Generalised Ramsey Numbers with respect to Classes of Graphs

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Abstract

Let $\mathcal{H}_1, ..., \mathcal{H}_t$ be classes of graphs. The class Ramsey number $R(\mathcal{H}_1, ..., \mathcal{H}_t)$ is the smallest integer n such that for each t-edge colouring $(G_1, ..., G_t)$ of K_n, there is at least one $i \in \{1, ..., t\}$ such that G_i contains a subgraph $H_i \in \mathcal{H}_i$. We take $t = 2$ and determine $R(\mathcal{G}_l^1, \mathcal{G}_m^1)$ for all $2 \leq l \leq m$ and $R(\mathcal{G}_l^2, \mathcal{G}_m^2)$ for all $3 \leq l \leq m$, where $\mathcal{G}_j^i$ consists of all edge-minimal graphs of order j and minimum degree i.

1 Introduction

Let $H_1, ..., H_t$ be graphs. The *generalised Ramsey number* $R(H_1, ..., H_t)$ is the smallest integer n such that for each t–edge colouring $(G_1, ..., G_t)$ of K_n, the graph H_i is a subgraph of G_i for at least one $i \in \{1, ..., t\}$. If $G_i = K_{s_i}$ for each i, then $R(H_1, ..., H_t) = r(s_1, ..., s_t)$, the classical Ramsey number. Generalised Ramsey numbers have been studied extensively for many years and an excellent survey of small values of generalised Ramsey numbers and a long list of references are given in [13].

Instead of graphs $H_1, ..., H_t$, we consider classes of graphs and define the corresponding class Ramsey numbers. This takes the generalisation of Ramsey numbers for graphs one step further.

Let $\mathcal{H}_1, ..., \mathcal{H}_t$ be classes of graphs. The *class Ramsey number* $R(\mathcal{H}_1, ..., \mathcal{H}_t)$ is the smallest integer n such that for each t–edge colouring $(G_1, ..., G_t)$ of K_n, there is at least one $i \in \{1, ..., t\}$ such that G_i contains a subgraph $H_i \in \mathcal{H}_i$. Note that if $\mathcal{H}_i$ contains a graph H_i of order p_i for $i = 1, ..., t$, then $R(\mathcal{H}_1, ..., \mathcal{H}_t) \leq r(p_1, ..., p_t)$, for if G_i contains a complete subgraph K_{p_i},

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then G_i also contains H_i. This also proves the existence of $R(\mathcal{H}_1, ..., \mathcal{H}_t)$ for all nonempty $\mathcal{H}_i$.

For $q \geq 0$ and $p \geq 1$, define

$$\mathcal{G}_p^q = \{G : |V(G)| = p,\ \delta(G) = q \text{ and } \delta(G-e) < q \text{ for all } e \in E(G)\}.$$

Then $\mathcal{G}_p^q \neq \emptyset$ if and only if $p > q$. For example, $\mathcal{G}_p^0 = \{\overline{K_p}\}$, $\mathcal{G}_p^1$ consists of the edge–minimal graphs of order p and minimum degree one, $\mathcal{G}_p^2$ consists of the edge–minimal graphs of order p and minimum degree two and $\mathcal{G}_p^{p-1} = \{K_p\}$. Hence

$$R(\mathcal{G}_{p_1}^{p_1-1}, ..., \mathcal{G}_{p_t}^{p_t-1}) = r(p_1, ..., p_t).$$

We consider the case $t = 2$ and determine the Ramsey numbers $R(\mathcal{G}_l^1, \mathcal{G}_m^1)$ for all $2 \leq l \leq m$ and $R(\mathcal{G}_l^2, \mathcal{G}_m^2)$ for all $3 \leq l \leq m$. We consider 2–edge colourings (R, B) of K_n in the colours red (R) and blue (B). Hence $R(\mathcal{G}_l^1, \mathcal{G}_m^1)$ is the smallest integer n such that in any 2–edge colouring (R, B) of K_n, the spanning subgraph R of K_n contains a subgraph of order l and minimum degree one, or the spanning subgraph B of K_n contains a subgraph of order m and minimum degree one. Similarly, $R(\mathcal{G}_l^2, \mathcal{G}_m^2)$ is the smallest integer n such that in any 2–edge colouring (R, B) of K_n, R contains a subgraph of order l and minimum degree two, or B contains a subgraph of order m and minimum degree two.

