Degree Sequences with Repeated Values

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Abstract

A given nonincreasing sequence $\mathcal{D} = (d_1, d_2, \cdots, d_n)$ is said to contain a (nonincreasing) repetition sequence $\mathcal{D}^* = (d_{i_1}, d_{i_2}, \cdots, d_{i_k})$ for some $k \leq n-2$ if all values of $\mathcal{D} - \mathcal{D}^*$ are distinct and for any $d_{i_l} \in \mathcal{D}^*$ there exists some $d_t \in \mathcal{D} - \mathcal{D}^*$ such that $d_{i_l} = d_t$. For any pair of integers n and k with $n \geq k+2$, we investigate the existence of a graphic sequence which contains a given repetition sequence. Our main theorem contains the known results for the special case $d_{i_1} = d_{i_k}$ if $k = 1$ or $k = 2$ (see [1, 5, 2]).

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1 Introduction

A nonincreasing sequence $(d_1, ..., d_n)$ with $0 \leq d_i \leq n-1$ for each i is called graphic if it gives the vertex degrees of some simple graph of order n. These sequences are well characterized by the classical result of Erdős and Gallai (see Theorem 5), but some questions cannot be answered directly from that result. An easier result is that a graphic sequence must contain at least one repetition. In this paper we study repetition patterns in degree sequences, that is, possible patterns of repeated degrees.

For example, it is known that if a graphic sequence contains precisely one repetition, then that repetition must lie between $\frac{n}{2} - 1$ and $\frac{n}{2}$ (see Theorem 1). Is there, for example, a graphic sequence with k repeated values of the integer $\frac{n}{2}$, for $3 \leq k \leq n-1$, with all other values distinct? Is it possible to have a graph such that two degrees are $\frac{3n}{4}$, two are $\frac{n}{2}$, two are $\frac{n}{4}$, and all others are distinct? The first question has been answered previously in Theorems 2 and 3 (in the affirmative). A consequence of the present work is that the answer to the second question is also yes.

Definition: Let $\mathcal{D} : d_1 \geq d_2 \geq \cdots \geq d_n$ be a finite sequence of non-negative integers. A subsequence $\mathcal{D}^* : d_{i_1} \geq d_{i_2} \geq \cdots \geq d_{i_k}$ is called the repetition subsequence, denoted by RS, if

(1) all values of $\mathcal{D} - \mathcal{D}^*$ are distinct, and

(2) for any $d_{i_l} \in \mathcal{D}^*$, there exists some $d_t \in \mathcal{D} - \mathcal{D}^*$ such that $d_{i_l} = d_t$, and $t < i_l$.

Given two integers n and k with $k \leq n-1$ and a sequence $\mathcal{D}^* : n > d_{i_1} \geq \cdots \geq d_{i_k} \geq 0$, we investigate the existence of a graphic sequence $\mathcal{D}$ with single valued repetitions having $\mathcal{D}^*$ as its RS. When $k = n-1$, this question asks about the existence of a d_1-regular graph on n vertices, and it is an elementary exercise to show that such a graph exists if and only if not both d_1 and n are odd; hence we assume $k \leq n-2$.

For the special case $d_{i_1} = d_{i_k}$, the following three results (see [1, 2, 5]) concern graphic sequences with single repetitions.

Theorem 1 (Behzad, Chartrand) *If $n \geq 2$, there exists a graphic sequence $(d_1, \ldots, d_n)$ with a repetition sequence $\mathcal{D}^* : d_{i_1} = j$ if and only if*

$$\frac{n}{2} - 1 \leq j \leq \frac{n}{2}.$$

Theorem 2 (Hutchinson) *If $n \geq 4$ then there exists a graphic sequence $(d_1, \ldots, d_n)$ with a repetition sequence $\mathcal{D}^* : d_{i_1} = d_{i_2} = j$ if and only if*

$$\frac{n-3}{4} \leq j \leq \frac{3n-1}{4}.$$

□

Theorem 3 (Chen, Piotrowski, Shreve) *If $n \geq k+2$, then there exists a graphic sequence having $\mathcal{D}^* : d_{i_1} = d_{i_2} = \cdots = d_{i_k} = j$ as its RS if and only if*

$$\frac{n-k-1}{2k} \leq j \leq n-1-\frac{n-k-1}{2k}.$$

□

Note that, with $j = \frac{n}{2}$ and $3 \leq k \leq n-1$, this last inequality holds, thus giving an affirmative answer to one question posed above.

