Edge-Magic Labelings of Generalized Petersen Graphs $P(n, 2)$

Yasuhiro Fukuchi
Department of Applied Mathematics
Science University of Tokyo
Shinjuku-ku, Tokyo 162-8601, JAPAN

Abstract

A graph G is called super-edge-magic if there exists a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that $f(u) + f(v) + f(uv) = C$ is a constant for any $uv \in E(G)$ and $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$. In this paper, we show that the generalized Petersen graph $P(n, k)$ is super-edge-magic if $n \geq 3$ is odd and $k = 2$.

1 Statement of the Main Result

Let G be a simple undirected graph, and let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. A bijection f from $V(G) \cup E(G)$ to $\{1, 2, \ldots, |V(G)| + |E(G)|\}$ is called an *edge-magic* labeling of G if there exists a constant C (called the *magic number* of f) such that $f(u) + f(v) + f(uv) = C$ for any edge $uv \in E(G)$ (Fig. 1). An edge-magic labeling f of G is called a *super-edge-magic* labeling if $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$ and $f(E(G)) = \{|V(G)| + 1, |V(G)| + 2, \ldots, |V(G)| + |E(G)|\}$ (Fig. 2). We say that G is edge-magic (resp. super-edge-magic) if there exists an edge-magic (resp. super-edge-magic) labeling of G.

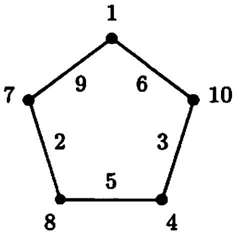

Fig. 1: edge-magic labeling of C_5

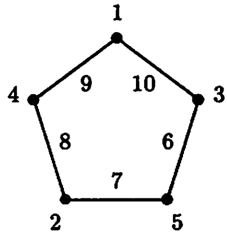

Fig. 2: super-edge-magic labeling of C_5

In [3], Kotzig and Rosa introduced the notion of edge-magic labelings (in [3], edge-magic labelings are called magic valuations). They proved that complete bipartite graphs, cycles and caterpillars are edge-magic, and that the complete graph K_n is edge-magic if and only if $n = 1, 2, 3, 5$ or 6. They also conjectured that trees are edge-magic (this conjecture remains open). In [1], Enomoto, Llado, Nakamigawa and Ringel introduced the notion of super-edge-magic labelings. They proved that the cycle C_n is super-edge-magic if and only if n is odd, that the complete bipartite graph $K_{m,n}$ is super-edge-magic if and only if $m = 1$ or $n = 1$, and that the complete graph K_n is super-edge-magic if and only if $n = 1, 2$ or 3. They also conjectured that trees are super-edge-magic (this conjecture also remains open). In addition, they proved that if $n \equiv 0 \pmod 4$, then the wheel graph W_n of order n is not edge-magic (in [2], it is proved that if $n \not\equiv 0 \pmod 4$, then W_n is edge-magic).

Let n, k be integers such that $n \geq 3$, $1 \leq k < n$ and $n \neq 2k$. For such n, k, the generalized Petersen graph $P(n,k)$ is defined by $V(P(n,k)) = \{u_j, v_j \mid 0 \leq j \leq n-1\}$ and $E(P(n,k)) = \{u_j u_{j+1}, v_j v_{j+k}, u_j v_j \mid 0 \leq j \leq n-1\}$ (subscripts are to be read modulo n). By definition, $P(n,k)$ is a 3-regular graph which has $2n$ vertices and $3n$ edges.

In [4], Tsuchiya and Yokomura constructed a super-edge-magic labeling of $P(n,k)$ in the case where n is odd and $k = 1$ (more generally, they constructed such a labeling for $P_m \times C_{2l-1}$). In this paper, we consider the case where n is odd and $k = 2$, and prove the following theorem:

Theorem *Let $n \geq 3$ be an odd integer. Then $P(n,2)$ is super-edge-magic.*

Note that $P(n,k_1) \cong P(n,k_2)$ if $k_1 + k_2 = n$ or $k_1 k_2 \equiv \pm 1 \pmod n$. Thus the theorem implies that for an odd integer n, $P(n,k)$ is also super-edge-magic in the case where $k = n-2$ or $k = \frac{1}{2}(n \pm 1)$.

