SUBSQUAGS AND NORMAL SUBSQUAGS

By

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Abstract. *Quackenbush [5] has studied the properties of squags or " Steiner quasigroups", that is a corresponding algebra of Steiner triple systems. He has proved that if a finite squag (P; .) contains two disjoint subsquags (P_1 ; .) and (P_2 ; .) with cardinality $|P_1| = |P_2| = 1/3\ |P|$, then the complement $P_3 = P - (P_1 \cup P_2)$ is also a subsquag and the three subsquags P_1, P_2 and P_3 are normal. Quackenbush then asks for an example of a finite squag of cardinality 3n with a subsquag of cardinality n, but not normal. In this paper, we construct an example of a squag of cardinality 3n with a subsquag of cardinality n, but it is not normal; for any positive integer $n \geq 7$ and $n \equiv 1$ or 3 (mod 6).*

Introduction:

A squag or " Steiner quasigroup " is a groupoid $\mathcal{S} = (S;\ .)$ satisfying the identities:

$$x.x = x \quad , \quad x.y = y.x \quad , \quad x.(x.y) = y \ .$$

A Steiner triple system is a pair (P; B), where P is a set of points and B is a set of 3-element subsets of P called blocks such that for distinct points p_1 , p_2 $\in$ P, there is a unique block b $\in$ B such that $\{p_1, p_2\} \subseteq b$. There is a one to one correspondence between the squags and the Steiner triple systems [1] [5]. We will denote the corresponding squag of the Steiner triple system (P ; B) by (P; .), where

$$x.y = z \Leftrightarrow \{x, y, z\} \in B \text{ ; for any two distinct elements x and y } \in P \text{ [5].}$$

The Steiner triple system (P ; B) is denoted by STS(n), if the cardinality of P is equal to n . It is well known that a necessary and sufficient condition for the existence of an STS(n) is $n \equiv 1$ or 3 (mod 6) [4].

A normal subsquag $\underline{P}$ = (P; .) of a squag $\underline{S}$ = (S; .) is defined as a congruence class of a congruence of $\underline{S}$. Quackenbush [5] has proved that if $\underline{P}_1$ = (P_1; .) and $\underline{P}_2$ = (P_2; .) are two subsquags of a finite squag $\underline{S}$ such that $P_1 \cap P_2 = \varnothing$ and $|S| = 3\,|P_1| = 3\,|P_2|$, then $\underline{P}_i$ for i =1,2,3 are normal subsquags, where $\underline{P}_3$ = (P_3 ; .) and $P_3 = S - (P_1 \cup P_2)$. And then Quackenbush in [5] asked for an example of a finite squag $\underline{S}$ =(S; .) with a subsquag $\underline{P}$ = (P; .) such that $|S| = 3\,|P|$, but P is not normal in $\underline{S}$. In fact, there is a one to one correspondence between the subsquags of the squag (P ; .) and the subSTSs or "subspaces" of the corresponding Steiner triple system (P ; B) [2]. Accordingly, the problem can be translated as follows:

Find a finite STS(3n) = (P; B) having a subspace STS(n) = (P_1; B_1) and the system $(P - P_1;\ B - B_1)$ has no more disjoint subspaces of cardinality n .

Construction of non-normal subsquags

Let $\underline{P}_i$ for i = 1 , 2 be the corresponding squags of the Steiner triple systems $\boldsymbol{P_1} = (P_1;\ B_1)$ and $\boldsymbol{P_2} = (P_2;\ B_2)$ respectivily, then the direct product of the two Steiner triple systems $\boldsymbol{P_1} \times \boldsymbol{P_2}$ is defined by the corresponding Steiner triple system of the direct product of the two squags $\underline{P}_1 \times \underline{P}_2$ [2]. Let $\boldsymbol{P_1} = (P_1;\ B_1)$ be a Steiner triple system of cardinality n , and let $P_1 = \{ a_1 , a_2 , \dots , a_n \}$. We consider the direct product $\boldsymbol{P_1} \times \boldsymbol{C_3}$, where $\boldsymbol{C_3}$ is the 3- element STS(3) on the set {1 ,2 ,3 }. The direct product $\boldsymbol{P_1} \times \boldsymbol{C_3} = (P; B)$ is formed by the usual tripling of $(P_1;\ B_1)$. The projections $P_1 \times \{ i \}$; for i = 1, 2 , 3 are three disjoint subSTS(n)s of the Steiner triple system STS(3n) = (P ; B).