Let n and k be positive integers with $n \geq k+2$, and for two nonincreasing sequences of nonnegative values $\mathcal{D} = (d_1, d_2, \cdots, d_n)$ and $\mathcal{D}^* = (d_{i_1}, d_{i_2}, \cdots, d_{i_k})$, we define

$$a = a(\mathcal{D}) = \min\{x : d_{x+1} \leq x\},$$

and for $0 \leq z \leq k \leq n-1$, we define

$$S(z) = (z-1)(z-k) + n(z-\frac{1}{2}) + \frac{1-k}{2},$$

and

$$f(z) = \sum_{l \leq z} d_{i_l} - \sum_{l > z} d_{i_l}.$$

Without loss of generality we will assume that $d_{i_k} \leq \frac{n-1}{2}$. (If necessary, take the complementary graph and let $d'_{i_l} = n-1-d_{i_{k+1-l}}$ for $1 \leq l \leq k$. Then one may use $\mathcal{D}' = \{d'_{i_1}, d'_{i_1}, \cdots, d'_{i_k}\}$.) Set

$$m = m(\mathcal{D}^*) = \min\{x : d_{i_{x+1}} \leq \left\lfloor \frac{n-k+2x}{2} \right\rfloor\}.$$

Thus, $0 \leq m \leq k-1$.

Example 1 Consider the complete bipartite graph $K_{\frac{2n}{3},\frac{n}{3}}$ with the smaller part completed to $K_{\frac{n}{3}}$. Thus, $\mathcal{D} = (n-1, \ldots, n-1, \frac{n}{3}, \ldots, \frac{n}{3})$ and

$\mathcal{D}^* = (d_2, \ldots, d_{\frac{n}{3}}, d_{\frac{n}{3}+2}, \ldots, d_n)$. That is, $(n-1)$ appears in the sequence $\mathcal{D}^*$ with multiplicity $\frac{n}{3} - 1$ and $\frac{n}{3}$ appears with multiplicity $\frac{2n}{3} - 1$ Thus, $k = n-2$, $a = \frac{n}{3}$, and $m = \frac{n}{3} - 1$.

In this paper, we will prove the following result.

Theorem 4 *(1) If there exists a graphic sequence $\mathcal{D} = (d_1, d_2, \cdots, d_n)$ having $\mathcal{D}^*$ as its RS, then there exists some q with $i_q \leq a(\mathcal{D}) < i_{q+1}$ ($i_l > a(\mathcal{D})$ for all $1 \leq l \leq k$ when $q = 0$) such that*

$$f(q) \leq S(q).$$

(2) Given n and $\mathcal{D}^* = d_{i_1}, d_{i_2}, \ldots, d_{i_k}$, a nonincreasing sequence with $0 \leq d_{i_l} \leq n-1$ for each entry, then if $d_{i_1} \leq n - k + m - 2$, $d_{i_k} \geq m$ and

$$f(m) \leq S(m),$$

then there is a graphic sequence having $\mathcal{D}^*$ as its RS.

□

In Example 1, we have $q = \frac{n}{3} - 1$ and $f(q) = \frac{n^2}{9} - n + 1 \leq \frac{n^2}{9} - \frac{n}{3} - \frac{1}{2} = S(q)$. Consider again the question of whether there is a graphic sequence in which two of the degrees are $\frac{3n}{4}$, two are $\frac{n}{2}$, two are $\frac{n}{4}$, and all others are distinct. Thus, $\mathcal{D}^* = (\frac{3n}{4}, \frac{n}{2}, \frac{n}{4})$, $k = 3$, $m = 2$, $f(2) = n$, $S(2) = \frac{3n}{2} - 2$ showing that $\mathcal{D}^*$ is an RS by Theorem 4 (2).

In the next section, we will prove Theorem 4 and show how it implies Theorems 1 and 2.

2 The Proof of Theorem 4

If $\mathcal{D} = (d_1, \ldots, d_n)$ is a nonincreasing sequence of integers such that $0 \leq d_i \leq n-1$ for all $1 \leq i \leq n$, we define

$$\Phi_{\mathcal{D}}(x) = x(x-1) + \sum_{i=x+1}^{n} \min\{x, d_i\} - \sum_{i=1}^{x} d_i. \tag{1}$$

In the proof of our result we will use the following classical theorem([4]).