We use Turán's Theorem [14] on the maximum number of edges in a triangle–free graph.

Theorem 1 *The maximum size of a triangle–free graph G of order n is $\lfloor n^2/4 \rfloor$, and if G has size exactly $\lfloor n^2/4 \rfloor$, then $G \cong K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.*

2 Known values of $R(\mathcal{G}_l^i, \mathcal{G}_m^j)$

As mentioned above, $\mathcal{G}_p^0 = \{\overline{K_p}\}$ and $\mathcal{G}_p^{p-1} = \{K_p\}$, hence

$$R(\mathcal{G}_l^0, \mathcal{G}_m^0) = R(\mathcal{G}_l^{l-1}, \mathcal{G}_m^{m-1}) = r(l, m)$$

and a recent list of known classical Ramsey numbers is given in [13]. Further, $\mathcal{G}_3^1 = \{K_3 - e\}$ and various small Ramsey numbers of the form $R(K_3 - e, H)$ are given in Table II of [13]. Similarly, $\mathcal{G}_4^2 = \{C_4\}$ and

$$R(\mathcal{G}_3^2, \mathcal{G}_4^2) = R(K_3, C_4) = 7 \ [1],$$
$$R(\mathcal{G}_4^2, \mathcal{G}_4^2) = R(C_4, C_4) = 6 \ [3].$$

Since $\mathcal{G}_5^3 = \{W_5\}$, where W_5 is the wheel on five vertices (K_5 with two independent edges removed), we also have

$$R(\mathcal{G}_3^2, \mathcal{G}_5^3) = R(K_3, W_5) = 11 \ [2],$$

$$R(\mathcal{G}_4^3, \mathcal{G}_5^3) = R(K_4, W_5) = 17 \quad [10],$$
$$R(\mathcal{G}_5^3, \mathcal{G}_5^3) = R(W_5, W_5) = 15 \quad [9, 11].$$

The bounds

$$27 \le R(\mathcal{G}_5^4, \mathcal{G}_5^3) = R(K_5, W_5) \le 29$$

were also obtained in [11].

3 The Ramsey numbers $R(\mathcal{G}_l^1, \mathcal{G}_m^1)$

We give a short proof of the elementary result $R(\mathcal{G}_l^1, \mathcal{G}_m^1) = m$ for all $2 \le l \le m$. We begin by observing that if G is a graph of order p, then G has a spanning subgraph with minimum degree one if and only if G has no isolated vertices, and if G has a universal vertex (a vertex adjacent to all other vertices of G), then G has a subgraph H with $\delta(H) = 1$ and order q for each $q \in \{2, ..., p\}$.

Proposition 2 *If* $2 \le l \le m$, *then* $R(\mathcal{G}_l^1, \mathcal{G}_m^1) = m$.

Proof. The edge colouring $(\overline{K_{m-1}}, K_{m-1})$ shows that $R(\mathcal{G}_l^1, \mathcal{G}_m^1) > m-1$. Consider any edge colouring (R, B) of K_m. Suppose B does not have a spanning subgraph with minimum degree one. Then as observed above, B has an isolated vertex, hence R has a universal vertex and therefore, since $l \le m$, a subgraph of order l and minimum degree one. ■

4 The Ramsey numbers $R(\mathcal{G}_l^2, \mathcal{G}_m^2)$

In most of this section we simplify the notation $\mathcal{G}_p^2$ to $\mathcal{G}_p$. We determine $R(\mathcal{G}_3, \mathcal{G}_5)$ separately, and then find formulas for $R(\mathcal{G}_3, \mathcal{G}_m)$, $m \ge 6$, $R(\mathcal{G}_m, \mathcal{G}_m)$, $m \ge 5$ and $R(\mathcal{G}_l, \mathcal{G}_m)$, $4 \le l < m$.

We denote the open and closed neighbourhoods of a vertex v of a graph G by $N(v)$ and $N[v]$ respectively, that is, $N(v) = \{u \in V(G) : uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$. For a vertex subset X of G, the subgraph of G induced by X is denoted by $\langle X \rangle$. When considering an edge colouring (R, B) of K_n we sometimes write $\langle X \rangle_R$ or $\langle X \rangle_B$ to emphasize the colour class containing the subgraph $\langle X \rangle$. We denote the neighbours of v in R (B, respectively) by R_v (B_v) and write $\deg_R v$ for $|R_v|$, $\deg_B v$ for $|B_v|$. Note that $\{v\}$, R_v and B_v form a partition of $V(K_n)$.