Without loss of generality, we can assume that { a , b , c } is a block in B_1 , then we have the following subset of blocks of B:

$$R = \{\{(a, 2), (a, 1), (a, 3)\},\{(a, 2), (b, 2), (c, 2)\},\{(a, 3), (b, 3), (c, 3)\},$$
$$\{(c, 2), (c, 1), (c, 3)\},\{(a, 1), (b, 3), (c, 2)\},\{(b, 2), (a, 1), (c, 3)\},$$
$$\{(c, 1), (b, 2), (a, 3)\},\{(b, 3), (c, 1), (a, 2)\}\},$$

defined on the subset $A \subseteq P$, where

$$A =\{(a, 1), (a, 2), (a, 3), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3)\}.$$

We consider the following set of blocks defined on the same set A:

$$H = \{\{(a, 1), (a, 3), (b, 3)\}, \{(a, 1), (a, 2), (b, 2)\},\{(b, 3), (c, 3), (c, 1)\},$$
$$\{(c, 1), (c, 2), (b, 2)\}, \{(a, 1), (c, 3), (c, 2)\}, \{(c, 1), (a, 2), (a, 3)\},$$
$$\{(b, 3), (c, 2), (a, 2)\}, \{(b, 2), (a, 3), (c, 3)\}\}.$$

If a 2-element set $\{x, y\} \subseteq A$ lies in a triple $t \in R$, then there is a triple $t' \in H$ containing { x , y } and $|R| = |H|$. Using the replacement property on the direct product $\boldsymbol{P_1} \times \boldsymbol{C_3}$, then we get the Steiner triple system STS (3n) = (P; $\underline{B}$), where $\underline{B} := (B - R) \cup H$ [2][4]. In fact, the subSTS formed by the direct product of { a, b, c} and {1, 2, 3} is replaced with an isomorphic copy on the same set but having the set of blocks H instead of the set of blocks R. In the same time, the Steiner triple system (P ; $\underline{B}$) still contains $\boldsymbol{P_1}$ as a subSTS(n).

To prove that the corresponding squag $\underline{P}_1$ of $\boldsymbol{P_1}$ is not a normal subsquag of the corresponding squag $\underline{P}$ of (P ; $\underline{B}$), assume that $\underline{P}_1$ is normal in $\underline{P}$. Then the set $x . P_1$, for each $x \in P$, forms a subsquag of $\underline{P}$. Then we deduce that $\underline{P}_1$ is not normal, if the set $x . P_1$ is not a subsquag of $\underline{P}$ for some selected element

$x \in P$. This means that it is enough to show that there is a translation set of the subSTS P_1 which is not a subSTS of (P; B̲).

Choose $d \in P_1 - \{a, b, c\}$, then the set of triples containing the element (d, 2) in the system (P; B̲) is the same as in the system (P; B). This means that the translation of the subSTS P_1 by the element (d, 2) is the set $\{ (a_i, 3) \mid a_i \in P_1 \}$. The triple $\{(a, 1), (a, 3), (b, 3)\}$ is a block in B̲. Therefore, the set $\{ (a_i, 3) \mid a_i \in P_1 \}$ is not a subSTS since the block containing (a, 3) and (b, 3) does not lie entirely inside $\{(a_i, 3) \mid a_i \in P_1 \}$. Hence, the corresponding subsquag $\underline{P}_1$ is not a normal subsquag of the corresponding squag P̲ of (P; B̲).

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A combinatorial approach to improved Bonferroni inequalities

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Abstract

We reprove an important case of a recent topological result on improved Bonferroni inequalities due to Naiman and Wynn in a purely combinatorial manner. Our statement and proof involves the combinatorial concept of non-evasiveness instead of the topological concept of contractibility. In contradistinction to the proof of Naiman and Wynn, our proof does not require knowledge of simplicial homology theory.

1 Introduction

Bonferroni inequalities (also known as inclusion-exclusion inequalities) are an important tool in probability theory, reliability theory and statistics. For any finite collection of sets, any measure μ on the algebra generated by these sets and any $n \in \mathbb{N}$, the traditional Bonferroni inequalities state that

$$\mu\left(\bigcup_{v\in V} A_v\right) \geq \sum_{\substack{I\in\mathcal{P}^*(V)\\|I|\leq n}} (-1)^{|I|-1}\mu\left(\bigcap_{i\in I} A_i\right) \quad (n \text{ even}),$$

$$\mu\left(\bigcup_{v\in V} A_v\right) \leq \sum_{\substack{I\in\mathcal{P}^*(V)\\|I|\leq n}} (-1)^{|I|-1}\mu\left(\bigcap_{i\in I} A_i\right) \quad (n \text{ odd}),$$

where $\mathcal{P}^*(V)$ is the set of non-empty subsets of V.