Theorem 5 (Erdős-Gallai) *The sequence $\mathcal{D}$ is graphic if and only if* $\sum_{i=1}^{n} d_i$ *is even, and*

$$\Phi_{\mathcal{D}}(x) \geq 0 \quad \text{for every } x,\ 1 \leq x \leq n. \tag{2}$$

□

See [3] for a simple proof of Theorem 5.

Observe that if $x \geq a$, then formula (1) simplifies to

$$\Phi_{\mathcal{D}}(x) = x(x-1) + \sum_{i=x+1}^{n} d_i - \sum_{i=1}^{x} d_i. \tag{3}$$

If in addition, $\sum_{i=1}^{n} d_i$ is odd, a simple parity argument shows that $\Phi_{\mathcal{D}}(x) \neq 0$.

Proposition 6 *If $\Phi_{\mathcal{D}}(a) \geq 0$, then $\Phi_{\mathcal{D}}(x) > 0$ for every $a < x \leq n$.*

Proof. For $x > a$, we have $d_{x+1} < x$, and $\Phi_{\mathcal{D}}(x+1) > \Phi_{\mathcal{D}}(x)$. In fact

$$\begin{aligned}
\Phi_{\mathcal{D}}(x+1) &= x(x+1) + \sum_{i=x+2}^{n} d_i - \sum_{i=1}^{x+1} d_i \\
&> x(x-1) + 2d_{x+1} + \sum_{i=x+2}^{n} d_i - \sum_{i=1}^{x+1} d_i = \Phi_{\mathcal{D}}(x).
\end{aligned}$$

A simple induction argument shows that $\Phi_{\mathcal{D}}(x) > \Phi_{\mathcal{D}}(a) \geq 0$ for every $a < x \leq n$. □

In the following, we assume $\mathcal{D} = (d_1, \ldots, d_n)$ is a sequence with $\mathcal{D}^* = (d_{i_1}, d_{i_2}, \cdots, d_{i_k})$ as its RS, and that there are q (p) indices in $\mathcal{D}^*$ which are less than or equal to (greater than) a. Let s and t be the the number of integers in the intervals $[0, a-1]$ and $[a, n-1]$ respectively, which do not appear in the sequence, where $a = a(\mathcal{D})$. Obviously, we have $q + p = k$. Since there are $n - k$ distinct values in $\mathcal{D}$, there are k values missed in $\mathcal{D}$. Thus, we also have $s + t = k$.

Proposition 7 *(i) $a = a(\mathcal{D}) = \frac{n+q-t}{2}$, if $a+1 \in \{i_1, i_2, \cdots, i_k\}$ or $d_{a+1} < a$.*

(ii) $a = \frac{n+q-t-1}{2}$, if $d_{a+1} = a$ and $a+1 \neq i_l$ for all $1 \leq l \leq k$.

Proof. (i) If $d_{a+1} < a$, then the number of distinct possible values, which d_i may take for $a+1 \le i \le n$, is a. Since there are s values missed and p values repeated, we have $n - a = a + p - s$. By $p = k - q$ and $s = k - t$, we obtain $a = \frac{n+q-t}{2}$.

If $d_{a+1} = a$ and $a + 1 \in \{i_1, i_2, \cdots, i_k\}$, then d_i may take $a+1$ distinct values for $a+1 \le i \le n$. Since there are s values missed and $p-1$ repeated (p are repeated except that now d_{a+1} does not count as a repeat), we have $n-a = a+1+p-1-s$. By $s+t = k$ and $p+q = k$, we obtain $a = \frac{n+q-t}{2}$.

(ii) If $d_{a+1} = a$ and $a+1 \ne i_l$ for all $1 \le l \le k$, for the same reason as above, we obtain $n - a = a + 1 + p - s$ and hence $a = \frac{n+q-t-1}{2}$. □

Proposition 8 *If $\mathcal{D}$ is a graphic sequence, then $\Phi_{\mathcal{D}}(a) \le S(q) - f(q)$.*

Proof. Let $a_i \ge a$ $(1 \le i \le t)$ and $b_l < a$ $(1 \le l \le s)$ be the missed values.

