THE NICHE CATEGORY OF DENSE GRAPHS

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ABSTRACT. The niche graph of a digraph D is the undirected graph defined on the same vertex set in which two vertices are adjacent if they share either a common in-neighbor or a common out-neighbor in D. We define a hierarchy of graphs depending on the condition of being the niche graph of a digraph having, respectively, no cycles, no cycles of length two, no loops, or loops. Our goal is to classify in this hierarchy all graphs of order $n \geq 3$ having a subgraph isomorphic to K_{n-2}.

1. Introduction and Preliminaries

A survey of the beginnings of the study of niche graphs can be found in [7]. Some more recent work has distinguished categories of niche graphs and attempted to identify the niche graphs category of the graphs belonging to various classes. (See, for example, [1],[3],[9]). The categories we define in this paper are the ones found in the literature, with the addition of one suggested by the recent complete description of the niche graphs of all tournaments in [4]. The purpose of the present work is to give the niche graph category of each dense graph. We begin with some definitions.

Notation. *The set of in-neighbors of a vertex x in a digraph D will be denoted $in_D(x)$. Similarly, the set of out-neighbors will be denoted $out_D(x)$. In case there is not danger of ambiguity, the subscript will be suppressed.*

Definition 1. *Given a digraph $D = (V, A)$, the* **niche graph** *of D is an (undirected) graph $G = (V, E)$ such that $[x, y] \in E$ if and only if either $in(x) \cap in(y)$ or $out(x) \cap out(y)$ is nonempty. We will also say that D is a niche digraph of G.*

Definition 2. *Among niche graphs, we identify the following categories. A graph G is:*

- *an* **acyclic** *niche graph if it has an acyclic niche digraph.*
- *an* **asymmetric** *niche graph if it has an asymmetric niche digraph (i.e., one without cycles of length 2).*

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- *a* ***cyclic*** *niche graph if it has a niche digraph without loops.*
- *a* ***loop*** *niche graph if it has an arbitrary niche digraph (possibly containing loops).*

It is evident that these categories are nested:

$$acyclic \Longrightarrow asymmetric \Longrightarrow cyclic \Longrightarrow loop$$

They also are distinct and do not exhaust all graphs, as is illustrated by the following list of examples. It is shown in [1] that $K_{1,5}$ is not a (loop) niche graph. The digraph D in Figure 1 demonstrates that $K_{1,3}$ is a loop niche graph, but it is shown in [3] that this graph is not a cyclic niche graph. In Section 3 we show that, for example, the graph G_1, depicted in Figure 3, is a cyclic niche graph which is not an asymmetric niche graph. It follows from Proposition 1 that K_4 is an asymmetric niche graph, but in [5] it is shown not to be an acyclic niche graph. Finally, the classification we provide in this paper shows that most dense graphs are acyclic niche graphs.

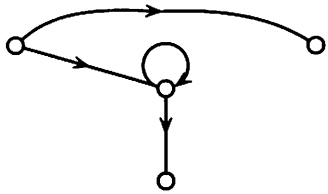

FIGURE 1. Niche digraph for $K_{1,3}$

Definition 3. *We will call a graph of order $n \geq 3$ dense if it contains a subgraph isomorphic to K_{n-2}.*

The goal is to determine the niche graph categories to which each dense graph belongs. We begin with two easy subclasses. It is proved in [5] that complete graphs are not acyclic niche graphs. On the other hand:

Proposition 1. *Every complete graph of order at least 4 is an asymmetric niche graph.*

Proof. Let $n \geq 4$ and G be complete with vertex set $\{x_1, \ldots, x_n\}$. Create the digraph D on the same vertex set and having arc set

$$A(D) = \{x_1 \to x_k : k = 2, \ldots, n\} \cup \{x_i \to x_{i+1} : i = 2, \ldots, n-1\} \cup \{x_n \to x_2\}.$$

It is routine to check that D is asymmetric and that its niche graph is G. □

Proposition 2. *Every graph of order $n \geq 4$ which is not complete and which contains a subgraph isomorphic to K_{n-1} is an acyclic niche graph.*

Proof. Let $n \geq 4$ and G be an incomplete graph with vertex set $\{x_1, \ldots, x_n\}$ and suppose that $\{x_2, \ldots, x_n\}$ generates a complete subgraph of G. Suppose moreover that the neighbors in G of x_1 are $x_2, \ldots, x_i$ (for $i < n$). The arc set

$$A(D) = \{x_1 \to x_k : k = 2, \ldots, n\} \cup \{x_k \to x_{k+1} : k = 2, \ldots, i\}$$

defines a digraph D on the same vertex set. (Set the second part of the definition of the arc set to the empty set if x_1 is isolated in G). It is routine to check that D is acyclic and that its niche graph is G. □