We conclude this section with comments on super-edge-magic labelings of regular graphs. In [1], Enomoto et al. proved the following lemma:

Lemma 1 ([1; Lemma 2.1]) If G is super-edge-magic, then $|E(G)| \leq 2|V(G)| - 3$.

In passing, we note that the condition $|E(G)| \leq 2|V(G)| - 3$ in Lemma 1 is not a sufficient condition for G to be super-edge-magic; for example, an even cycle C_{2n} has $2n$ vertices and $2n$ edges, so C_{2n} satisfies $|E(C_{2n})| \leq 2|V(C_{2n})| - 3$, but C_{2n} is not super-edge-magic ([1; Theorem 2.2]). Now it follows from Lemma 1 that if an r-regular graph is super-edge-magic, then $r \leq 3$. Since the generalized Petersen graphs $P(n,k)$ form an important class of 3-regular graphs, it is therefore desirable that one should determine which of the $P(n,k)$ are super-edge-magic, and our theorem can be regarded as an initial step toward this end.

We also prove the following lemma:

Lemma 2 Let r be an odd integer. Let n be an integer, and let G be an r-regular graph such that $|V(G)| = n$. (i) If $n \equiv 4 \pmod 8$, then G is not edge-magic. (ii) If $n \equiv 0 \pmod 4$, then G is not super-edge-magic.

Proof. Suppose that there exists an edge-magic labeling f of G with magic number C. Since $|E(G)| = \frac{1}{2}rn$, $|V(G)| + |E(G)| = n + \frac{1}{2}rn$. Hence

$$\begin{aligned}\frac{1}{2}rnC &= \sum_{uv\in E(G)} \{f(u) + f(v) + f(uv)\} \\ &= \sum_{i=1}^{n+\frac{1}{2}rn} i + (r-1)\sum_{v\in V(G)} f(v) \\ &= \frac{1}{2}(n + \frac{1}{2}rn)(n + \frac{1}{2}rn + 1) + (r-1)\sum_{v\in V(G)} f(v). \qquad (*)\end{aligned}$$

If $n \equiv 4 \pmod 8$, then both $\frac{1}{2}rnC$ and $(r-1)\sum_{v\in V(G)} f(v)$ are even, but $\frac{1}{2}(n + \frac{1}{2}rn)(n + \frac{1}{2}rn + 1)$ is odd, which is a contradiction. Suppose now that f is a super-edge-magic labeling of G and $n \equiv 0 \pmod 4$, and write $n = 4m$ ($m \geq 1$). Then $\sum_{v\in V(G)} f(v) = \sum_{j=1}^{n} j = \frac{1}{2}n(n+1)$, and hence by $(*)$,

$$2rmC = (r+2)m(2(r+2)m+1) + 2(r-1)m(4m+1).$$

Consequently,

$$2rC = (r+2)(2(r+2)m+1) + 2(r-1)(4m+1).$$

But both $2rC$ and $2(r-1)(4m+1)$ are even, and $(r+2)(2(r+2)m+1)$ is odd, which is a contradiction. □

It follows from Lemma 2 that if n is even, then $P(n,k)$ is not super-edge-magic.

2 Proof of Theorem

We give a constructive proof of the theorem. Since $n \geq 3$ is odd, we can write $n = 2m-1$ ($m \geq 2$). Thus $|V(P(n,2))| + |E(P(n,2))| = 5n = 10m-5$.

For labelings of u_j and u_ju_{j+1} $(0 \le j \le 2m-2)$, define

$$\begin{aligned}
f(u_{2j}) &= 1+j & (0 \le j \le m-1),\\
f(u_{2j+1}) &= m+1+j & (0 \le j \le m-2),\\
f(u_{2j}u_{2j+1}) &= 10m-6-2j & (0 \le j \le m-2),\\
f(u_{2j+1}u_{2j+2}) &= 10m-7-2j & (0 \le j \le m-2),\\
f(u_{2m-2}u_0) &= 10m-5.
\end{aligned}$$

Then

$$\{f(u_j) \mid 0 \le j \le 2m-2\} = \{1, 2, \ldots, 2m-1\},$$
$$\{f(u_ju_{j+1}) \mid 0 \le j \le 2m-2\} = \{8m-3, 8m-2, \ldots, 10m-5\}.$$

For labelings of v_j, v_jv_{j+2} and u_jv_j $(0 \le j \le 2m-2)$, we consider two cases.