If $d_{a+1} \le a - 1$ or $a + 1 = i_l$ for some $1 \le l \le k$, then $a = \frac{n+q-t}{2}$, and

$$\begin{aligned}
\Phi_{\mathcal{D}}(a) &= a(a-1) + \sum_{i=a+1}^{n} d_i - \sum_{i=1}^{a} d_i \\
&= a(a-1) + \sum_{l>q} d_{i_l} - \sum_{l\le q} d_{i_l} + \sum_{i=1}^{a-1} i - \sum_{i=1}^{s} b_i - \sum_{i=a}^{n-1} i + \sum_{i=1}^{t} a_i \\
&\le a(a-1) - f(q) + \frac{a(a-1)}{2} - \frac{s(s-1)}{2} - \frac{n(n-1)}{2} + \frac{a(a-1)}{2} \\
&\quad + tn - \frac{t(t+1)}{2} \\
&= n(q - \frac{1}{2}) + h_1(t) - f(q),
\end{aligned}$$

where

$$\begin{aligned}
h_1(t) &= \frac{(q-t)^2}{2} - (q-t) - \frac{(t+1)t}{2} - \frac{(k-t)(k-t-1)}{2} \\
&= \frac{-t^2 + 2(k-q)t + q^2 - k^2 - 2q + k}{2}.
\end{aligned}$$

Observe that $h_1(t)$ is a quadratic function with respect to t, and that $h_1'(t) = 0$ if $t = k - q$. Thus, $h_1(k-q) = \max\{h(t) : 0 \le t \le k-1\}$ and

$$h_1(k-q) = \frac{(k-q)^2 + q^2 - 2q - k^2 + k}{2}$$

$$= \quad (1-q)(k-q) - \frac{k}{2}$$

Hence, we have $\Phi_{\mathcal{D}}(a) \leq S(q) - f(q)$.

If $d_{a+1} = a$ and $a + 1 \neq i_l$ for all $1 \leq l \leq k$, then by Proposition 7, $a = \frac{n+q-t-1}{2}$ and

$$\begin{aligned}
\Phi_{\mathcal{D}}(a) &= a(a-1) + \sum_{i=a+1}^{n} d_i - \sum_{i=1}^{a} d_i \\
&= a(a-1) + \sum_{l>q} d_{i_l} - \sum_{l \leq q} d_{i_l} + \sum_{i=1}^{a} i - \sum_{i=1}^{s} b_i - \sum_{i=a+1}^{n-1} i + \sum_{i=1}^{t} a_i \\
&\leq a(a-1) - f(q) + \frac{a(a+1)}{2} - \frac{s(s-1)}{2} - \frac{n(n-1)}{2} + \frac{(a+1)a}{2} \\
&\quad + tn - \frac{t(t+1)}{2} = n(q - \frac{1}{2}) + h_2(t) - f(q),
\end{aligned}$$

where

$$\begin{aligned}
h_2(t) &= \frac{(q-t-1)^2 - t^2 - (k-t)^2 + k - 2t}{2} \\
&= \frac{-t^2 + 2(k-q)t + q^2 - k^2 - 2q + k + 1}{2}.
\end{aligned}$$

Since $h_2(t) = h_1(t) + \frac{1}{2}$, reasoning as above, we have

$$h_2(t) \leq \max\{h_2(t) : 0 \leq t \leq k\} = h_2(k-q) = h_1(k-q) + \frac{1}{2}.$$

Hence, $\Phi_{\mathcal{D}}(a) \leq S(q) - f(q)$. □

Let $\mathcal{D}^* = \{d_{i_1}, d_{i_2}, \cdots, d_{i_k}\}$ be a nonincreasing sequence satisfying $d_{i_1} \leq n - k + m - 2$, $d_{i_k} \geq m$ and $f(m) \leq S(m)$. We define $\mathcal{E} = (e_1, \ldots, e_n)$ to be the following sequence.