For any l, m and n, an edge colouring of K_n such that R does not contain a subgraph in $\mathcal{G}_l$ and B does not contain a subgraph in $\mathcal{G}_m$ is said to be an (l, m) *colouring* of K_n. We now consider the sets $\mathcal{G}_3 = \{K_3\}$ and $\mathcal{G}_m$, where $m \ge 5$ and begin with two lemmas.

Lemma 3 *For $m = 5$ and $n \geq 9$, or $m \geq 6$ and $n \geq m+2$, and any $(3,m)$ colouring (R,B) of K_n, $|B_v| \leq m-2$ for every vertex v.*

Proof. Suppose to the contrary that $\deg_B v \geq m-1$ for some vertex v and let $X_0 = \{u \in B_v : u \text{ is not isolated in } \langle B_v \rangle_B\}$. We first prove that $|X_0| \leq m-2$. If $|X_0| = m-1$, then $\delta(\langle X_0 \cup \{v\}\rangle_B) \geq 2$ and $|X_0 \cup \{v\}| = m$. Hence $\langle X_0 \cup \{v\}\rangle$ has a spanning subgraph $H \in \mathcal{G}_m$, a contradiction. If $|X_0| \geq m$ and each vertex in X_0 is adjacent to an endvertex of $\langle X_0 \rangle_B$, then $\langle X_0 \rangle_B \cong pK_2$ for some p. Then $m \geq 6$ and $\langle X_0 \rangle_B$ has at least three components, so that obviously $\langle X_0 \rangle_R$ has a K_3, a contradiction. Thus there exists $x_0 \in X_0$ which is not adjacent to an endvertex and hence $X_1 = X_0 - \{x_0\}$ satisfies $|X_1| < |X_0|$ and $\delta(\langle X_1 \rangle_B) \geq 1$. Continuing we obtain a set $X = X_k$ for some k with $|X| = m-1$ and $\delta(\langle X \rangle_B) \geq 1$. Again $\langle X \cup \{v\}\rangle$ has an m–vertex spanning subgraph with minimum degree two, contradicting the fact that (R,B) is a $(3,m)$ colouring of K_n.

Hence $|X_0| \leq m-2$ and there exists an isolated vertex u in $\langle B_v \rangle_B$. Let $S = B_v - \{u\}$ and note that $|S| \geq m-2$ and each edge from u to S is red. Thus to avoid triangles in R, $\langle S \rangle_B$ is complete, hence $S = X_0$, *i.e.* $\langle S \rangle_B \cong K_{m-2}$ and

$$\deg_B v = m-1. \tag{1}$$

Since $n \geq m+2$, $q = |R_v| \geq 2$ and if $m = 5$ then $q \geq 4$. Since R is triangle–free, $\langle R_v \rangle_R \cong \overline{K_q}$, *i.e.* $\langle R_v \rangle_B \cong K_q$. Moreover, there is at most one blue edge from any vertex $r \in R_v$ to S, for otherwise $\langle S \cup \{r,v\}\rangle_B$ has a spanning subgraph with minimum degree two. Thus there are red edges from any vertex in R_v to S, and to avoid red triangles, ru is blue for any $r \in R_v$. Hence if $m = 5$ it follows that $\deg_B u \geq |R_v \cup \{v\}| \geq 5$. But this contradicts (1), which holds for any vertex x with $\deg_B x \geq m-1$. Therefore $m \geq 6$ and so for any two vertices $r, s \in R_v$ and any $Y \subseteq S$ with $|Y| = m-3$, $\langle \{r,s,u\} \cup Y \rangle_B$ has minimum degree at least two and so contains a spanning subgraph $H \in \mathcal{G}_m$, a contradiction. ■

Lemma 4 *For any $m \geq 6$ and any $(3,m)$ colouring (R,B) of K_{m+2}, $\deg_B v \geq 4$ for some vertex v.*