Case 1 $m \equiv 0 \pmod 2$
Write $m = 2l$ $(l \ge 1)$. Thus $n = 4l-1$, $|V(P(n,2))| + |E(P(n,2))| = 20l-5$. Define

$$\begin{aligned}
f(v_{4j}) &= 6l-1-j & (0 \le j \le l-1),\\
f(v_{4j+1}) &= 5l-1-j & (0 \le j \le l-1),\\
f(v_{4j+2}) &= 8l-2-j & (0 \le j \le l-1),\\
f(v_{4j+3}) &= 7l-2-j & (0 \le j \le l-2),
\end{aligned}$$

$$\begin{aligned}
f(v_{4j}v_{4j+2}) &= 8l-1+2j & (0 \le j \le l-1),\\
f(v_{4j+2}v_{4j+4}) &= 8l+2j & (0 \le j \le l-2),\\
f(v_{4j+1}v_{4j+3}) &= 10l-1+2j & (0 \le j \le l-2),\\
f(v_{4j+3}v_{4j+5}) &= 10l+2j & (0 \le j \le l-2),\\
f(v_{4l-3}v_0) &= 12l-3,\\
f(v_{4l-2}v_1) &= 10l-2,
\end{aligned}$$

$$\begin{aligned}
f(u_{4j}v_{4j}) &= 16l-4-j & (0 \le j \le l-1),\\
f(u_{4j+1}v_{4j+1}) &= 15l-4-j & (0 \le j \le l-1),\\
f(u_{4j+2}v_{4j+2}) &= 14l-4-j & (0 \le j \le l-1),\\
f(u_{4j+3}v_{4j+3}) &= 13l-4-j & (0 \le j \le l-2).
\end{aligned}$$

Case 2 $m \equiv 1 \pmod 2$
Write $m = 2l+1$ $(l \ge 1)$. Thus $n = 4l+1$, $|V(P(n,2))| + |E(P(n,2))| = 20l+5$. Define

$$\begin{aligned}
f(v_{4j}) &= 6l+2-j & (0 \le j \le l),\\
f(v_{4j+1}) &= 7l+2-j & (0 \le j \le l-1),\\
f(v_{4j+2}) &= 8l+2-j & (0 \le j \le l-1),\\
f(v_{4j+3}) &= 5l+1-j & (0 \le j \le l-1),
\end{aligned}$$

$$\begin{aligned}
f(v_{4j}v_{4j+2}) &= 8l+3+2j & (0 \le j \le l-1),\\
f(v_{4j+2}v_{4j+4}) &= 8l+4+2j & (0 \le j \le l-1),\\
f(v_{4j+1}v_{4j+3}) &= 10l+4+2j & (0 \le j \le l-1),\\
f(v_{4j+3}v_{4j+5}) &= 10l+5+2j & (0 \le j \le l-2),\\
f(v_{4l-1}v_0) &= 12l+3, &\\
f(v_{4l}v_1) &= 10l+3, &
\end{aligned}$$

$$\begin{aligned}
f(u_{4j}v_{4j}) &= 16l+4-j & (0 \le j \le l),\\
f(u_{4j+1}v_{4j+1}) &= 13l+3-j & (0 \le j \le l-1),\\
f(u_{4j+2}v_{4j+2}) &= 14l+3-j & (0 \le j \le l-1),\\
f(u_{4j+3}v_{4j+3}) &= 15l+3-j & (0 \le j \le l-1).
\end{aligned}$$

Then in both cases,

$$\begin{aligned}
&\{f(v_j) \mid 0 \le j \le 2m-2\} = \{2m, 2m+1, \ldots, 4m-2\},\\
&\{f(v_jv_{j+2}) \mid 0 \le j \le 2m-2\} = \{4m-1, 4m-2, \ldots, 6m-3\},\\
&\{f(u_jv_j) \mid 0 \le j \le 2m-2\} = \{6m-2, 6m-1, \ldots, 8m-4\}.
\end{aligned}$$

Consequently, f is a super-edge-magic labeling of $P(n, 2)$ with magic number $11m-4$. □

References

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