The dense graphs of order 3 and 4 which are not covered by these two results are easily seen to be acyclic niche graphs with the exception of K_3, $K_{1,3}$, and C_4. As already pointed out, $K_{1,3}$ is a loop niche graph which is not a cyclic niche graph. In [5] it is shown that neither of the other two is an acyclic niche graph. It follows easily that K_3 and, with only a little more work, that C_4 cannot be asymmetric either. The digraphs in Figure 2 show that K_3 and C_4 are cyclic niche graphs.

FIGURE 2. Niche digraphs for K_3 and C_4

The rest of the paper will be devoted to dense graphs of order at least 5. We say that a graph G of order n *satisfies condition* ★ if:

1. $n \geq 5$,
2. G is not complete, and
3. $V(G) = W \cup \{r, s\}$, where $\langle W \rangle \cong K_{n-2}$.

The symbols r and s will be reserved througout the rest of the paper for the two vertices of G outside of W. Our determination of niche category of graphs satisfying condition ★ will be split into the case in which r is adjacent to s in G, undertaken in Section 2, and the case in which r is not adjacent to s, covered in Section 3.

2. VERTICES r AND s ADJACENT IN THE NICHE GRAPH

Throughout this section we assume that G is a graph satisfying ★ and that vertices r and s are adjacent. We will show that all these graphs are acyclic niche graphs by providing appropriate digraphs.

Notation. *The following abbreviations will be useful:*

$$nbr'(r) = nbr_G(r) - \{s\}$$
$$nbr'(s) = nbr_G(s) - \{r\}.$$

If V is the vertex set of a digraph, $B \subset V$ and $x \in V$, then

$$\{x \to B\} = \{x \to y | y \in B\} \quad \text{and}$$
$$\{B \to x\} = \{y \to x | y \in B\}.$$

Lemma 1. *If $nbr'(r) \neq nbr'(s)$ and $nbr'(r) \cap nbr'(s) \neq \emptyset$, then G is an acyclic niche graph.*

Proof. Suppose, without loss of generality, that $nbr'(s) - nbr'(r) \neq \emptyset$ and fix $u \in nbr'(r) \cap nbr'(s)$ and $v \in nbr'(s) - nbr'(r)$. Let D be the digraph with the following arc set:

$$\{r \to W, u \to s, s \to v\} \cup \{y \to u | y \in nbr'(r) - nbr'(s)\}$$
$$\cup \{u \to y | y \in nbr'(s) - \{u\}\} \cup \{y \to v | y \in nbr'(r) \cap nbr'(s)\}.$$

We must show that D is acyclic and has niche graph G. First, notice that in checking a digraph for cycles, one may remove vertices with either indegree or outdegree equal to zero. Accordingly, remove from D the vertex r (since its indegree is zero) and all vertices of $W - nbr'(r)$ (since each has outdegree equal to zero). In the result, no vertex other than u has positive indegree and positive outdegree, so D is acyclic. All the vertices of W are adjacent in the niche graph of D by virtue of the common in-neighbor r. The vertex s is adjacent to each of its neighbors in G except u and r by virtue of the common in-neighbor u and is adjacent to u and to r since v is a common out-neighbor. Next, the edges between r and the vertices of $nbr'(r) - nbr'(s)$ arise from the common out-neighbor u and the edges between r and the vertices of $nbr'(r) \cap nbr'(s)$ are due to the common out-neighbor v. It remains to check that the niche graph of D has no edges not in G. To this end, it is only necessary to check that the in-neighbors of each vertex with indegree greater than one are mutually adjacent in G and that the same is true of the out-neighbors of each vertex with outdegree greater than one. In the digraph under consideration, only r and u have out-degree greater than one and only the vertices of $nbr'(s)$ have indegree greater than one. But each of the sets: $out_D(r)$, $out_D(u)$, $in_D(u)$, and $in_D(v)$ generates a complete subgraph of G and, if $x \in nbr'(s) - \{u, v\}$, then $in_D(x) = \{u, r\}$. The proof is therefore complete. □

The details showing that the digraph in the preceeding proof is acyclic and has the desired niche graph are typical. In the rest of this section and the next, we will provide appropriate arc sets and omit the remaining details.