Recently, Naiman and Wynn [8] established a wide ranging generalization and improvement of these inequalities. Their main result involves the topological concept of contractibility, and in their proof a key role is played by the simplicial homology groups of an abstract simplicial complex.

In this paper, we consider a special but still important case of the main result of Naiman and Wynn [8], which is obtained by replacing the topological notion of contractibility by the combinatorial concept of non-evasiveness, which was introduced by Kahn, Saks and Sturtevant [6] in the context of graph complexity questions, and which was further investigated by Kurzweil [7] in the study of order complexes of finite groups. For this special but important case, we present a new and purely combinatorial proof, which, in contradistinction to the original proof of Naiman and Wynn [8] for the general case, does not require knowledge of simplicial homology theory. We finally deduce some recent results from this special case that already proved useful in the context of network reliability analysis.

2 Preliminaries

We need some definitions and facts from combinatorial topology. For a detailed exposition, we recommend the textbook of Harzheim [5].

An *abstract simplicial complex* $\mathcal{S}$ is a set of non-empty subsets of some finite set such that $I \in \mathcal{S}$ and $\emptyset \neq J \subset I$ imply $J \in \mathcal{S}$. The elements of $\mathcal{S}$ resp. $\operatorname{Vert}(\mathcal{S}) := \bigcup_{I \in \mathcal{S}} I$ are the *faces* resp. *vertices* of $\mathcal{S}$. The *dimension* of a face I of $\mathcal{S}$ is one less than its cardinality and denoted by $\dim I$. The *Euler characteristic of* $\mathcal{S}$ is defined by

$$\chi(\mathcal{S}) := \sum_{I \in \mathcal{S}} (-1)^{\dim I} .$$

With any abstract simplicial complex $\mathcal{S}$ and any $k \in \mathbb{N} \cup \{0\}$ we associate its *k-skeleton* $\mathcal{S}^k$, that is, the abstract simplicial complex consisting of all faces of $\mathcal{S}$ whose dimension is less than or equal to k. A *geometric realization* of an abstract simplicial complex $\mathcal{S}$ is any topological space homeomorphic to

$$\bigcup_{I \in \mathcal{S}} \left\{ \sum_{i \in I} t_i \mathbf{e}_i \,\middle|\, \text{each } t_i \geq 0 \text{ and } \sum_{i \in I} t_i = 1 \right\} ,$$

where $\{\mathbf{e}_v\}_{v \in V}$ is the standard basis of the vector space $\mathbb{R}^V$ and $V = \operatorname{Vert}(\mathcal{S})$. Recall that two topological spaces X and Y are *homeomorphic* if there is a bijective mapping $\phi : X \to Y$ such that both ϕ and its inverse ϕ^{-1} are continuous. A topological space X is *contractible* if there is a continuous mapping $\Phi : X \times [0,1] \to X$ such that $\Phi(x,0) = x$ for any $x \in X$ and $\Phi(x,1) = x_0$ for any $x \in X$ and some fixed $x_0 \in X$. Since contractibility is known to be a homeomorphism invariant, we may call an abstract simplicial complex *contractible* if it has a contractible geometric realization. For instance, the abstract simplicial complex $\mathcal{P}^*(V)$ consisting of all non-empty subsets of some finite set V is contractible.

3 Improvements via contractibility

The main result of Naiman and Wynn [8] is the following:

Theorem 3.1 (Naiman-Wynn) *Let $\{A_v\}_{v \in V}$ be a finite family of sets and $\mathcal{S} \subseteq \mathcal{P}^*(V)$ an abstract simplicial complex such that*

$$\mathcal{S}(\omega) = \left\{ I \in \mathcal{S} \,\middle|\, \omega \in \bigcap_{i \in I} A_i \right\}$$

is contractible for any $\omega \in \bigcup_{v \in V} A_v$. Then, for any $n \in \mathbb{N}$ and any measure μ on the algebra generated by $\{A_v\}_{v \in V}$, the following inequalities hold:

$$\mu\left(\bigcup_{v \in V} A_v\right) \geq \sum_{\substack{I \in \mathcal{S} \\ |I| \leq n}} (-1)^{|I|-1} \mu\left(\bigcap_{i \in I} A_i\right) \quad \textit{(n even)},$$

$$\mu\left(\bigcup_{v \in V} A_v\right) \leq \sum_{\substack{I \in \mathcal{S} \\ |I| \leq n}} (-1)^{|I|-1} \mu\left(\bigcap_{i \in I} A_i\right) \quad \textit{(n odd)}.$$

Remark. For $\mathcal{S} = \mathcal{P}^*(V)$, Theorem 3.1 specializes to the traditional Bonferroni inequalities. By Theorem 5 of Naiman and Wynn [8], the bounds provided by Theorem 3.1 are at least as sharp as the traditional ones, although less computational effort is required to compute them.

