ON THE GENERALIZED FIBONACCI MATRIX OF ORDER 2^k SEQUENCE $\{G_n^{(2^k)}\}$

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1. Introduction

In this paper, we construct the generalized Fibonacci matrix of order 2^k sequence $\{G_n^{(2^k)}\}$ by use of the generalized Fibonacci sequence $\{G_n\}$, and study its properties.

The generalized sequence $\{W_n\}$ is defined for all integers n by the recurrence relation (see[1])

$$W_{n+2} = pW_{n+1} + qW_n \tag{1}$$

in which $W_0 = a$, $W_1 = b$, where a, b, p, q are arbitrary integers. Special cases of $\{W_n\}$ which interest us are

1> The Fibonacci sequence $\{F_n\}$: p=1, q=1, a=0, b=1

2> The Lucas sequence $\{L_n\}$: p=1, q=1, a=2, b=1 (2)

3> The Pell sequence $\{P_n\}$: p=2, q=1, a=0, b=1

4> The Pell-Lucas sequence $\{Q_n\}$: p=2, q=1, a=2, b=2

They have something in common, this is q=1. When q=1, the formual (1) can be written

$$G_{n+1} = pG_n + G_{n-1} \tag{3}$$

in which $G_0 = a$, $G_1 = b$.

Then, this $\{G_n\}$ is called the generalized Fibonacci sequence, and it is defined for all integers n by the recurrence relation (3). Rule (3) can be used to extend the sequence backwards, thus

$G_{-1} = G_1 - pG_0$, $G_{-2} = G_0 - pG_{-1}$, ⋯⋯ and so that

$$G_{-(n+1)} = G_{-(n-1)} - pG_{-n} \tag{4}$$

This produces

n :	……	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	……
F_n :	……	-8	5	-3	2	-1	1	0	1	1	2	3	5	8	……
L_n :	……	18	-11	7	-4	3	-1	2	1	3	4	7	11	18	……
P_n :	……	-70	29	-12	5	-2	1	0	1	2	5	12	29	70	……
Q_n :	……	198	-82	34	-14	6	-2	2	2	6	14	34	82	198	……

and generally

$$G_{-n} = \begin{cases} (-1)^{n+1} G_n , & \text{when} \quad a = 0 \\ \\ (-1)^n G_n , & \text{when} \quad a \neq 0 \end{cases} \tag{5}$$

2. To Construct the Generalized Fibonacci Matrix of Order 2^k Sequence $\{G_n^{(2^k)}\}$

Now we construct the generalized Fibonacci matrix of order 2^k sequence $\{G_n^{(2^k)}\}$ by use of the generalized Fibonacci matrixes. We let $G_n^{(2)}$ is a matrix of order 2 as follows:

$$G_n^{(2)} = \begin{pmatrix} G_{n+1} & G_n \\ G_n & G_{n-1} \end{pmatrix} \qquad \text{(where } n \geqslant 0 \text{)} \tag{6}$$

then

$$pG_n^{(2)} + G_{n-1}^{(2)} = p\begin{pmatrix} G_{n+1} & G_n \\ G_n & G_{n-1} \end{pmatrix} + \begin{pmatrix} G_n & G_{n-1} \\ G_{n-1} & G_{n-2} \end{pmatrix} = \begin{pmatrix} G_{n+2} & G_{n+1} \\ G_{n+1} & G_n \end{pmatrix} = G_{n+1}^{(2)}$$

Hence, we obtain the generalized Fibonacci matrix of order 2 sequence $\{G_n^{(2)}\}$. It is defined for all $n \geqslant 1$ by the recurrence relation as follows:

$$G_{n+1}^{(2)} = pG_n^{(2)} + G_{n-1}^{(2)} \qquad (n \geqslant 1) \tag{7}$$

where

$$G_0^{(2)} = \begin{pmatrix} G_1 & G_0 \\ G_0 & G_{-1} \end{pmatrix} , \qquad G_1^{(2)} = \begin{pmatrix} G_2 & G_1 \\ G_1 & G_0 \end{pmatrix}$$