$$e_i = \begin{cases} n-i-k+m & \text{if } 1 \le i \le n-d_{i_1}-k+m, \\ d_{i_1} & \text{if } i = n-d_{i_1}-k+m+1, \\ n-i-k+m+1 & \text{if } d_{i_2} \le n-i-k+m+1 \le d_{i_1}-1, \\ d_{i_2} & \text{if } i = n-k-d_{i_2}+m+2, \\ & \vdots \\ n-i-k+m+l-1 & \text{if } d_{i_l}+1 \le n-i-k+m+l \le d_{i_{l-1}}, \\ d_{i_l} & \text{if } i = n-d_{i_l}-k+m+l, \\ n-i-k+m+l & \text{if } d_{i_{l+1}} \le n-i-k+m+l \le d_{i_l}-1, \\ & \vdots \\ n-i+m-1 & \text{if } d_{i_k}+1 \le n-i+m \le d_{i_{k-1}}, \\ d_{i_k} & \text{if } i = n-d_{i_k}+m, \\ n-i+m & \text{if } n-d_{i_k}+m+1 \le i \le n. \end{cases} \tag{4}$$

In other words, $\mathcal{E}$ is the sequence with $\mathcal{D}^*$ as its RS, and such that the $k-m$ largest and the m smallest integers of the interval $[0, n-1]$ do not appear in the sequence.

Proposition 9 *Let $e_{h_l} = d_{i_l}$ for $1 \le l \le k$ and $a = a(\mathcal{E})$.*

(i) If $m = 0$, then $h_l > a$ for all $1 \le l \le k$. (Thus $q_\mathcal{E} = 0$, by Proposition 7.)

(ii) If $m \ge 1$, then either, $h_{m-1} \le a < h_m$ and $f_\mathcal{E}(m-1) \le S_\mathcal{E}(m-1)$, or $h_m \le a < h_{m+1}$, where $h_0 = h_1 - 1$. (Thus $q_\mathcal{E} = m-1$ or m, respectively, by Proposition 7.)

Proof. (i) Since $m = 0$, $d_{i_1} \le \lfloor \frac{n-k}{2} \rfloor$. By the construction of $\mathcal{E}$, $h_1 = n-k-d_{i_1}+1 \ge d_{i_1}+1 = e_{h_1}+1$. Thus, $h_1 - 1 \ge a$. Hence, $h_l > a$ for all $1 \le l \le k$.

(ii) If $m \ge 1$, by the construction of $\mathcal{E}$, we have $h_m = n - d_{i_m} - k + 2m$. If $n-k$ is even, $d_{i_m} > \frac{n-k+2(m-1)}{2}$. Thus,

$$h_m - 1 < \frac{n-k+2m}{2} \le d_{i_m} = e_{h_m}.$$

By the definition of a, $h_m - 1 < a$. Hence $h_m \le a$. Similarly, since $h_{m+1} = n - d_{i_{m+1}} - k + 2m + 1$ and $d_{i_{m+1}} \le \frac{n-k+2m}{2}$, we obtain

$$h_{m+1} \ge \frac{n-k+2m}{2} + 1 \ge d_{i_{m+1}} + 1 = e_{h_{m+1}} + 1.$$

Hence, $h_{m+1} - 1 \geq a$, that is, $h_{m+1} > a$. If $n-k$ is odd, then $d_{i_m} \geq \frac{n-k+2m-1}{2}$. Thus,

$$h_m - 1 \leq \frac{n-k+2m-1}{2} \leq d_{i_m} = e_{h_m}.$$

By the definition of $a(\mathcal{E})$, we have $h_m - 1 \leq a(\mathcal{E})$. When $h_m \leq a$, for the same reason as above it follows that $h_{m+1} > a$. Thus, the proposition holds.

When $h_m = a+1$, by the construction of $\mathcal{E}$, we obtain $a = \frac{n-k+2m-1}{2} = e_{h_m} = d_{i_m}$. Hence, $h_{m-1} \leq a < h_m$ and

$$f_{\mathcal{E}}(m-1) = f_{\mathcal{E}}(m) - 2e_{h_m} \leq S_{\mathcal{E}}(m) - (n-k+2m-1) \leq S_{\mathcal{E}}(m-1).$$

□

Proposition 10 *If $\Phi_{\mathcal{E}}(a) \geq 0$ then $\Phi_{\mathcal{E}}(x) \geq 0$ for every $1 \leq x \leq n$.*

Proof. Notice by the construction of $\mathcal{E}$ that $e_i - e_{i+1} \leq 1$ for any i such that $1 \leq i \leq n$. By Proposition 6, it is enough to show that $\Phi_{\mathcal{E}}(x) \geq 0$ for every $1 \leq x < a$; i.e., we may assume that $e_{x+1} > x$. Then

$$\begin{aligned}\Phi_{\mathcal{E}}(x) - \Phi_{\mathcal{E}}(x+1) &= e_{x+1} - x - \sum_{i=x+2}^{n} (\min\{x+1, e_i\} - \min\{x, e_i\}) \\ &= e_{x+1} - x - |I|,\end{aligned}$$

where $I = \{e_i \ : \ e_i \geq x+1, \ i \geq x+2\}$.