4 Improvements via non-evasiveness

Following Kahn, Saks and Sturtevant [6], by induction on the number of vertices we now define which abstract simplicial complexes are non-evasive:

1. Any abstract simplicial complex having only one vertex is non-evasive.

2. If $\mathcal{S}$ is an abstract simplicial complex such that both

$$\begin{aligned} \mathcal{S}\backslash v &:= \{I \mid I \in \mathcal{S}, v \notin I\} \quad \text{and} \\ \mathcal{S}/v &:= \{I \setminus \{v\} \mid I \in \mathcal{S}, v \in I, I \neq \{v\}\} \end{aligned}$$

are non-evasive for *some* vertex v of $\mathcal{S}$, then $\mathcal{S}$ is non-evasive, too.

Note that both $\mathcal{S}\backslash v$ and $\mathcal{S}/v$ are abstract simplicial complexes having fewer vertices than $\mathcal{S}$. Therefore, the class of non-evasive complexes is well defined.

Example 4.1 A *tree* is a non-empty abstract simplicial complex $\mathcal{T}$ whose faces are of dimension 0 or 1 and which is cycle-free in the usual graph-theoretic sense. By induction on the number of vertices it follows that trees are non-evasive: Any tree having only one vertex is non-evasive by means of Property 1 in the definition of non-evasiveness. Now, let $\mathcal{T}$ be a tree having more than one vertex, and let v be a leave of $\mathcal{T}$. Then, $\mathcal{T}\backslash v$ and $\mathcal{T}/v$ are again trees, and by the induction hypothesis, they both are non-evasive. By this and Property 2, $\mathcal{T}$ is non-evasive as well.

Example 4.2 A *cone* is an abstract simplicial complex $\mathcal{C}$ having a vertex a which is contained in every maximal face of $\mathcal{C}$. We then refer to a as an *apex of* $\mathcal{C}$ and to $\mathcal{C}$ as a *cone with apex* a. For instance, $\mathcal{P}^*(V)$ is a cone with apex v for any $v \in V$. By induction on the number of vertices we prove that cones are non-evasive: Any cone having only one vertex is non-evasive by means of Property 1. Now, let $\mathcal{C}$ be a cone having more than one vertex, and let a be an apex of $\mathcal{C}$. Then, for any $v \neq a$, $\mathcal{C}\backslash v$ and $\mathcal{C}/v$ are cones with apex a, and by the induction hypothesis, they both are non-evasive. Therefore, by Property 2, $\mathcal{C}$ is non-evasive as well.

Example 4.3 The complex $\mathcal{S} = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$ (a triangle without its interior) is easily seen to be not non-evasive.

Kahn, Saks and Sturtevant [6] proved that non-evasiveness entrains contractibility. Thus, the following theorem is a specialization of Theorem 3.1.

Theorem 4.4 *Let $\{A_v\}_{v \in V}$ be a finite family of sets and $\mathcal{S} \subseteq \mathcal{P}^*(V)$ an abstract simplicial complex such that*

$$\mathcal{S}(\omega) = \left\{ I \in \mathcal{S} \,\middle|\, \omega \in \bigcap_{i \in I} A_i \right\}$$

is non-evasive for any $\omega \in \bigcup_{v \in V} A_v$. Then, for any $n \in \mathbb{N}$ and any measure μ on the algebra generated by $\{A_v\}_{v\in V}$, the following inequalities hold:

$$\mu\left(\bigcup_{v\in V} A_v\right) \geq \sum_{\substack{I\in \mathcal{S}\\ |I|\leq n}} (-1)^{|I|-1}\mu\left(\bigcap_{i\in I} A_i\right) \quad \textit{(n even)},$$

$$\mu\left(\bigcup_{v\in V} A_v\right) \leq \sum_{\substack{I\in \mathcal{S}\\ |I|\leq n}} (-1)^{|I|-1}\mu\left(\bigcap_{i\in I} A_i\right) \quad \textit{(n odd)}.$$

In the following, a new and combinatorial proof of Theorem 4.4 is presented, which does not make use of results from simplicial homology theory:

Proof. The proof is based on the recurrence

(1) $$\chi(\mathcal{S}) = 1 + \chi(\mathcal{S}\backslash v) - \chi(\mathcal{S}/v) \quad (v \in \mathrm{Vert}(\mathcal{S})),$$

which follows by distincting the cases $v \in I$ and $v \notin I$, and the identities