Rule (7) can be used to extend the sequence backwards, thus

$G_{-1}^{(2)} = G_1^{(2)} - pG_0^{(2)}$, $G_{-2}^{(2)} = G_0^{(2)} - pG_{-1}^{(2)}$, …… , and so that

$$G^{(2)}_{-(n+1)} = G^{(2)}_{-(n-1)} - pG^{(2)}_{-n} \qquad (n \geqslant 0) \qquad (8)$$

This produces

$$G^{(2)}_{-1} = \begin{pmatrix} G_0 & G_{-1} \\ G_{-1} & G_{-2} \end{pmatrix} \quad , \quad G^{(2)}_{-2} = \begin{pmatrix} G_{-1} & G_{-2} \\ G_{-2} & G_{-3} \end{pmatrix} \quad ,\cdots\cdots$$

and generally

$$G^{(2)}_{-n} = \begin{pmatrix} G_{-(n-1)} & G_{-n} \\ G_{-n} & G_{-(n+1)} \end{pmatrix} \qquad (n \geqslant 0) \qquad (9)$$

Again, we let the generalized Fibonacci matrix of order 4 $G^{(4)}_n$ is equal to a partitioned matrix :

$$G^{(4)}_n = \begin{pmatrix} G^{(2)}_{n+1} & G^{(2)}_n \\ G^{(2)}_n & G^{(2)}_{n-1} \end{pmatrix} \qquad (n \geqslant 0) \qquad (10)$$

then

$$pG^{(4)}_n + G^{(4)}_{n-1} = p\begin{pmatrix} G^{(2)}_{n+1} & G^{(2)}_n \\ G^{(2)}_n & G^{(2)}_{n-1} \end{pmatrix} + \begin{pmatrix} G^{(2)}_n & G^{(2)}_{n-1} \\ G^{(2)}_{n-1} & G^{(2)}_{n-2} \end{pmatrix} = \begin{pmatrix} G^{(2)}_{n+2} & G^{(2)}_{n+1} \\ G^{(2)}_{n+1} & G^{(2)}_n \end{pmatrix} = G^{(4)}_{n+1}$$

Hence, we obtain the generalized Fibonacci matrix of order 4 sequence $\{G^{(4)}_n\}$. It is defined for all $n \geqslant 1$ by the recurrence relation as follows:

$$G^{(4)}_{n+1} = pG^{(4)}_n + G^{(4)}_{n-1} \qquad (n \geqslant 1) \qquad (11)$$

where $$G^{(4)}_0 = \begin{pmatrix} G^{(2)}_1 & G^{(2)}_0 \\ G^{(2)}_0 & G^{(2)}_{-1} \end{pmatrix} \quad , \quad G^{(4)}_1 = \begin{pmatrix} G^{(2)}_2 & G^{(2)}_1 \\ G^{(2)}_1 & G^{(2)}_0 \end{pmatrix}$$

Rule (11) can be used to extend the sequence backwards, thus

$G^{(4)}_{-1} = G^{(4)}_1 - pG^{(4)}_0$, $G^{(4)}_{-2} = G^{(4)}_0 - pG^{(4)}_{-1}$,$\cdots\cdots$ and so that

$$G^{(4)}_{-(n+1)} = G^{(4)}_{-(n-1)} + pG^{(4)}_{-n} \qquad (n \geqslant 0) \qquad (12)$$

This produces

$$G_{-1}^{(4)}=\begin{pmatrix} G_0^{(2)} & G_{-1}^{(2)} \\ G_{-1}^{(2)} & G_{-2}^{(2)} \end{pmatrix} \quad , \quad G_{-2}^{(4)}=\begin{pmatrix} G_{-1}^{(2)} & G_{-2}^{(2)} \\ G_{-2}^{(2)} & G_{-3}^{(2)} \end{pmatrix} \quad , \cdots\cdots$$

and generally

$$G_{-n}^{(4)}=\begin{pmatrix} G_{-(n-1)}^{(2)} & G_{-n}^{(2)} \\ G_{-n}^{(2)} & G_{-(n+1)}^{(2)} \end{pmatrix} \qquad (n\geqslant 0) \qquad (13)$$

