4], [2,8,6], [2,9,5], [3,6,8], [3,7,9], [3,8,7], [3,9,6], [4,8,9], [4,9,8], [5,7,8], [5,9,7]}. Now, N_1 comes from T_1 with {[1,4,5], [1,5,4], [1,6,9], [1,7,8], [1,8,7], [1,9,6], [5,5,6], [6,6,7], [8,8,5], [9,9,4]} replaced by {[1,4,9]. [1,5,8], [1,6, 7], [1,7,6], [1,8,5], [1,9,4], [5,5,4], [6,6,5], [8,8,7], [9,9,6]}. N_2 comes from T_2 with {[3,4,8], [3,6,9], [3,8,4], [3,9,6], [4,4,6], [7,7,8], [8,8,9], [9,9,7]} replaced by {[3,4,6], [3,6,4], [3,8,9], [3,9,8], [4,4,8], [7,7,9], [8,8,7], [9,9, 6]}. By $\langle 1,2,3,1\rangle \subseteq T_1$,$\langle 1,2,3,4,1\rangle \subseteq T_4$, $\langle 1,2,3,4,5,1\rangle \subseteq T_3$, $\langle 4,8,5,6,7,9,4\rangle \subseteq T_1$, and $\langle 1,2,3,4,5,6,7,1\rangle \subseteq T_5$, we have 20, 21, 22, 23, 24 $\in J[9,0]$. From $|T_1 \cap N_1| = 17$, $|T_2 \cap N_2| = 19$ and Table 2, we have $J[9,0] = I[9,0]$.

Table 2

Intersection	Size	Intersection	Size
$T_1 \cap (34)(678)T_1$	1	$T_1 \cap (6798)T_1$	10
$T_1 \cap (34)(789)T_1$	2	$T_1 \cap (67)(89)T_1$	11
$T_1 \cap (56)(79)T_1$	4	$T_1 \cap (23)(6798)T_1$	13
$T_1 \cap (6789)T_1$	5	$T_1 \cap (24)(35)T_1$	14
$T_1 \cap (57)(689)T_1$	7	$T_1 \cap (45)(6798)T_1$	16
$T_1 \cap (789)T_1$	8		

For $v = 12$, using a similiar argment to Lemma 3.1, we obtain $J[12,0] \supseteq I[12,0] \setminus \{34,35\}$. $T_1 = B \cup C \cup D \cup$ {[1,6,10], [1,10,6], [1,7,11], [1,11,7], [1,8,12], [1,12,8], [2,5,10], [2,10,5], [2,8,11], [2,11,8], [3,8,10], [3,10,8], [4,7,10], [4,10,7], [4,8,9], [4,9,8], [1,1,5], [5,5,9], [9,9,1], [6,6,5], [7,7,6], [8,8,6], [5,7,8] [5,8,7], [10,10,9], [11,11,10], [12,12,10], [9,11,12], [9,12, 11]}, where B = {[2,6,12], [2,12,6], [2,7,9], [2,9,7], [3,6,9], [3,9,6], [3,7, 12], [3,12,7]}, C = {[3,5,11], [3,11,5], [4,5,12], [4,12,5], [4,6,11], [4,11,6]} and D = {[2,2,1], [3,3,2], [4,4,2], [1,3,4], [1,4,3]} Now, N_1 comes from T_1 by removing the blocks $B \cup C$ and replacing them with {[2,6,9], [2,9,6], [2,7,12], [2,12,7], [3,5,12], [3,12,5], [3,6,11], [3,11,6], [3,7,9], [3,9,7], [4,5,11], [4,11,5], [4,6,12], [4,12,6]}. N_2 comes from T_1 by removing the blocks $B \cup D$ and replacing them with {[2,6,9], [2,9,6], [2,7,12], [2,12,7], [3,6,12], [3,12,6], [3,7,9], [3,9,7], [3,3,1], [2,2,3], [4,4,3], [1,2,4], [1,4,2]}. Then $|T_1 \cap N_1| = 34$ and $|T_1 \cap N_2| = 35$. Thus, $J[12,0] = I[12,0]$.

