Diagonal flips of pseudo triangulations on the sphere

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Abstract

A plane graph is an embedding of a planar graph into the sphere which may have multiple edges and loops. A face of a plane graph is said to be a pseudo triangle if either the boundary of it has three distinct edges or the boundary of it consists of a loop and a pendant edge. A plane pseudo triangulation is a connected plane graph of which each face is a pseudo triangle. If a plane pseudo triangulation has neither a multiple edge nor a loop, then it is a plane triangulation. As a generalization of the diagonal flip of a plane triangulation, the diagonal flip of a plane pseudo triangulation is naturally defined. In this paper we show that any two plane pseudo triangulations of order n can be transformed into each other, up to ambient isotopy, by at most $14n - 64$ diagonal flips if $n \geq 7$. We also show that for a positive integer $n \geq 5$, there are two plane pseudo triangulations with n vertices such that at least $4n - 15$ diagonal flips are needed to transform into each other.

1. Introduction

In this paper we consider finite and planar graphs which are not necessary simple, i.e., which may have mutiple edges

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and loops. Let G be a graph. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. An edge other than a loop is called a *link* of G. For a vertex x of G, we denote the set of links incident to x and the set of loops incident to x by $Link(x; G)$ and $Loop(x; G)$, respectively. The sum of the number of links and 2 times of the number of loops incident to x is called the *degree* of x, and is denoted by $deg_G(x)$. Namely, $deg_G(x) = |Link(x; G)| + 2|Loop(x; G)|$. A graph is said to be regular if each vertex of it has the same degree. A link is called a *pendant edge* if one end vertex of it has degree 1. A loop incident to x is somtimes simply called an x-loop.

A *plane graph* is an embedding of a planar graph into the sphere. From now on, we consider plane graphs. Let G be a connected plane graph. When a face has a bridge in its boundary, then the bridge is counted twice in its boundary. A face of G is called a *pseudo triangle* if the number of edges in its boundary is 3. We observe that the boundary of a pseudo triangle consists of either 3 distinct edges or a loop and a pendant edge.

A connected plane graph is said to be a *plane pseudo triangulation* if each face of it is a pseudo triangle. We note that each plane pseudo triangulation has at least 3 vertices and if a plane pseudo triangulation is simple, then it is a triangulation on the sphere. Also, we observe that each plane pseudo triangulation has no bridges other than pendant edges. There are 2 plane pseudo triangulations of order 3 and 6 plane pseudo triangulations of order 4. (Fig. 1)

We can define a diagonal flip on a plane pseudo triangulation as follows; it is a natural generalization of a flip on a triangulation. Let G be a plane pseudo triangulation and e be an edge of G which is not pendant. Then e is contained in the boundaries of two faces, say F_1 and F_2.

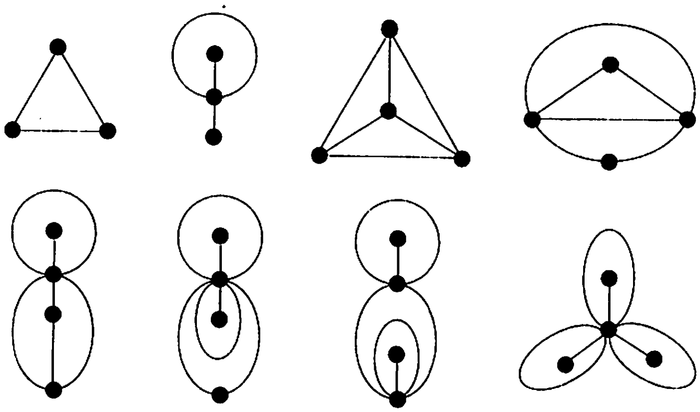

Fig.-1

Assume F_1 is bounded by the edges a, b, and e and F_2 is bounded by the edges c, d, and e. Then $G - e$ has a face F bounded by the 4 edges a, b, c, and d. We assume that the 4 edges a, b, c, and d are located in this order in the boundary of F. In this situation, we define a local operation on G called a diagonal flip around e as follows.

(1) Remove the edge e from G.

(2) Add a new edge e' so that the resulting graph has two new pseudo triangles the boundaries of which consist of $\{a, d, e'\}$ and $\{b, c, e'\}$, respectively. Fig. 2 ilustrates an example of a diagonal flip.