Thus, and so on and so forth, let the generalized Fibonacci matrix of order 2^k $G_n^{(2^k)}$ is equal to a partitioned matrix:

$$G_n^{(2^k)}=\begin{pmatrix} G_{n+1}^{(2^{k-1})} & G_n^{(2^{k-1})} \\ G_n^{(2^{k-1})} & G_{n-1}^{(2^{k-1})} \end{pmatrix} \qquad (n\geqslant 0, k\geqslant 1) \qquad (14)$$

then

$$pG_n^{(2^k)}+G_{n-1}^{(2^k)}=p\begin{pmatrix} G_{n+1}^{(2^{k-1})} & G_n^{(2^{k-1})} \\ G_n^{(2^{k-1})} & G_{n-1}^{(2^{k-1})} \end{pmatrix}+\begin{pmatrix} G_n^{(2^{k-1})} & G_{n-1}^{(2^{k-1})} \\ G_{n-1}^{(2^{k-1})} & G_{n-2}^{(2^{k-1})} \end{pmatrix}=\begin{pmatrix} G_{n+2}^{(2^{k-1})} & G_{n+1}^{(2^{k-1})} \\ G_{n+1}^{(2^{k-1})} & G_n^{(2^{k-1})} \end{pmatrix}=G_{n+1}^{(2^k)}$$

Hence, we obtain the generalized Fibonacci matrix of order 2^k sequence $\{G_n^{(2^k)}\}$. It is defined for all $n\geqslant 1$ by the recurrence relation as follows:

$$G_{n+1}^{(2^k)}=pG_n^{(2^k)}+G_{n-1}^{(2^k)} \qquad (n\geqslant 1,\ k\geqslant 1) \qquad (15)$$

where $$G_0^{(2^k)}=\begin{pmatrix} G_1^{(2^{k-1})} & G_0^{(2^{k-1})} \\ G_0^{(2^{k-1})} & G_{-1}^{(2^{k-1})} \end{pmatrix} \quad , \quad G_1^{(2^k)}=\begin{pmatrix} G_2^{(2^{k-1})} & G_1^{(2^{k-1})} \\ G_1^{(2^{k-1})} & G_0^{(2^{k-1})} \end{pmatrix} .$$

Rule (15) can be used to extend the sequence backwards, thus

$G_{-1}^{(2^k)}=G_1^{(2^k)}-pG_0^{(2^k)}$, $G_{-2}^{(2^k)}=G_0^{(2^k)}-pG_{-1}^{(2^k)}$,…… and so that

$$G_{-(n+1)}^{(2^k)}=G_{-(n-1)}^{(2^k)}-pG_{-n}^{(2^k)} \qquad (n\geqslant 0,\ k\geqslant 1) \qquad (16)$$

This produces

$$G_{-1}^{(2^k)}=\begin{pmatrix} G_0^{(2^{k-1})} & G_{-1}^{(2^{k-1})} \\ G_{-1}^{(2^{k-1})} & G_{-2}^{(2^{k-1})} \end{pmatrix} \quad , \quad G_{-2}^{(2^k)}=\begin{pmatrix} G_{-1}^{(2^{k-1})} & G_{-2}^{(2^{k-1})} \\ G_{-2}^{(2^{k-1})} & G_{-3}^{(2^{k-1})} \end{pmatrix} \quad , \cdots\cdots$$

and generally

$$G_{-n}^{(2^k)} = \begin{pmatrix} G_{-(n-1)}^{(2^{k-1})} & G_{-n}^{(2^{k-1})} \\ G_{-n}^{(2^{k-1})} & G_{-(n+1)}^{(2^{k-1})} \end{pmatrix} \qquad (n \geq 0, k \geq 1) \qquad (17)$$

Special cases of $\{G_n^{(2^k)}\}$ which interest us are

1> The Fibonacci matrix of order 2^k sequence $\{F_n^{(2^k)}\}$, when p=1, a=0, b=1.

2> The Lucas matrix of order 2^k sequence $\{L_n^{(2^k)}\}$, when p=1, a=2, b=1.

3> The Pell matrix of order 2^k sequence $\{P_n^{(2^k)}\}$, when p=2, a=0, b=1.