Combining the above Lemmas 3.1, 3.2 and 3.3, we obtained the following results:

Theorem 3.4 *$J[v,0] = I[v,0]$, for $v \equiv 0 \pmod 3$.*

4 For the Class of EMTS(v, 1)

The spectrum of EMTS(v, 1) is $v \not\equiv 0 \pmod 3$. Since the smallest possible mutually balanced subsets of an EMTS(v, 1) are $\langle x, y\rangle$ (which can be changed to $\langle y, x\rangle$), it follows that $J[v,1] \subseteq I[v,1]$.

By using a similar proof to Lemmas 3.1 and 3.2 and replacing EMTS(n, 0), $I[n,0]$ and $J[v,0]$ by EMTS(n, 1), $I[n,1]$ and $J[v,1]$, respectively, for each $n = v, 2v, 2v+3$, we have

Lemma 4.1 *If $J[v,1] = I[v,1]$ and v is an integer ≥ 7 then $J[2v,1] = I[2v,1]$.*

Lemma 4.2 *If $J[v,1] = I[v,1]$ and v is an integer ≥ 5 then $J[2v+3,1] = I[2v+3,1]$.*

Hoffman and Lindner [4] proved that there exist two MTS(v) intersecting in r triples if and only if $r \in S_v = \{0, 1, 2, \ldots, v(v-1)/3 - 6, v(v-1)/3 - 4, v(v-1)/3\}$, for $v \equiv 0, 1 \pmod 3$, $v \neq 6$. If $v \not\equiv 0 \pmod 3$ and $v \neq 7$, there exist two MTS($v-1$), say (V, B_1) and (V, B_2), intersecting in r triples, where $r \in S_{v-1}$ and $V = \{1, 2, 3, \ldots, v-1\}$. Let $V^* = V \cup \{v\}$, $N = \{[2,2,v], [3,3,v], \ldots, [v-1, v-1, v], [v,v,1], [1,1,1]\}$, $B_1^* = B_1 \cup N$ and $B_2^* = B_2 \cup N$. Then (V^*, B_1^*) and (V^*, B_2^*) are EMTS(v, 1) and they have $r + v$ common triples. So,

$$v + S_{v-1} = \{v, v+1, \ldots, b_{v,1} - 6, b_{v,1} - 4, b_{v,1}\} \subseteq J[v,1]. \qquad (1)$$

By $\langle 2, v, 1\rangle \subseteq B_1^*$, we have $b_{v,1} - 2, b_{v,1} - 3 \in J[v,1]$. Thus we have

$$\{v, v+1, \ldots, b_{v,1} - 6, b_{v,1} - 4, b_{v,1} - 3, b_{v,1} - 2, b_{v,1}\} \subseteq J[v,1] \qquad (2)$$

, for $v \not\equiv 0 \pmod 3$ and $v \neq 7$.

Lemma 4.3 *$J[v,1] = I[v,1]$, for v =4, 5, 7, 8, 10, 11.*

Proof. For $v = 4$, let $T_1 = \{[3,3,2], [2,2,1], [1,1,1], [4,4,2], [1,3,4], [1,4,3]\}$. Now, N_1 comes from T_1 with $\{[3,3,2], [4,4,2], [1,3,4], [1,4,3]\}$ replaced by

$\{[3,3,1],[4,4,1],[2,3,4],[2,4,3]\}$. $3,2 \in J[4,1]$ follow by $\langle 3,2,1\rangle \subset T_1$. From $|T_1 \cap N_1| = 2$, $|T_1 \cap (23)T_1| = 1$ and $|T_1 \cap (12)T_1| = 0$, we have $J[4,1] = I[4,1]$.