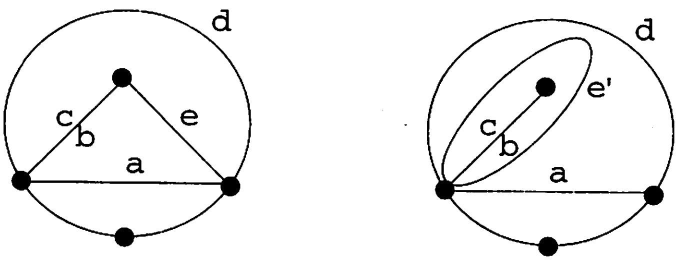

Fig.-2

The new edge e' is called the successor of e, and e is said to be the predecessor of e'. We also say that e and e' correspond to each other by a diagonal flip. Note that we do not apply this operation around a pendant edge.

For triangulations on the sphere, the following problem was studied.

Problem For any given two triangulations G and G' on the sphere of the same order n, find a finite sequence of diagonal flips which transforms G into G' up to ambient isotopy.

If an algorithm for this problem produces sequences of diagonal flips of length $O(f(n))$, then we say that it is an $O(f(n))$ algorithm. The following result has been known for many years.

Theorem A(Wagner[2]) Any two triangulations on the sphere of the same order can be transformed into each other, up to ambient isotopy, by a finite sequence of diagonal flips.

From the proof of Theorem A, one can derive an $O(n^2)$ algorithm for the problem. Recently one of the authors proved the following result and gave an $O(n)$ algorithm for the problem.

Theorem B (Komuro[1]) Any two triangulations on the sphere of the same order n can be transformed into each other, up to ambient isotopy, by at most $8n - 54$ diagonal flips if $n \geq 13$ and by at most $8n - 48$ diagonal flips if $n \geq 7$.

The purpose of this paper is to give an answer to the problem for plane pseudo triangulations. Our main results are the following two Theorems.

Theorem 1 Any two plane pseudo triangulations of the same order n can be transformed into each other, up to ambient isotopy, by at most $14n - 64$ diagonal flips if $n \geq 7$.

Theorem 2 For any integer $n \geq 4$, there are plane pseudo triangulations G and G' of order n such that at least $\max\{4n - 15, 3n - 8\}$ diagonal flips are needed to transform G into G'.

Before closing this section, we introduce a family of plane triangle graphs called flowers which plays an important role in the proof of our Theorems.

A plane pseudo triangulation is said to be a *flower* if it has a vertex x such that each vertex other than x has degree 1. The vertex x is called the *center* of the flower.

There is a flower of order 3 whose edge set consists of a loop and 2 pendant edges. And there is a flower of order 4 whose edge set consists of 3 loops and 3 pendant edges(See Fig.1.). Note that we can construct all flowers inductively as follows. Let G be a flower of order n and let x be the center of it. At first, remove a vertex other than x from G and next, into the face of the removed vertex, add two x-loops each of which contains a pendant edge in it. Then the resulting graph is a flower of order $n + 1$.

2. Proof of Theorem 1

In this section we prove Theorem 1. Before giving the proof, we need to introduce some notation. We denote the open neighbourhood of x by $N_G(x)$. Note that $N_G(x)$ does not include x. Let G be a connected plane graph and let F be a face of G. We denote the boundary of F by ∂F. We write $V(F)$ and $E(F)$ for $V(\partial F)$ and $E(\partial F)$, respectively. A face F whose boundary consists of a loop and a pendant edge is called a reduced triangle. We classify pseudo triangles according to the number of vertices of their boundaries as follows. We call a pseudo triangle F a 3-triangle, a 2-triangle, and a 1-triangle if $|V(F)| = 3$, $|V(F)| = 2$, and $|V(F)| = 1$, respectively. By this definition, a reduced triangle is a 2-triangle. We denote the type of a face F by $type(F)$. We observe that a 3-triangle is bounded by 3 links, a 2-triangle other than a reduced triangle is bounded by 2 links and 1 loop, and a 1-triangle is bounded by 3 loops. Note that there is no pseudo triangle bounded by 1 link and 2 loops. In Fig.3, each face F_i is an i-triangle.