If $\{x+2, x+3, \cdots, e_{x+3}\} \cap \mathcal{D}^* = \emptyset$, then $|I| = e_{x+2} - x$, and

$$\Phi_{\mathcal{E}}(x) - \Phi_{\mathcal{E}}(x+1) = e_{x+1} - e_{x+2} \geq 0.$$

If $\{x+2, x+3, \cdots, e_{x+3}\} \cap \mathcal{D}^* \neq \emptyset$, then, $|I| \geq e_{x+2} - x + 1$ and

$$\Phi_{\mathcal{E}}(x) - \Phi_{\mathcal{E}}(x+1) = e_{x+1} - e_{x+2} - 1 \leq 0.$$

Trivially, $\Phi_{\mathcal{E}}(1) \geq 0$; a simple induction argument completes the proof. □

Now, we are ready to prove the main theorem.

Proof of Theorem 4, Part (1)

Let $\mathcal{D} = (d_1, \ldots, d_n)$ be a sequence with $\mathcal{D}^* = \{d_{i_1}, d_{i_2}, \cdots, d_{i_k}\}$, and let q be the number of indices of $\mathcal{D}^*$ which are less than or equal to $a = a(\mathcal{D})$.

Then $i_q \leq a < i_{q+1}$ or $i_l > a$ for all $1 \leq l \leq k$ when $q = 0$. By Theorem 5 and Proposition 8, we have

$$0 \leq \Phi_{\mathcal{D}}(a) \leq S(q) - f(q), \text{ that is, } f(q) \leq S(q).$$

Proof of Theorem 4, Part (2)

Recall that $m = \min\{x : d_{i_{x+1}} \leq \lfloor \frac{n-k+2x}{2} \rfloor\}$ and $f(m) \leq S(m)$. We will show that the sequence $\mathcal{E}$ defined before is graphic, where e_1 is chosen in such a way that $\sum_{i=1}^{n} e_i$ is even; i.e. we take a sequence $\mathcal{F}$ where $f_1 = e_1 + 1$ and all other elements of both sequences are identical. (Notice that $e_1 \notin \mathcal{D}^*$.) By the Erdős-Gallai theorem and Proposition 10, to prove that $\mathcal{E}$ is a graphic sequence, it is enough to show that $\Phi_{\mathcal{E}}(a) \geq 0$, where $a = a(\mathcal{E})$. By Propositions 7 and 9, it follows that $a = \frac{n-k+2m}{2}$ or $a = \frac{n-k+2m-1}{2}$.

Case 1. $a = \frac{n-k+2m}{2}$. In this case, we have by Proposition 9, $m = 0$ or $h_m \leq a < h_{m+1}$.

If $e_1 = n - k + m - 1$,

$$\begin{aligned}
\Phi_{\mathcal{E}}(a) &= a(a-1) + \sum_{i=a+1}^{n} d_i - \sum_{i=1}^{a} d_i \\
&= a(a-1) + \sum_{l>m} d_{i_l} - \sum_{l\leq m} d_{i_l} + \sum_{i=1}^{a-1} i - \sum_{i=1}^{m-1} i - \sum_{i=a}^{n-1} i + \sum_{n-k+m}^{n} i \\
&= a(a-1) - f(m) + \frac{a(a-1)}{2} - \frac{m(m-1)}{2} - an + \frac{(a+1)a}{2} \\
&\quad +(k-m)n - \frac{(k-m)(k-m+1)}{2} \\
&= n(q - \frac{1}{2}) + (1-m)(k-m) - \frac{k}{2} - f(m).
\end{aligned}$$

Since $-f(m) \geq -S(m)$ and $\Phi_{\mathcal{E}}(a)$ is an integer, we have $\Phi_{\mathcal{E}}(a) \geq -\frac{1}{2}$, making $\Phi_{\mathcal{E}}(a) \geq 0$.