Proof. Suppose $\Delta(B) \leq 3$. Then $\delta(R) \geq m-2$ and so R has at least $\frac{1}{2}(m^2-4)$ edges. But

$$\frac{m^2-4}{2} > \frac{(m+2)^2}{4}$$

for all $m > 6$. Hence by Turán's Theorem (Theorem 1) R contains a K_3, a contradiction. If $m = 6$, then

$$\frac{m^2-4}{2} = \frac{(m+2)^2}{4}$$

and thus $R = K_{4,4}$, *i.e.* $B = 2K_4$ and B has $2K_3$ as subgraph, a contradiction. ■

It is now easy to determine $R(\mathcal{G}_3, \mathcal{G}_5)$, that is, the smallest integer n such that for every edge colouring (R, B) of K_n, R has a triangle or B has a subgraph of order five with minimum degree (at least) two.

Proposition 5 $R(\mathcal{G}_3, \mathcal{G}_5) = 9$.

Proof. To see that $R(\mathcal{G}_3, \mathcal{G}_5) > 8$, let (R, B) be the edge colouring of K_8 with $R \cong K_{4,4}$ and $B \cong 2K_4$. Then R is triangle–free and any subgraph H of B with five vertices has a component with at most two vertices, hence $\delta(H) \leq 1$.

Now suppose there exists an edge colouring (R, B) of K_9 such that R is triangle–free and B does not contain a 5–vertex subgraph with minimum degree two. By Lemma 3, $\Delta(B) \leq 3$ so that $\delta(R) \geq 5$. Hence R has at least 23 edges and it follows from Turán's Theorem (Theorem 1) that R contains a triangle, a contradiction. The result follows. ■

We next consider the general case $R(\mathcal{G}_3, \mathcal{G}_m)$, where $m \geq 6$.

Theorem 6 *For any* $m \geq 6$, $R(\mathcal{G}_3, \mathcal{G}_m) = m + 2$.

Proof. The edge colouring (R, B) with $R \cong K_{2,m-1}$ and $B \cong K_2 \cup K_{m-1}$ is a $(3, m)$ colouring of K_{m+1}.

Suppose that (R, B) is a $(3, m)$ colouring of K_{m+2} and consider a vertex v with $\deg_B v = \Delta(B)$. By Lemmas 3 and 4, $4 \leq p = \deg_B v \leq m - 2$ and hence $q = |R_v| = m + 1 - p \geq 3$. To avoid triangles in R, $\langle R_v \rangle_B = K_q$. If $\langle B_v \rangle_B$ contains an isolated vertex u and $S = B_v - \{u\}$, then (to avoid red triangles with u) $\langle S \rangle_B = K_{p-1}$. Since $p \geq 4$, $\langle S \cup R_v \rangle_B$ is an m–vertex graph with minimum degree at least two and hence contains a graph in $\mathcal{G}_m$, a contradiction.

Therefore $\delta(\langle B_v \rangle_B) \geq 1$. If K_2 is a component of $\langle B_v \rangle_B$ and $T = \langle B_v \rangle_B - K_2$, then $\langle \{v\} \cup T \cup R_v \rangle_B$ contains $K_{p-1} \cup K_{m+1-p}$ and thus a graph in $\mathcal{G}_m$ as subgraph, a contradiction. If K_2 is not a component of $\langle B_v \rangle_B$, then B_v also contains two vertices u and w such that $\delta(\langle B_v - \{u, w\} \rangle_B) \geq 1$ and we obtain a contradiction as before. ■

Thus we see that $R(\mathcal{G}_3, \mathcal{G}_6) = 8$ while $R(\mathcal{G}_3, \mathcal{G}_5) = 9$ and so $R(\mathcal{G}_3, \mathcal{G}_m)$ is not monotone. As indicated in Section 2, $R(\mathcal{G}_4, \mathcal{G}_4) = R(C_4, C_4) = 6$. We generalise this result to show that $R(\mathcal{G}_m, \mathcal{G}_m) = m + 2$ for all $m \geq 4$.