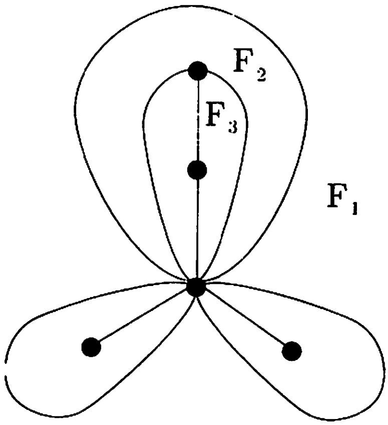

Fig.3

Let e be an edge of G which is in the boundary of two distinct faces, say F_1 and F_2. Then e is said to be a ($type(F_1)$, $type(F_2)$)-edge. Moreover, if e is a link, then e is called a ($type(F_1)$,$type(F_2)$)-link and if e is a loop, then e is called a ($type(F_1)$,$type(F_2)$)-loop.

Again let e be an edge of G which is in the boundary of two distinct faces F_1 and F_2. Let F be the face of $G - e$ bounded by 4 edges arising from F_1 and F_2. We consider the subgraph of G whose vertex set is $V(F)$ and whose edge set is $E(F)$. If this subgraph is regular, then e is said to be regular. If an edge

e is not regular, then e is said to be irregular. If e is a (1,1)-loop, then $V(F)$ consists of the only one vertex. Hence each (1,1)-loop is regular. On the other hand, we observe that each (3,2)-link, each (2,2)-loop, and each (2,1)-loop are irregular. There may be regular (3,3)-links and irregular (3,3)-links in a pseudo triangulation. Also, there may be regular (2,2)-links and irregular (2,2)-links. Since a 3-triangle has no loops in its boundary, and a 1-triangle has no links in its boundary, we observe that there can be the following 8 possible edge types, a regular (3,3)-link, an irregular (3,3)-link, an irregular (3,2)-link, a regular (2,2)-link, an irregular (2,2)-link, an irregular (2,2)-loop, an irregular (2,1)-loop, and a regular (1,1)-loop. Figure 4 ilustrates the 8 edge types.

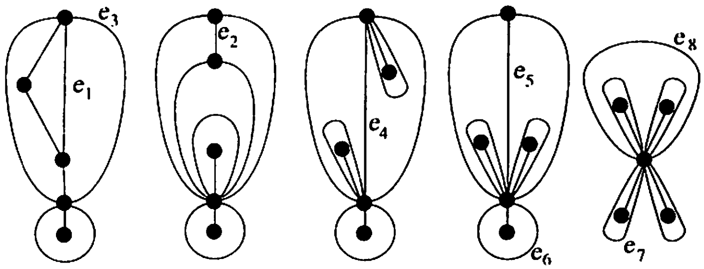

Fig.4

In Fig.4, e_1 is a regular (3,3)-link, e_2 is an irregular (3,3)-link, e_3 is an irregular (3,2)-link, e_4 is a regular (2,2)-link, e_5 is an irregular (2,2)-link, e_6 is an irregular (2,2)-loop, e_7 is an irregular (2,1)-loop, and e_8 is a regular (1,1)-loop.

If there is no ambiguity, we omit the words "regular" and "irregular". For example, we simply write a (3,2)-link for an irregular (3,2)-link since there is no regular (3,2)-link. We do not consider the edge type of a pendant edge in this paper.

The following Lemma follows from an easy observation.

Lemma 1 (1) The successor of a regular edge has the same type as its predecessor.

(2) The successor of an irregular (3,2)-link is also an irregular (3,2)-link.

(3) An irregular (3,3)-link and a (2,2)-loop correspond to each other by a diagonal flip.

(4) An irregular (3,2)-link and a (2,1)-loop correspond to each other by a diagonal flip.