Lemma 2. *If $nbr'(r) = nbr'(s)$, then G is an acyclic niche graph.*

Proof. Suppose G satisfies the stated conditions, let $N = nbr'(r) = nbr'(s)$ and fix $w \in W - N$ (nonempty for the G under consideration). A suitable arc set is

$$A(D) = \{r \to W, s \to w, N \to w\}$$

□

Lemma 3. *If $nbr'(r) \cap nbr'(s) = \emptyset$, then G is an acyclic niche graph.*

Proof. Suppose G satisfies the stated conditions. Assume, without loss of generality, that $|nbr'(s)| \leq |nbr'(r)|$. In case $|nbr'(s)| = |nbr'(r)| = 0$, the conclusion follows from Lemma 2. If $|nbr'(s)| = 0 < |nbr'(r)|$, set

$$A(D) = \{r \to W, s \to r, s \to nbr'(r)\}.$$

If $nbr'(r) = \{u\}$ and $nbr'(s) = \{v\}$ (so $u \neq v$), then fix $w \in W - \{u, v\}$ (such a w must exist since $|G| > 4$) and let

$$A(D) = \{r \to W, u \to s, u \to v, s \to w\}.$$

Finally, if $|nbr'(r)| > 1$ and $|nbr'(s)| \geq 1$, fix distinct vertices u and w from $nbr'(r)$ and v from $nbr'(s)$. A suitable arc set is

$$A(D) = \{r \to W, u \to nbr'(s), nbr'(r) \to v, s \to w, u \to s\}.$$

□

The list of conditions on $nbr'(r)$ and $nbr'(s)$ from the last three lemmas is exhaustive:

Theorem 1. *If G satisfies ★ and r is adjacent to s, then G is an acyclic niche graph.*

3. Vertices r and s not adjacent in the niche graph

In this section we consider the niche category of graphs satisfying ★ in which vertices r and s are not adjacent. We will show that all such graphs are acyclic niche graphs with the exception of four specific graphs and an infinite family (namely those for which $nbr(r) = nbr(s) = \{u\}$). The investigation of these graphs is undertaken in a number of lemmas depending on the interaction of the sets W, $nbr(r)$, and $nbr(s)$. Throughout this section we employ the notation of ★ and assume that r is not adjacent to s.

Lemma 4. *If neither of the sets $nbr(r)$ nor $nbr(s)$ is a subset of the other, then G is an acyclic niche graph*

Proof. Fix $u \in nbr(r) - nbr(s)$ and $w \in nbr(s) - nbr(r)$ and let

$$A(D) = \{r \to W, u \to nbr(s), nbr(r) \to w, u \to s\}.$$

□

Lemma 5. *If $nbr(s) = \emptyset$, then G is an acyclic niche graph.*

Proof. If $nbr(r) = W$, then $G \cong K_{n-1} \cup K_1$, where n is the order of G. Such a G is an acyclic niche graph [CJLS]. Otherwise, fix $u \in W - nbr(r)$ and let

$$A(D) = \{r \to W\} \cup \{nbr(r) \to u\}.$$

□

Lemma 6. *If $nbr(s) \subseteq nbr(r) = W$, then G is an acyclic niche graph.*

Proof. If $nbr(s) = nbr(r)$, let

$$A(D) = \{r \to s, r \to W, W \to s\}.$$

Otherwise, fix $u \in W - nbr(s)$ and set

$$A(D) = \{s \to W, s \to r, nbr(s) \to u\}.$$

□

Lemma 7. *If $\emptyset \neq nbr(s) \subsetneq nbr(r) \subsetneq W$, then G is an acyclic niche graph.*

Proof. Fix $u \in nbr(r) - nbr(s)$, and $w \in W - nbr(r)$ and set

$$A(D) = \{r \to W, u \to nbr(s), u \to s, (nbr(r) - \{u\}) \to w\}.$$

□

The results of this section so far leave only the case in which $nbr(r)$ and $nbr(s)$ are equal and nonempty.