Lemma 7 *For* $m \geq 5$ *and any* (m, m) *colouring* (R, B) *of* K_{m+2}, R *and* B *have at least three vertices of degree at most three.*

Proof. We prove the result for R; the result for B follows by symmetry. Let $V(R) = \{v_1, ..., v_{m+2}\}$ with $\deg_R v_i \leq \deg_R v_j$ if $i < j$. Define $U =$

$V(R) - \{v_1, v_2\}$ and $R_1 = \langle U \rangle_R$. By hypothesis $\delta(R_1) \le 1$ and so if v_k is a vertex of R_1 of minimum degree, then $\deg_R v_k \le 3$. But $\deg_R v_1 \le \deg_R v_2 \le \deg_R v_k$ and the result follows. ■

Lemma 8 *If (R, B) is an (m, m) colouring of K_{m+2}, then in R and in B, $|N[u] \cup N[v]| \ge 5$ for any two vertices u and v.*

Proof. Suppose to the contrary that (say) in R, $|N[u] \cup N[v]| \le 4$ and let U be any subset of $V(R) - (N[u] \cup N[v])$ with $|U| = m - 2$. Then $\langle U \cup \{u, v\} \rangle_B$ contains $K_{2,m-2}$ as spanning subgraph, a contradiction. ■

Theorem 9 *For any $m \ge 5$, $R(\mathcal{G}_m, \mathcal{G}_m) = m + 2$.*

Proof. Consider the edge colouring (R, B) of K_{m+1} where R consists of K_{m-1} together with two pendant edges. Then both R and B have at most $m - 1$ vertices of degree two or more. Thus $R(\mathcal{G}_m, \mathcal{G}_m) > m + 1$.

Suppose (R, B) is an (m, m) colouring of K_{m+2}. Let $V(R) = \{v_1, \ldots, v_{m+2}\}$ with $\deg_R v_i \le \deg_R v_j$ if $i < j$. Define

$$\begin{aligned} W_1 &= \{v_i : \deg_R v_i = \deg_R v_1\}, \\ k &= \max\{i : v_i \in W_1\} \quad \text{and} \\ W_2 &= \{v_i : \deg_R v_i = \deg_R v_{k+1}\}. \end{aligned}$$

If $|W_1| \ge 2$, then if possible let u and v be adjacent vertices in W_1; otherwise let u and v be any vertices in W_1. If $|W_1| = 1$, let $u = v_1$ and if possible let $v \in W_2$ be adjacent to u; otherwise let $v \in W_2$ be arbitrary. By Lemma 7, $\deg_R u \le \deg_R v \le 3$. Let $N(u) = N_R(u)$ and $N(v) = N_R(v)$. We first prove a lemma.

Lemma 9.1 $N(u) \cap N(v) \ne \emptyset$.

Proof. Suppose to the contrary that $N(u) \cap N(v) = \emptyset$ and consider $R_{uv} = \langle V(R) - \{u, v\} \rangle_R$. By hypothesis $\delta(R_{uv}) \le 1$. Let w be a vertex of R_{uv} with $\deg_{R_{uv}} w = \delta(R_{uv})$. If $w \notin N(u) \cup N(v)$, then $\deg_R w \le 1$ and so by the choice of u and v, $\deg_R u \le \deg_R v \le 1$, contradicting Lemma 8. Hence $w \in N(u)$ or $w \in N(v)$. Since $w \notin N(u) \cap N(v)$ it follows that $\deg_R w \le 2$ and so by the choice of u and v, $\deg_R u \le \deg_R v \le 2$. But since w is adjacent to u or to v it follows that $|N[u] \cup N[w]| \le 4$ or $|N[v] \cup N[w]| \le 4$, a contradiction. □

In particular, Lemmas 8 and 9.1 together imply that R has no isolated vertices and $\deg_R u + \deg_R v \ge 4$. We show that a stronger result holds.

Lemma 9.2 $\deg_R u = \deg_R v = 3$ *and* $uv \in E(R)$, *or* $\deg_R u = \deg_R v = 2$ *and* $uv \in E(B)$.

Proof. Suppose firstly that $uv \in E(B)$ but $\deg_R v = 3$. As above let w be a vertex of R_{uv} with $\deg_R w = \delta(R_{uv}) \leq 1$. By the choice of v, $\deg_R w \geq \deg_R v = 3$ and so $\deg_R w = 3$ and $w \in N(u) \cap N(v)$, *i.e.* $w \in W_1 \cup W_2$ and w is adjacent to u while v is not adjacent (in R) to u. This contradicts the choice of v. Thus $\deg_R v \leq 2$ and so by Lemmas 8 and 9.1, $\deg_R u = \deg_R v = 2$.