4> The Pell-Lucas matrix of order 2^k sequence $\{Q_n^{(2^k)}\}$, when p=2, a=2, b=2.

Now we obtain a basic property of the generalized Fibonacci matrix of order 2^k sequence $\{G_n^{(2^k)}\}$ by the equation (17) and (5).

Theorem 1 : The generalized Fibonacci matrix of order 2^k sequence $\{G_n^{(2^k)}\}$ is satisfied with

1> if $G_0 = a = 0$, then $G_{-n}^{(2^k)} = (-1)^{n+1} E_{2^k} G_n^{(2^k)} E_{2^k}$, (18)

2> if $G_0 = a \neq 0$, then $G_{-n}^{(2^k)} = (-1)^n E_{2^k} G_n^{(2^k)} E_{2^k}$, (19)

where E_{2^k} is equal to a partitioned matrix

$$E_{2^k} = \begin{pmatrix} O_{2^{k-1}} & E_{2^{k-1}} \\ -E_{2^{k-1}} & O_{2^{k-1}} \end{pmatrix}, \text{ when } k=1 \ E_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$O_{2^{k-1}}$ is a zero matrix of order 2^{k-1}.

Proof: We prove (18) by induction.

If $G_0 = a = 0$, and when k=1, we have

$$G_{-n}^{(2)} = \begin{pmatrix} G_{-(n-1)} & G_{-n} \\ G_{-n} & G_{-(n+1)} \end{pmatrix} = (-1)^{n+1} \begin{pmatrix} -G_{n-1} & G_n \\ G_n & -G_{n+1} \end{pmatrix}$$

$$= (-1)^{n+1} \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} G_{n+1} & G_n \\ G_n & G_{n-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] = (-1)^{n+1} E_2 G_n^{(2)} E_2$$

Then, when k=1, the formula (18) is true. When k=2, we have

$$G_{-n}^{(4)} = \begin{pmatrix} G_{-(n-1)}^{(2)} & G_{-n}^{(2)} \\ G_{-n}^{(2)} & G_{-(n+1)}^{(2)} \end{pmatrix} = (-1)^{n+1} \begin{pmatrix} -E_2 G_{n-1}^{(2)} E_2 & E_2 G_n^{(2)} E_2 \\ E_2 G_n^{(2)} E_2 & -E_2 G_{n+1}^{(2)} E_2 \end{pmatrix}$$

$$= (-1)^{n+1} \left[\begin{pmatrix} 0_2 & E_2 \\ -E_2 & 0_2 \end{pmatrix} \begin{pmatrix} G_{n+1}^{(2)} & G_n^{(2)} \\ G_n^{(2)} & G_{n-1}^{(2)} \end{pmatrix} \begin{pmatrix} 0_2 & E_2 \\ -E_2 & 0_2 \end{pmatrix} \right]$$

$$= (-1)^{n+1} E_4 G_n^{(4)} E_4 , \qquad \text{where } E_4 = \begin{pmatrix} 0_2 & E_2 \\ -E_2 & 0_2 \end{pmatrix} .$$

Then, when k=2, the formula (18) is true. Assuume the formula (18) to be true for k=m-1. In a similar manner, we can prove that the formula (18) is true for k=m.

To sum up, the formula (18) is proved.

The formula (19) can be proved with the same method.

Corollary 1: When p=1, a=0 and b=1, the formula (18) can be written

$$F_{-n}^{(2^k)} = (-1)^{n+1} E_{2^k} F_n^{(2^k)} E_{2^k} \tag{20}$$

Corollary 2: When p=1, a=2 and b=1, the formula (19) can be written

$$L_{-n}^{(2^k)} = (-1)^n E_{2^k} L_n^{(2^k)} E_{2^k} \tag{21}$$

Corollary 3: When p=2, a=0 and b=1, the formula (18) can be written

$$P_{-n}^{(2^k)} = (-1)^{n+1} E_{2^k} P_n^{(2^k)} E_{2^k} \tag{22}$$

Corollary 4: When p=2, a=2 and b=2, the formula (19) can be written

$$Q_{-n}^{(2^k)} = (-1)^n E_{2^k} Q_n^{(2^k)} E_{2^k} \tag{23}$$

3. The sum formula of $\{G_n^{(2^k)}\}$

Theorem 2: The sum formula of $\{G_n^{(2^k)}\}$ is as follows:

$$\sum_{i=1}^{n} G_i^{(2^k)} = \frac{1}{p}\left(G_n^{(2^k)} + G_{n+1}^{(2^k)} - G_0^{(2^k)} - G_1^{(2^k)} \right) \quad (n \geqslant 1, k \geqslant 1) \tag{24}$$