(2) $$\mathcal{S}^k\backslash v = (\mathcal{S}\backslash v)^k, \quad \mathcal{S}^k/v = (\mathcal{S}/v)^{k-1} \quad (k > 0).$$

By induction on the number of vertices in $\mathcal{S}$ we show that for any $n \in \mathbb{N}$,

(3) $$\chi(\mathcal{S}^{n-1}) \leq 1 \quad (n \text{ even}),$$

(4) $$\chi(\mathcal{S}^{n-1}) \geq 1 \quad (n \text{ odd}).$$

If $\mathcal{S}$ has exactly one vertex, then $\chi(\mathcal{S}^{n-1}) = 1$ for any $n \in \mathbb{N}$, and in this case, the statement is clear. Now, assume that $\mathcal{S}$ has more than one vertex. Choose $v \in \mathrm{Vert}(\mathcal{S})$ such that both $\mathcal{S}\backslash v$ and $\mathcal{S}/v$ are non-evasive. Let n be even (the case where n is odd is treated in a similar way). From the recurrence (1), the identities (2) and the induction hypothesis we obtain

$$\chi(\mathcal{S}^{n-1}) = 1 + \chi((\mathcal{S}\backslash v)^{n-1}) - \chi((\mathcal{S}/v)^{n-2}) \leq 1 + 1 - 1 = 1,$$

thus proving (3) and (4). Now, from (3) and (4) we conclude that

$$1_{\bigcup_{v\in V} A_v} \geq \sum_{\substack{I\in \mathcal{S}\\ |I|\leq n}} (-1)^{|I|-1}\, 1_{\bigcap_{i\in I} A_i} \quad (n \text{ even}),$$

$$1_{\bigcup_{v\in V} A_v} \leq \sum_{\substack{I\in \mathcal{S}\\ |I|\leq n}} (-1)^{|I|-1}\, 1_{\bigcap_{i\in I} A_i} \quad (n \text{ odd}).$$

By integrating these inequalities with respect to μ, the proof is finished. □

As a consequence of Theorem 4.4, we now deduce the main result of [1], which includes the main results of [2], [3] and [9] as special cases. For applications to network reliability analysis, the reader is referred to [3] and [4].

Theorem 4.5 *Let $\{A_v\}_{v\in V}$ be a finite collection of sets, and let $\mathcal{X} \subseteq \mathcal{P}^*(V)$ be a union-closed set system such that $\bigcap_{x\in X} A_x \subseteq \bigcup_{v\notin X} A_v$ for any $X \in \mathcal{X}$. Then, for any measure μ on the algebra generated by $\{A_v\}_{v\in V}$,*

$$\mu\left(\bigcup_{v\in V} A_v\right) \geq \sum_{\substack{I\in \mathcal{I}(V,\mathcal{X}) \\ |I|\leq n}} (-1)^{|I|-1} \mu\left(\bigcap_{i\in I} A_i\right) \quad (n \text{ even}),$$

$$\mu\left(\bigcup_{v\in V} A_v\right) \leq \sum_{\substack{I\in \mathcal{I}(V,\mathcal{X}) \\ |I|\leq n}} (-1)^{|I|-1} \mu\left(\bigcap_{i\in I} A_i\right) \quad (n \text{ odd}),$$

where in both cases $n \in \mathbb{N}$ and

$$\mathcal{I}(V,\mathcal{X}) := \{I \in \mathcal{P}^*(V) \,|\, I \not\supseteq X \,\forall X \in \mathcal{X}\}.$$

Proof. Assume that the inequalities of Theorem 4.4 do not hold for $\mathcal{S} = \mathcal{I}(V,\mathcal{X})$. Then, for some $\omega \in \bigcup_{v\in V} A_v$, $\mathcal{I}(V,\mathcal{X})(\omega)$ is not non-evasive and hence not a cone in view of Example 4.2. From this, it follows that $\mathcal{X}\cap\mathcal{P}^*(V_\omega)$ is a covering of V_ω, where $V_\omega := \{v \in V \,|\, \omega \in A_v\}$. From this and the union-closedness of $\mathcal{X}$ we conclude that $V_\omega \in \mathcal{X}$ and consequently,

$$\omega \in \bigcap_{v\in V_\omega} A_v \subseteq \bigcup_{v\notin V_\omega} A_v .$$

Thus, $\omega \in A_v$ for some $v \notin V_\omega$, which contradicts the definition of V_ω. □

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