Now suppose that $uv \in E(R)$. By Lemma 9.1, $\deg_R u \geq 2$. But if $\deg_R u = 2$, then by Lemma 8, $\deg_R v = 3$. Also, $N[u] \subseteq N[v]$ (since u is adjacent to v and $N(u) \cap N(v) \neq \emptyset$). This implies that $|N[u] \cup N[v]| = 4$, contradicting Lemma 8 and the result follows. □

Lemmas 9.1 and 9.2 imply that $|N[u] \cup N[v]| \leq 5$ and so $|N[u] \cup N[v]| = 5$ by Lemma 8. Therefore $|N(u) \cap N(v)| = 1$; say $N(u) \cap N(v) = \{w\}$. Then also $|N(u) - \{v, w\}| = |N(v) - \{u, w\}| = 1$; say $N(u) - \{v, w\} = \{x\}$, $N(v) - \{u, w\} = \{y\}$. Let w' be a vertex of degree at most one in R_{uv}. By the choice of u and Lemma 9.2, $\deg_R w' \geq 2$ and so $w' \in N(u) \cup N(v)$. However, if $w' \in \{x, y\}$, then $\deg_R w' = \deg_{R_{uv}} w' + 1 \leq 2$ and so $\deg_R w' = 2$. But then $\deg_R u = \deg_R v = 2$ and R has two adjacent vertices (w' and one of u and v) of degree two, contradicting Lemma 8. Hence $w' = w$ and $\deg_R w \leq 3$; thus w is adjacent to at most one vertex in $S = V(R) - N[\{u, v\}]$. Say $N(w) \cap S \subseteq \{z\}$.

Let G be the subgraph of B induced by $S \cup \{u, v, w\}$ and note that $K_{3,m-3} - wz$ is a spanning subgraph of G. Hence (except in the single case where $|S| = 2$, *i.e.*, $m = 5$, and $N(w) \cap S = \{z\}$) $\delta(G) \geq 2$ and G contains a graph in $\mathcal{G}_m$. In the exceptional case, since R does not contain a 5–cycle, $xy \notin E(R)$ and so xy is blue. But then $xyuzvx$ is a 5–cycle in B, a contradiction.

■

Our last result determines $R(\mathcal{G}_l, \mathcal{G}_m)$ for all $4 \leq l < m$.

Theorem 10 *For all* $4 \leq l < m$, $R(\mathcal{G}_l, \mathcal{G}_m) = m + 1$.

Proof. The edge colouring (R, B) of K_m with $R \cong K_{1,m-1}$ and $B \cong K_1 \cup K_{m-1}$ is an (l, m) colouring of K_m.

Suppose there exists an (l, m) colouring (R, B) of K_{m+1} and let u be a vertex of B with $\deg_B u = \delta(B)$. Then $\delta(B - u) \leq 1$ (by assumption) and so there exists a vertex v with $\deg_{B-u} v \leq 1$. Hence $\deg_B u \leq \deg_B v \leq 2$. Moreover, if v is not adjacent to u, then $\deg_B u \leq \deg_B v \leq 1$. In either case $|N[\{u, v\}]| \leq 4$ and so $|V(B) - N[\{u, v\}]| \geq m - 3 \geq l - 2 \geq 2$. Let S be any subset of $V(B) - N[\{u, v\}]$ with $|S| = l - 2$. Then $K_{2,l-2}$ is a spanning subgraph of $\langle S \cup \{u, v\}\rangle_R$ with minimum degree two, a contradiction. ■

To summarise, the known values of $R(\mathcal{G}_l^i, \mathcal{G}_m^i)$, where $1 \leq i \leq 2$, are as follows:

$$R(\mathcal{G}_l^1, \mathcal{G}_m^1) = m, \quad 2 \leq l \leq m$$

$$
\begin{aligned}
R(\mathcal{G}_3^2,\mathcal{G}_3^2) &= R(K_3,K_3)=6\\
R(\mathcal{G}_3^2,\mathcal{G}_4^2) &= R(K_3,C_4)=7\\
R(\mathcal{G}_3^2,\mathcal{G}_5^2) &= 9\\
R(\mathcal{G}_3^2,\mathcal{G}_m^2) &= m+2, \quad m\geq 6\\
R(\mathcal{G}_l^2,\mathcal{G}_m^2) &= m+1, \quad 4\leq l<m\\
R(\mathcal{G}_m^2,\mathcal{G}_m^2) &= m+2, \quad m\geq 4.
\end{aligned}
$$

5 Irredundant Ramsey numbers

We close with some brief remarks about the relevance of the class Ramsey numbers $R(\mathcal{G}_l^i,\mathcal{G}_m^i)$, where $1\leq i\leq 2$, to generalised irredundant Ramsey numbers.