Proof: Because

$$p\sum_{i=1}^{n} G_i^{(2^k)} = \sum_{i=2}^{n+1} G_i^{(2^k)} - \sum_{i=0}^{n-1} G_i^{(2^k)} = G_n^{(2^k)} + G_{n+1}^{(2^k)} - G_0^{(2^k)} - G_1^{(2^k)}$$

therefore $$\sum_{i=1}^{n} G_i^{(2^k)} = \frac{1}{p}\left(G_n^{(2^k)} + G_{n+1}^{(2^k)} - G_0^{(2^k)} - G_1^{(2^k)} \right)$$

Corollary 5 : When p=1, a=0 and b=1, the formula (24) can be written

$$\sum_{i=1}^{n} F_i^{(2^k)} = F_{n+2}^{(2^k)} - F_2^{(2^k)} \qquad (n\geqslant 1, k\geqslant 1) \qquad (25)$$

Corollary 6: When p=1, a=2 and b=1, the formula (24) can be written

$$\sum_{i=1}^{n} L_i^{(2^k)} = L_{n+2}^{(2^k)} - L_2^{(2^k)} \qquad (n\geqslant 1, k\geqslant 1) \qquad (26)$$

Corollary 7: When p=2, a=0 and b=1, the formula (24) can be written

$$\sum_{i=1}^{n} P_i^{(2^k)} = \frac{1}{2}\left(P_n^{(2^k)} + P_{n+1}^{(2^k)} - P_0^{(2^k)} - P_1^{(2^k)} \right) \qquad (n\geqslant 1, k\geqslant 1) \qquad (27)$$

Corollary 8: When p=2, a=2 and b=2, the formula (24) can be written

$$\sum_{i=1}^{n} Q_i^{(2^k)} = \frac{1}{2}\left(Q_n^{(2^k)} + Q_{n+1}^{(2^k)} - Q_0^{(2^k)} - Q_1^{(2^k)} \right) \qquad (n\geqslant 1, k\geqslant 1) \qquad (28)$$

4. Other properties of $\{G_n^{(2^k)}\}$

Theorem 3 :

$$G_n^{(2^{k+1})} = G_1^{(2^{k+1})} \begin{pmatrix} pI_{2^k} & I_{2^k} \\ I_{2^k} & 0_{2^k} \end{pmatrix}^{n-1} \qquad (n\geqslant 1, k\geqslant 1) \qquad (29)$$

where I_{2^k} is a unit matrix of order 2^k, 0_{2^k} is a zero matrix of order 2^k.

Proof :

$$G_n^{(2^{k+1})} = \begin{pmatrix} G_{n+1}^{(2^{k-1})} & G_n^{(2^{k-1})} \\ G_n^{(2^{k-1})} & G_{n-1}^{(2^{k-1})} \end{pmatrix} = \begin{pmatrix} G_n^{(2^{k-1})} & G_{n-1}^{(2^{k-1})} \\ G_{n-1}^{(2^{k-1})} & G_{n-2}^{(2^{k-1})} \end{pmatrix} \begin{pmatrix} pI_{2^k} & I_{2^k} \\ I_{2^k} & 0_{2^k} \end{pmatrix} = \cdots\cdots$$

$$= \begin{pmatrix} G_3^{(2^k)} & G_2^{(2^k)} \\ G_2^{(2^k)} & G_1^{(2^k)} \end{pmatrix} \begin{pmatrix} pI_{2^k} & I_{2^k} \\ I_{2^k} & 0_{2^k} \end{pmatrix}^{n-2}$$

$$= G_1^{(2^{k+1})} \begin{pmatrix} pI_{2^k} & I_{2^k} \\ I_{2^k} & 0_{2^k} \end{pmatrix}^{n-1}$$