∎

Let G be a plane pseudo triangulation. We denote the number of loops in G by $\ell(G)$. Also, we denote the number of 1-triangles in G by $t(G)$. Note that, if $\ell(G) = 0$, then $t(G) = 0$. For a pair of distinct vertices $x, y \in V(G)$, we denote the number of edges between x and y in G by $\mu(xy; G)$. An edge e is called a multiple edge if $\mu(V(e); G) \geq 2$. Let

$$\mu(G) = \sum_{\mu(xy;G)\geq 1} (\mu(xy; G) - 1).$$

Clearly G has no multiple edges if and only if $\mu(G) = 0$. Now we introduce an invariant $\xi(G)$ of a plane pseudo triangulation G as follows:

$$\xi(G) = \mu(G) + \ell(G) + t(G).$$

We note that G is a triangulation if and only if $\xi(G) = 0$. The invariant $\xi(G)$ plays a key role in this section. It is important to investigate the effect of the application of a diagonal flip to the invariants of a plane pseudo triangulation.

Lemma 2 Let G be a plane pseudo triangulation and let e be an edge of G and let G' be the graph obtained from G by the diagonal flip around e. We write μ, ℓ, t, and ξ for $\mu(G)$, $\ell(G)$, $t(G)$, and $\xi(G)$, respectively. We also write μ', ℓ',

t', and ξ' for $\mu(G')$, $\ell(G')$, $t(G')$, and $\xi(G')$, respectively. Then we get the following table.

	type of e	$\mu' - \mu$	$\ell' - \ell$	$t' - t$	$\xi' - \xi$
(1)	reg. (3,3)-link	*	0	0	*
(2)	irreg. (3,3)-link	0	+1	0	+1
(3)	(3,2)-link	0	0	0	0
(4)	reg. (2,2)-link	0	0	0	0
(5)	irreg. (2,2)-link	-1	+1	+1	+1
(6)	(2,2)-loop	0	-1	0	-1
(7)	(2,1)-loop	+1	-1	-1	-1
(8)	(1,1)-loop	0	0	0	0

* means 0 or ± 1.

Proof: Let f be the successor of e.

(1) Assume e is a regular (3,3)-link. Suppose that e is a multiple edge in G. Write $V(e) = \{x, y\}$ and $V(f) = \{u, v\}$. Since $\mu(V(e); G) \geq 2$, there is a link e' between x and y other than e. Then we observe that u and v are separated by the 2-cycle $e \cup e'$ in G; this implies that there is no edge between u and v in G. Hence we have

$$\mu' - \mu = \begin{cases} 0 & \text{if} \quad \mu(V(e); G) = \mu(V(f); G') = 1 \\ 1 & \text{if} \quad \mu(V(e); G) = 1 \quad \text{and} \quad \mu(V(f); G') \geq 2 \\ -1 & \text{if} \quad \mu(V(e); G) \geq 2 \quad \text{and} \quad \mu(V(f); G') = 1 \end{cases}$$

Since f is also a regular (3,3)-link, $\ell' - \ell = t' - t = 0$.

(2),(6) Assume that e is an irregular (3,3)-link. In this case, f is a (2,2)-loop. We observe that $\mu(V(e); G) = 1$ and this implies that $\mu' - \mu = 0$. Since e is a link and f is a loop, $\ell' - \ell = 1$. The two new faces are both 2-triangles. Hence $t' - t = 0$.

(3) Assume that e is a (3,2)-link. Then f is also a (3,2)-link. We observe that $\mu(V(e); G) = \mu(V(f); G') \geq 2$; this

implies that $\mu' - \mu = 0$. Since both e and f are links, $\ell' - \ell = t' - t = 0$.

(4) Assume that e is a regular (2,2)-link. Then f is also a regular (2,2)-link. We observe that $\mu(V(e); G) = \mu(V(f); G') \geq 3$; this implies that $\mu' - \mu = 0$. Since both e and f are links, $\ell' - \ell = t' - t = 0$.

(5),(7) Assume that e is an irregular (2,2)-link. Then f is a (2,1)-loop. In this case we observe that $\mu(V(e); G) \geq 3$. This, together with the fact that f is a loop, implies that $\mu' - \mu = -1$. Since e is a link and f is a loop, $\ell' - \ell = 1$. Since e is a (2,2) type and f is a (2,1) type, $t' - t = 1$.

(8) Assume e is a (1,1)-loop. In this case, f is also a (1,1)-loop and no invariant changes. Hence $\mu' - \mu = \ell' - \ell = t' - t = 0$.