For $v = 5$, let $T_1 = \{[2,2,1], [3,3,1], [5,5,1], [1,1,4], [4,4,4], [2,3,5], [2,4,3], [2,5,4], [3,4,5]\}$. and let N_1 comes from T_1 with $\{[4,2,5], [4,5,3], [4,3,2], [2,2,1], [3,3,1], [5,5,1]\}$ replaced by $\{[1,2,5], [1,5,3], [1,3,5], [2,2,4], [3,3,4], [5,5,4]\}$. $5,6,7 \in J[5,1]$ follow by the equation (2). By $|T_1 \cap N_1| = 4$, $|T_1 \cap (12)(45)T_1| = 0$, $|T_1 \cap (12)(345)T_1| = 1$ and $|T_1 \cap (45)T_1| = 2$,we have $J[5,1] = I[5,1]$.

For $v = 7$, let $T_1 = \{[1,1,1], [2,2,1], [3,3,2], [4,4,2], [5,5,2], [6,6,3], [7,7,4], [1,3,7], [1,4,3], [1,5,6], [1,6,4], [1,7,5], [2,6,7], [2,7,6], [3,4,5], [3,5,7], [4,6,5]\}$ and $T_2 = \{[1,1,1], [2,2,7], [7,7,4], [4,4,3], [3,3,6], [6,6,5], [5,5,2], [1,2,3], [1,3,2], [1,4,5], [1,5,4], [1,6,7], [1,7,6], [2,4,6], [2,6,4], [3,5,7], [3,7,5]\}$. By $\langle 2,7,4,3,6,5,2\rangle \subseteq T_2$ and $\langle 6,3,2,1\rangle \subseteq T_1$, we have 11, 13, 14, $15 \in J[7,1]$. From Table 3, we obtain $J[7,1] = I[7,1]$.

Table 3

Intersection	Size	Intersection	Size
$T_1 \cap (12)(45)(67)T_1$	0	$T_1 \cap (567)T_1$	6
$T_1 \cap (23)(67)T_1$	1	$T_1 \cap (56)T_1$	7
$T_1 \cap (23)(57)T_1$	2	$T_1 \cap (345)T_1$	8
$T_1 \cap (46)(57)T_1$	3	$T_1 \cap (69)T_1$	9
$T_1 \cap (467)T_1$	4	$T_1 \cap (35476)T_1$	10
$T_1 \cap (4567)T_1$	5	$T_1 \cap (345)(67)T_1$	12

Using a similar argument to Lemma 4.1 and 4.2, for $v = 8$, 10 or 11, we obtain $J[8,1] = I[8,1] \setminus \{13,15\}$, $J[10,1] = I[10,1] \setminus \{23\}$ and $J[11,1] = I[11,1] \setminus \{5,12,19,26,33\}$. From equation (2), we have $13,15 \in J[8,1]$, $23 \in J[10,1]$ and $12,19,26,33 \in J[11,1]$. For the remaining data 5 in the case of $v = 11$, let $T_1 = \{[1,1,5], [1,2,7], [1,3,4], [1,4,3], [1,6,9], [1,7,2], [1,8,11], [1,9,6], [1,11,8], [2,2,11], [2,3,6], [2,4,10], [2,5,9], [2,6,3], [2,9,5], [2,10,4], [3,3,8], [3,5,10], [3,7,9], [3,9,7], [3,10,5], [4,4,7], [4,5,8], [4,8,5], [4,9,11], [4,11,9], [5,5,6], [5,7,11], [5,11,7], [6,6,4], [6,7,8], [6,8,7], [6,10,11], [6,11,10], [7,7,10], [8,8,2], [8,9,10], [8,10,9], [9,9,9], [10,10,1], [11,11,3]\}$. We can obtain $|T_1 \cap (6,7)(8,10)(9,11)T_1| = 5$. Thus $J[i,1] = I[i,1]$, for $i = 8,10,11$.

Combining the above Lemmas 4.1, 4.2 and 4.3, we obtained the following results:

Theorem 4.4 $J[v,1] = I[v,1]$, *for* $v \not\equiv 0 \pmod 3$.

5 Conclusions.

From Theorems 3.4 and 4.4, we obtained the following results:

Main Theorem $J[v, 0] = I[v, 0]$, for $v \equiv 0 (\text{mod } 3)$, and $J[v, 1] = I[v, 1]$, for $v \not\equiv 0 (\text{mod } 3)$.

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