Corollary 9: When p=1, a=0 and b=1, the formula (29) can be written

$$F_n^{(2^{k+1})} = F_1^{(2^{k+1})}\begin{pmatrix} I_{2^k} & I_{2^k} \\ I_{2^k} & 0_{2^k} \end{pmatrix}^{n-1} \qquad (n \geqslant 1, k \geqslant 1) \qquad (30)$$

Corollary 10: When p=1, a=2 and b=1, the formula (29) can be written

$$L_n^{(2^{k+1})} = L_1^{(2^{k+1})}\begin{pmatrix} I_{2^k} & I_{2^k} \\ I_{2^k} & 0_{2^k} \end{pmatrix}^{n-1} \qquad (n \geqslant 1, k \geqslant 1) \qquad (31)$$

Corollary 11: When p=2, a=0 and b=1, the formula (29) can be written

$$P_n^{(2^{k+1})} = P_1^{(2^{k+1})}\begin{pmatrix} 2I_{2^k} & I_{2^k} \\ I_{2^k} & 0_{2^k} \end{pmatrix}^{n-1} \qquad (n \geqslant 1, k \geqslant 1) \qquad (32)$$

Corollary 12: When p=2, a=2 and b=2, the formula (29) can be written

$$Q_n^{(2^{k+1})} = Q_1^{(2^{k+1})}\begin{pmatrix} 2I_{2^k} & I_{2^k} \\ I_{2^k} & 0_{2^k} \end{pmatrix}^{n-1} \qquad (n \geqslant 1, k \geqslant 1) \qquad (33)$$

Theorem 4 :

$$\sum_{i=1}^{n} G_{2i-1}^{(2^k)} = \frac{1}{p}\left(G_{2n}^{(2^k)} - G_0^{(2^k)}\right) \qquad (n \geqslant 1, k \geqslant 1) \qquad (34)$$

Proof :

$$p\sum_{i=1}^{n} G_{2i-1}^{(2^k)} = \sum_{i=1}^{n} G_{2i}^{(2^k)} - \sum_{i=1}^{n} G_{2i-2}^{(2^k)}$$

$$= \sum_{i=1}^{n} G_{2i}^{(2^k)} - \sum_{i=0}^{n-1} G_{2i}^{(2^k)}$$

$$= G_{2n}^{(2^k)} - G_0^{(2^k)}$$

Hence

$$\sum_{i=1}^{n} G_{2i-1}^{(2^k)} = \frac{1}{p}\left(G_{2n}^{(2^k)} - G_0^{(2^k)}\right)$$

Corollary 13: When p=1, a=0 and b=1, the formula (34) can be writte

$$\sum_{i=1}^{n} F_{2i-1}^{(2^k)} = F_{2n}^{(2^k)} - F_{0}^{(2^k)} \qquad (n\geqslant 1, k\geqslant 1) \qquad (35)$$

Corollary 14: When p=1, a=2 and b=1, the formula (34) can be written

$$\sum_{i=1}^{n} L_{2i-1}^{(2^k)} = L_{2n}^{(2^k)} - L_{0}^{(2^k)} \qquad (n\geqslant 1, k\geqslant 1) \qquad (36)$$

Corollary 15: When p=2, a=0 and b=1, the formula (34) can be written

$$\sum_{i=1}^{n} P_{2i-1}^{(2^k)} = \frac{1}{2}(P_{2n}^{(2^k)} - P_{0}^{(2^k)}) \qquad (n\geqslant 1, k\geqslant 1) \qquad (37)$$

Corollary 16: When p=2, a=2 and b=2, the formula (34) can be written

$$\sum_{i=1}^{n} Q_{2i-1}^{(2^k)} = \frac{1}{2}(Q_{2n}^{(2^k)} - Q_{0}^{(2^k)}) \qquad (n\geqslant 1, k\geqslant 1) \qquad (38)$$