Now the proof of Lemma 2 is completed. ∎

Lemma 3. Let G be a plane pseudo triangulation of order n. Then at least $\xi(G)$ diagonal flips are needed to transform G into a triangulation on the sphere.

Conversely, any connected plane pseudo triangulation G of order n can be transformed into a triangulation on the sphere by $\xi(G)$ diagonal flips.

Proof: Let G be a plane pseudo triangulation of order n. By Lemma 2, we know that each diagonal flip decreases the invariant $\xi(G)$ by at most 1. The first statement follows from this, together with the fact that a plane pseudo triangulation G is a trinagulation if and only if $\xi(G) = 0$.

Next we prove the second statement. Let G be a plane pseudo triangulation with $\xi(G) > 0$. Let e be one of a multiple (3,3)-link, a (2,2)-loop, and a (2,1)-loop. Then, by Lemma 2, a diagonal flip around e decreases $\xi(G)$ by 1. Hence it suffices to show that G has one of a multiple (3,3)-link, a (2,2)-loop, and a (2,1)-loop.

First, assume that G has a 1-triangle. We consider an inner 1-triangle. Then it must be adjacent to a 2-triangle along a (2,1)-loop. So we may assume that G has no 1-triangle. In this situation, a loop of G must be a (2,2)-loop. If G has no loop, then since $\xi(G) > 0$, G has a multiple (3,3)-link.

Now the second statement is established and the proof of Lemma 3 is completed. ∎

Lemma 4 Let G be a connected plane pseudo triangulation of order n. Then

$$\xi(G) \leq 3n - 8.$$

Moreover, there is a plane pseudo triangulation for which the equality holds.

Proof: Since $\xi(G) = \mu(G) + \ell(G) + t(G)$, it suffices to prove the following two inequalities.

$$\begin{aligned} &(1) \quad \mu(G) + \ell(G) \leq 2n - 5 \\ &(2) \quad t(G) \leq n - 3 \end{aligned}$$

Since G is connected

$$\sum_{x \in V(G)} |N_G(x)| \geq 2n - 2.$$

Hence

$$\begin{aligned} 2\mu(G) &= \sum_{x \in V(G)} (|Link(x; G)| - |N_G(x)|) \\ &\leq \sum_{x \in V(G)} deg_G(x) - 2\ell(G) - \sum_{x \in V(G)} |N_G(x)| \\ &\leq (6n - 12) - 2\ell(G) - (2n - 2). \end{aligned}$$

So we get inequality (1).

Next we show that inequality (2) holds. We remove all links from G. Then we remove isolated vertices from the resulting graph. We write the resulting graph as $\tilde{G}$. We denote the number of 1-triangles in $\tilde{G}$ by $t(\tilde{G})$. Then, since each 1-triangle in G still remains a 1-triangle in $\tilde{G}$, the inequality $t(G) \leq t(\tilde{G})$ holds. Let T be the dual graph of $\tilde{G}$. Then we observe that T is a tree. Since the number of faces of $\tilde{G}$ is not greater than that of G, we get $|V(T)| \leq 2n - 4$. Let $V_i(T)$ denote the set of vertices of T whose degree is i. Then, since a 1-triangle of $\tilde{G}$ is corresponds to a vertex of T whose degree is 3, the inequality $t(\tilde{G}) \leq |V_3(T)|$ holds. Now we prove the inequality $|V_3(T)| \leq |V_1(T)| - 2$ for any tree T other than K_1 by mathmatical induction on $|V(T)|$. The fact that K_2 has two leaves establishes the initial step. So we consider the induction step. Let x be a leaf of T and let y be the neighbourhood of x. Remove x from T and denote the resulting tree by T'. In the case that $deg_T(y) \neq 3$, we observe that $|V_3(T')| \geq |V_3(T)|$ and $|V_1(T')| \leq |V_1(T)|$. In the case that $deg_T(y) = 3$, we observe that $|V_3(T')| = |V_3(T)| - 1$ and $|V_1(T')| = |V_1(T)| - 1$. Now the induction step is proved. Hence

$$|V_3(T)| \leq |V_1(T)| - 2 \leq |V(T)| - |V_3(T)| - 2.$$
$$|V_3(T)| \leq \frac{1}{2}(|V(T)| - 2) \leq n - 3.$$

Finally, we get

$$t(G) \leq t(\tilde{G}) \leq |V_3(T)| \leq n - 3.$$

which is the desired inequality (2).