If $e_1 = n - k + m$, that is, the sum of all values of $\mathcal{E}$ is odd, then $\Phi_{\mathcal{E}}(a) \neq 0$. Since $\Phi_{\mathcal{F}} = \Phi_{\mathcal{E}} - 1$ and $\Phi_{\mathcal{E}}(a) > 0$, we have $\Phi_{\mathcal{F}}(a) \geq 0$.

Case 2. $a = \frac{n-k+2m-1}{2}$. In this case, by Proposition 9, we have $m = 0$ or $h_{m-1} \leq a < h_m$, and $f_{\mathcal{E}}(m-1) \leq S_{\mathcal{E}}(m-1)$. Using the same method as in Case 1, either $\Phi_{\mathcal{E}}(a) \geq 0$ or $\Phi_{\mathcal{F}}(a) \geq 0$.

Hence, the proof of Theorem 4 is complete. □

Now, we prove Theorems 1 and 2 by applying Theorem 4.

Suppose without loss in generality that $\mathcal{D}$ contains only one repeated value $j \leq \lfloor \frac{n-1}{2} \rfloor$. Let k, t, q and m be defined as before. By symmetry we need only to check the lower bound for j.

If $k = 1$, then $a(\mathcal{D}) \geq \lfloor \frac{n-1}{2} \rfloor \geq j$. Thus, $q = 0$. When $\mathcal{D}$ is a graphic sequence, by Proposition 8, $-j \leq -\frac{n}{2} + 1$, that is, $j \geq \frac{n}{2} - 1$. Conversely, given $\mathcal{D}^* = (d_{i_1})$ with $d_{i_1} = j$ and $\frac{n}{2} - 1 \leq j \leq \frac{n}{2}$, we may assume without loss of generality that $j \leq \frac{n-1}{2}$. That is, $d_{i_1} = j \leq \lfloor \frac{n-1}{2} \rfloor$. By definition $m = 0$. Thus, $j \geq \frac{n}{2} - 1$. When $n \geq 5$, it follows that $j \leq n - 3$. Thus, by Theorem 4(2), there exists a graphic sequence with j as its RS. For the case, $2 \leq n \leq 4$, we can directly verify that Theorem 1 holds.

If $k = 2$, then $a(\mathcal{D}) \geq \lfloor \frac{n-2+q}{2} \rfloor$. Since $j \leq \lfloor \frac{n-1}{2} \rfloor$, we have $q \leq 1$. When $q = 0$, by Theorem 4, we have $-2j \leq -\frac{n-3}{2}$. Thus $j \geq \frac{n-3}{4}$. For $q = 1$, it follows that $j = \lfloor \frac{n-1}{2} \rfloor \geq \frac{n-3}{4}$.

Conversely, given $\mathcal{D}^* = (d_{i_1}, d_{1_2})$ with $d_{i_1} = d_{1_2} = j$ and $\frac{n-3}{4} \leq j \leq \frac{3n-1}{4}$, we may assume with no loss of generality that $j \leq \frac{n-1}{2}$. First we note that for $n \leq 6$, a graphical sequence with appropriate RS can be shown by direct verification. Since $0 \leq m \leq k - 1$ and $k = 2$, we only need consider two cases. If $m = 0$, then we only need show that $0 \leq j \leq n - 4$ and $f(0) \leq S(0)$. The former is true for all $n \geq 7$ and the latter is a restatement of the hypothesis that $j \geq \frac{n-3}{4}$. If $m = 1$, then we need show that $1 \leq d_{i_k} = j = d_{i_1} \leq n - 3$, and $f(1) \leq S(1)$. The former is clearly satisfied for all $n \geq 5$ since $j \leq \frac{n-1}{2}$. The latter is $0 \leq \frac{n-1}{2}$, true for all appropriate n.

Thus, we have proven Theorems 1 and 2 by using Theorem 4. Unfortunately, we are unable to prove Theorem 3 by using Theorem 4 directly. We conclude this section by raising the following conjecture which is a generalization of Theorem 3.

Conjecture Let $\mathcal{D}^* = (d_{i_1}, d_{i_2}, \cdots, d_{i_k})$ be a nonincreasing sequence with d distinct values and $n \geq k + d + 1$. Then, there exists a graphic sequence with $\mathcal{D}^*$ as its RS if and only if $f(0) \leq S(0)$ and $f(k) \leq S(k)$.

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