Notation. *For the rest of this section, let $N = nbr(r) = nbr(s)$ and suppose that $N \neq \emptyset$. By Lemma 6, we may assume that N is a proper subset of W.*

Lemma 8. *If $|N| \geq 4$, then G is an acyclic niche graph.*

Proof. Fix distinct vertices $u \in W - N$ and $w, x, y, z \in N$ and let

$$A(D) = \{w \to s, w \to (N - \{w, x\}), x \to (W - \{x\}), (N - \{w\}) \to u,$$
$$r \to \{u, x, y\}, s \to z, y \to z, (W - N - \{u\}) \to w\}.$$

Since the arc set in this case is fairly involved, we will include the details of this proof. To check D for cycles, note that $indeg_D(r) = outdeg_D(u) = 0$ so they may be removed from consideration. In the resulting digraph, x has no in-neighbors and z has no out-neighbors, so they may be removed.

This leaves a digraph in which only w has both in-neighbors and out-neighbors, thus D is acyclic. The vertices of $W - \{x\}$ are out-neighbors of x and so are adjacent in the niche graph of D. On the other hand, u is a common out-neighbor of x and the vertices of $N - \{x, w\}$ and w is a common out-neighbor of x and the vertices of $W - N$. Finally, y is a common out-neighbor of x and w. Thus the vertices of W are mutually adjacent in the niche graph. Vertex r is adjacent to w due to the common out-neighbor y and r is adjacent to the rest of the vertices of N due to the common out-neighbor u. Edges between s and the vertices of $N - \{w, x\}$ arise from the common in-neighbor w and those between s, w, and x arise from the common out-neighbor z. It remains to check that the niche graph of D contains no edges not in G. For this, it is sufficient to examine only the vertices adjacent in D to r and s. Specifically, $out(w)$ and $in(z)$ are subsets of $\{s\} \cup N$ and $in(x)$, $in(y)$, and $in(u)$ are all subsets of $\{r\} \cup N$. Since both $\{s\} \cup N$ and $\{r\} \cup N$ generate complete subgraphs of G, the proof is complete. □

Lemma 9. *If $|N| \geq 2$ and $|W - N| \geq 3$, then G is an acyclic niche graph.*

Proof. Fix distinct vertices $u, w \in N$ and $x, y, z \in W - N$ and let

$$A(D) = \{s \to x, N \to x, w \to (W - \{w\}), (W - N - \{x\}) \to u, \\ s \to w, y \to r, r \to z, y \to (N - \{w\})\}.$$

□

Lemma 10. *If $|N| = 1$ and $|W - N| \geq 4$, then G is an asymmetric niche graph, but not an acyclic niche graph.*

Proof. If $|N| = 1$, then G belongs to a class of graphs called *novas* . In [5] it is proved that no such graph is an acyclic niche graph. On the other hand, if $N = \{u\}$ and v, x, y, and z and distinct vertices in $W - N$, then the set

$$A(D) = \{u \to z, v \to r, x \to s, z \to x, (W - \{u, z\}) \to u, y \to (W - \{y\})\}$$

defines an asymmetric digraph with niche graph G. □

The six remaining graphs satisfying ★ in which r and s are not adjacent are shown in the Figure 3. (The vertices within the rectangles generate complete subgraphs whose edges are not shown.)

Below is an arcset A_i defining a digraph with niche graph isomorphic to G_i for each integer i from 1 to 6. Refering to Figure 3, for each i between 1 and 6, the vertices outside the rectangle are denoted r and s and those within the rectangle are denoted (top to bottom) $x_1, \ldots, x_n$, where $n = |G_i| - 2$.

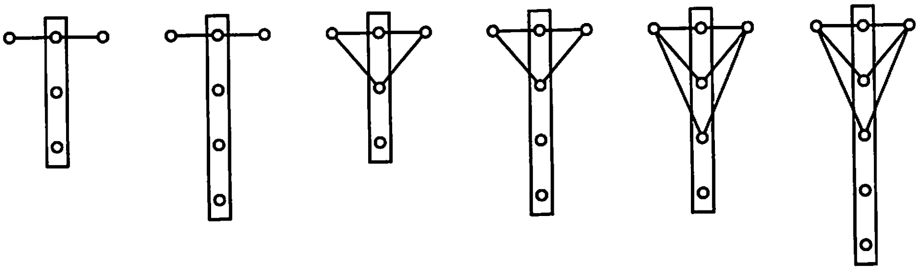

FIGURE 3. Exceptional Dense Graphs

$A_1 = \{r \to x_2, x_1 \to \{x_2, x_3\}, x_2 \to \{x_1, x_3\}, x_3 \to \{x_1, s\}\}$
$A_2 = A_1 \cup \{x_1 \to x_4, x_2 \to x_4\}$
$A_3 = \{r \to x_i | i = 1, 2, 3\} \cup \{x_1 \to s, x_2 \to s, x_1 \to x_2, x_2 \to x_1\}$
$A_4 = A_3 \cup \{r \to x_4\}$
$A_5 = \{r \to x_i | i = 1, 2, 3, 4\} \cup \{\{x_1, x_2, x_3\} \to s, x_1 \to x_2, x_2 \to x_3, x_3 \to x_1\}$
$A_6 = A_5 \cup \{r \to x_5\}$