To complete the proof of this Lemma, we show that for each flower G of order n, the equality $\xi(G) = 3n - 8$ holds. Since G has no multiple edges, $\mu(G) = 0$. We observe that G has $n - 1$ links each of which is pendant and also G has the same number of reduced triangles. This fact, together with the equality $E(G) = 3n - 6$, implies that G has $2n - 5$ loops, i.e., $\ell(G) = 2n - 5$. Since the number of triangles of G is $2n - 4$ and since each triangle, other than a reduced one, is a 1-triangle, we have $t(G) = (2n - 4) - (n - 1) = n - 3$. Finally, we get the desired equality

$$\xi(G) = \mu(G) + \ell(G) + t(G) = 3n - 8,$$

and the proof of Lemma 4 is completed. ∎

Combining Lemma 3 and Lemma 4, we get Corollary 4.1.

Corollary 4.1 Any plane pseudo triangulation of order n can be transformed into a triangulation on the sphere by at most $3n - 8$ diagonal flips. ∎

This Corollary 4.1, together with Theorem B, proves Theorem 1.

3. Proof of Theorem 2

Let G be a flower of order n with center x. Let H be a triangulation on the sphere of order n whose maximum degree is 6. We show that at least $\max\{4n - 15, 3n - 8\}$ diagonal flips are needed to transform G into H. Let $G = G_0, G_1, \cdots, G_m = H$ be a sequence of plane pseudo triangulations from G to H such that G_i is obtained from G_{i-1} by a diagonal flip for $1 \leq i \leq m$. Choose such a sequence so that m is minimum. If $n \leq 7$, then Lemma 3 assures us that $m \geq 3n - 8 = max\{4n - 15, 3n - 8\}$. Hence we may assume that $n \geq 8$, and we show $m \geq 4n - 15$. Let $e_i \in E(G_i)$ be the edge such that G_{i+1} is obtained from G_i by a diagonal flip around e_i. We denote the set of these edges by Y, namely, $Y = \{e_0, e_1, \cdots, e_{m-1}\}$. We denote the successor of e_i by f_{i+1}. Write $Z = \{f_1, f_2, \cdots, f_m\}$. Even if $V(e_i) = V(f_j)$ and e_i and f_j are the same edge, we regard them as different. Hence, neither Y nor Z is a multiset and $Z \cap E(G) = \phi$. We denote the set of x-loops in Y by L. Furthermore, we write

$$\tilde{L} = \{e_i \in L \mid e_i \text{ is a } (2,1)\text{-loop of } G_i\}$$

and

$$\tilde{Z} = \{f_i \in Z \mid e_{i-1} \in \tilde{L}\}.$$

Lemma 5.

$$m \geq |L| + |Link(x;G)| + |\tilde{Z}| - 6.$$

Proof: By Lemma 2, we observe that each link $f_i \in \tilde{Z}$ is incident to x. Hence, since $deg_H(x) \leq 6$, $|(Link(x;G) \cup \tilde{Z}) - Y| \leq 6$. This inequality, together with the fact that $Link(x;G)$, $\tilde{Z}$, and L are mutually disjoint, implies the desired inequality. ∎

By Lemma 3, we have $t(G_{i+1}) \geq t(G_i) - 1$. Moreover, $t(G_{i+1}) = t(G_i) - 1$ if and only if e_i is a $(2,1)$-loop. This fact, together with the fact that H has no 1-triangles, implies the inequality $t(G) \leq |\tilde{L}| = |\tilde{Z}|$. Since H has no loops, $Loop(x;G) \subset L$. Recall that $|Loop(x;G)| = 2n - 5$, $|Link(x;G)| = n - 1$, and $t(G) = n - 3$. Hence by Lemma 5,

$$\begin{aligned} m &\geq |L| + |Link(x;G)| + |\tilde{Z}| - 6 \\ &\geq |Loop(x;G)| + |Link(x;G)| + t(G) - 6 \\ &= (2n-5) + (n-1) + (n-3) - 6 \\ &= 4n - 15. \end{aligned}$$

Now Theorem 2 is proved. ∎

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