For a graph $G=(V,E)$, a set $S\subseteq V$ and a vertex $s\in S$, we define three types of S–*private neighbours* (S–*pns*) t of s (see [4, 5]). The vertex t is an (i) S–*self private neighbour* (S–*spn*) of s if $t=s$ and s is an isolated vertex of $\langle S\rangle$; (ii) S–*internal private neighbour* (S–*ipn*) of s if $t\in S-\{s\}$ and $N(t)\cap S=\{s\}$; (iii) S–*external private neighbour* (S–*epn*) of s if $t\in V-S$ and $N(t)\cap S=\{s\}$. Notice that each such t is an element of $N[s]-N(S-\{s\})$ and that no $s\in S$ can have S–pns of both type (i) and type (ii).

The negation of a Boolean variable x is denoted by $\overline{x}$. For $s\in S$, let $p(s,S)$, $q(s,S)$ and $r(s,S)$ be Boolean variables which take the value 1 if and only if s has an S–pn of type (i), (ii) or (iii) respectively. We abbreviate these variables to p, q and r. Note that for each $s\in S$, $p\wedge q=0$, *i.e.* p,q,r are not independent variables. Let $S(s)=(p,q,r)$. The condition $p\wedge q=0$ implies that $S(s)$ is never (1,1,0) or (1,1,1). Using this observation together with conjunction, disjunction and negation, a set $\mathcal{F}$ of 62 distinct Boolean functions $f=f(p,q,r)$ with $f\neq 0$, $f\neq 1$ are defined in [4]. The set S is called an f–*set* of G if for all $s\in S$, $f(S(s))=1$.

For example, if $f=p$, then S is an f–set if and only if each vertex $s\in S$ is an isolated vertex of $\langle S\rangle$, *i.e.* S is independent in G. If $f=p\vee r$, then S is an f–set if and only if each $s\in S$ is isolated in $\langle S\rangle$ or has an S–epn, *i.e.*, S is an *irredundant set* of G.

Further, if $f=\overline{p}$, then S is an f–set if and only if no element $s\in S$ is isolated in $\langle S\rangle$, *i.e.* $\delta(\langle S\rangle)\geq 1$; if $f=\overline{p}\wedge\overline{q}$, then S is an f–set if and only if no element $s\in S$ is isolated or has an S–ipn in S, *i.e.* $\delta(\langle S\rangle)\geq 2$. (Note that in the notation of [5], $\overline{p}=f_{60}$ and $\overline{p}\wedge\overline{q}=f_{48}$.)

Using independent sets instead of complete graphs, the classical Ramsey number, here denoted by $R_p(s_1,...,s_t)$, can now be defined as the smallest integer n such that for each t–edge colouring $(G_1,...,G_t)$ of K_n, there exists $i\in\{1,...,t\}$ such that the complement $\overline{G_i}$ contains an independent set (*i.e.* a p–set) of cardinality s_i. By replacing "p" by a different Boolean function f from $\mathcal{F}$, this concept can be generalised — see [4, 5]. For example, if p is replaced by $p \vee r$ (respectively $p \vee q \vee r$, $p \vee q$), we define *irredundant Ramsey numbers* (see [12]) (respectively *CO-irredundant Ramsey numbers* [6, 8], 1–*dependent Ramsey numbers* [7]).

Similarly, if p is replaced by $\overline{p}$ and $t = 2$, it is easy to see that the generalised Ramsey number $R_{\overline{p}}(l, m)$ is precisely the class Ramsey number $R\left(\mathcal{G}_l^1, \mathcal{G}_m^1\right)$, while if p is replaced by $\overline{p} \wedge \overline{q}$, then $R_{\overline{p}\wedge\overline{q}}\,(l, m)$ is the same as $R\left(\mathcal{G}_l^2, \mathcal{G}_m^2\right)$.

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