These arc sets demonstrate that G_1, G_2, G_3, and G_4 are cyclic niche graphs and that G_5 and G_6 are asymmetric niche graphs. To complete the classification of dense graphs, it remains to show that one can do no better in each case. We provide complete details to show that neither G_1 nor G_2 has an asymmetric niche digraph in Lemma 11. We have omitted the details of a similar, but much longer, argument showing that the same is true of G_3 and G_4. Finally, an exhaustive computer search revealed that neither G_5 nor G_6 has an acyclic niche digraph. (The results of this computer search confirm those of an independent one reported in [6])

Lemma 11. *Neither G_1 nor G_2 is the niche graph of any asymmetric digraph.*

Proof. Suppose, to the contrary, that D is an asymmetric digraph with niche graph G (either G_1 or G_2) and, moreover, that any arc whose removal does not affect the niche graph has been removed. Observe first that this implies that r and s are not adjacent in D. Indeed, if $r \to s$ in D, then without loss of generality, we may assume that $x_1 \to s$ in D. Now the edge $[s, x_1]$ in G requires $r \to x_1$ but this leaves no way to produce the edge $[x_1, x_2]$ in the niche graph. Thus D contains no arc connecting r and s. There is no loss of generality in assuming then that x_2 is a common in-neighbor of r and x_1. The edge $[s, x_1]$ can only be achieved by x_3 as either a common in-neighbor or a common out-neighbor. In the first case, the edge $[x_1, x_3]$ requires x_4 as a common in-neighbor, as which stage, the edge $[x_1, x_4]$ is impossible in an asymmetric digraph. Similaraly, in the second case, the edge $[x_1, x_3]$ requires x_4 as a common out-neighbor, but again, the edge $[x_1, x_4]$ is impossible. Hence, the niche graph of D cannot be G, a contradiction. □

4. Summary

This completes the classification of the niche category of all dense graphs. All such graphs are acyclic niche graphs except the dense novas, the complete graphs, C_4, and the graphs G_3, G_4, G_5, and G_6 from Figure 3. The dense novas of order at least 7, the complete graphs of order at least 4, C_4, and the graphs G_5 and G_6 from Figure 3 are asymmetric niche graphs. The graphs G_i in Figure 3 for $1 \leq i \leq 4$ are not asymmetric niche graphs, but are cyclic niche graphs. Finally, $K_{1,3}$, the only remaining dense nova, is not a cyclic niche graph, but is a loop niche graph.

References

[1] C. A. Anderson, *Loop and cyclic niche graphs*, Linear Alg. Applic. **217** (1995), 5–13.

[2] Stephen Bowser and Charles Cable, *Cliques and niche graphs*, Congressus Numeratium **76** (1990), 151–156.

[3] Stephen Bowser and Charles Cable, *Cyclic and loop niche numbers of novas and wheels*, Congressus Numeratium **103** (1994), 215–218.

[4] Stephen Bowser, Charles Cable and Richard Lundgren, *Niche Graphs and Mixed Pair Graphs of Tournaments*, to appear in J. Graph Theory.

[5] Charles Cable, Kathryn F. Jones, J. R. Lundgren, and Suzanne Seager, *Niche Graphs*, Discrete Appl. Math. **23** (1989), 231–241.

[6] W. V. Gehrlein and P. C. Fishburn, *The smallest graphs with niche number three*, Computers & Mathematics with Applications **27**(1994), 53–57.

[7] J. R. Lundgren, *Food webs, competition graphs, competition-common enemy graphs, and niche graphs*, in F. S. Roberts, editor, *Applications of Combinatorics and Graph Theory to the Biological and Social Sciences*, Springer-Verlag, 1989. In *Volumes in Mathematics and Its Applications, Vol. 17*, 221–244.

[8] Suzanne Seager, *Niche Graph Properties of Trees*, Congressus Numeratium **81** (1991), 149–156.

[9] Suzanne Seager, *Relaxations of niche graphs*, Congressus Numeratium **96** (1993), 205–214.

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