Theorem 5:

$$\sum_{i=1}^{n} G_{2i}^{(2^k)} = \frac{1}{p}(G_{2n+1}^{(2^k)} - G_{1}^{(2^k)}) \qquad (n\geqslant 1, k\geqslant 1) \qquad (39)$$

Proof:

$$p\sum_{i=1}^{n} G_{2i}^{(2^k)} = \sum_{i=1}^{n} G_{2i+1}^{(2^k)} - \sum_{i=1}^{n} G_{2i-1}^{(2^k)}$$

$$= \sum_{i=1}^{n} G_{2i+1}^{(2^k)} - \sum_{i=0}^{n-1} G_{2i+1}^{(2^k)}$$

$$= G_{2n+1}^{(2^k)} - G_{1}^{(2^k)}$$

Hence

$$\sum_{i=1}^{n} G_{2i}^{(2^k)} = \frac{1}{p}(G_{2n+1}^{(2^k)} - G_{1}^{(2^k)}) .$$

Corollary 17: When p=1, a=0and b=1, the formula (39) can be written

$$\sum_{i=1}^{n} F_{2i}^{(2^k)} = F_{2n+1}^{(2^k)} - F_{1}^{(2^k)} \qquad (n\geqslant 1, k\geqslant 1) \qquad (40)$$

Corollary 18: When p=1, a=2and b=1, the formula (39) can be written

$$\sum_{i=1}^{n} L_{2i}^{(2^k)} = L_{2n+1}^{(2^k)} - L_{1}^{(2^k)} \qquad (n\geqslant 1, k\geqslant 1) \qquad (41)$$

Corollary 19: When p=2, a=0and b=1, the formula (39) can be written

$$\sum_{i=1}^{n} P_{2i}^{(2^k)} = \frac{1}{2}(P_{2n+1}^{(2^k)} - P_{1}^{(2^k)}) \qquad (n\geqslant 1, k\geqslant 1) \qquad (42)$$

Corollary 20: When p=2, a=2and b=2, the formula (39) can be written

$$\sum_{i=1}^{n} Q_{2i}^{(2^k)} = \frac{1}{2}(Q_{2n+1}^{(2^k)} - Q_1^{(2^k)}) \qquad (n\geq 1, k\geq 1) \qquad (43)$$

Theorem 6:

$$\sum_{i=1}^{2n} (-1)^i G_i^{(2^k)} = \frac{1}{p}(G_{2n+1}^{(2^k)} - G_{2n}^{(2^k)} - G_1^{(2^k)} + G_0^{(2^k)}) \quad (n\geq 1, k\geq 1) \quad (44)$$

Proof: Subtraction of (34) from (39) produces (44).

Corollary 21: When p=1, a=0 and b=1, the formula (44) can be written

$$\sum_{i=1}^{2n}(-1)^i F_i^{(2^k)} = F_{2n-1}^{(2^k)} - F_1^{(2^k)} + F_0^{(2^k)} \qquad (n\geq 1, k\geq 1) \qquad (45)$$

Corollary 22: When p=1, a=2 and b=1, the formula (44) can be written

$$\sum_{i=1}^{2n} (-1)^i L_i^{(2^k)} = L_{2n-1}^{(2^k)} - L_1^{(2^k)} + L_0^{(2^k)} \qquad (n\geq 1, k\geq 1) \qquad (46)$$

Corollary 23: When p=2, a=0 and b=1, the formula (44) can be written

$$\sum_{i=1}^{2n} (-1)^i P_i^{(2^k)} = \frac{1}{2}(P_{2n+1}^{(2^k)} - P_{2n}^{(2^k)} - P_1^{(2^k)} + P_0^{(2^k)}) \quad (n\geq 1, k\geq 1) \quad (47)$$

Corollary 24: When p=2, a=2 and b=2, the formula (44) can be written

$$\sum_{i=1}^{2n} (-1)^i Q_i^{(2^k)} = \frac{1}{2}(Q_{2n+1}^{(2^k)} - Q_{2n}^{(2^k)} - Q_1^{(2^k)} + Q_0^{(2^k)}) \quad (n\geq 1, k\geq 1) \quad (48)$$

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