ON THE FIBONACCI MATRIX OF ORDER 2^k SEQUENCE $\{F_n^{(2^k)}\}$

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1. Introduction

In this paper, we construct the Fibonacci matrix of order 2^k sequence $\{F_n^{(2^k)}\}$ by use of the Fibonacci matrixes, and make a study of its properties. Owing to its recurrence relation is similar to the recurrence relation of the Fibonacci sequence, its properties are similar to some properties of the Fibonacci sequence too.

It is well known that the Fibonacci sequence $\{F_n\}$ is defined for all $n \geqslant 0$ by the recurrence relation

$$F_{n+1} = F_n + F_{n-1} \qquad (\text{where } F_0 = 0,\ F_1 = 1) \qquad (1)$$

The rule (1) can be used in extending the sequence backwards, thus

$$F_{-1} = F_1 - F_0, \quad F_{-2} = F_0 - F_{-1}, \ \ldots\ldots$$

and so that

$$F_{-(n+1)} = F_{-(n-1)} - F_{-n} \qquad (2)$$

This produces (see[1])

n	0	1	2	3	4	5	……
F_{-n}	0	1	-1	2	-3	5	……

and generally

$$F_{-n} = (-1)^{n+1} F_n \qquad (3)$$

2. To Construct the Fibonacci matrix of Order 2^k Sequence $\{F_n^{(2^k)}\}$

Now we construct the Fibonacci matrix of order 2^k sequence $\{F_n^{(2^k)}\}$ by use of the Fibonacci matrixes. We let

$$F_n^{(2)} = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \qquad (\text{where } n \geqslant 0) \qquad (4)$$

Then

$$F_n^{(2)} + F_{n-1}^{(2)} = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} + \begin{pmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{pmatrix} = \begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix} = F_{n+1}^{(2)}$$

Hence, we obtain the Fibonacci matrix of order 2 sequence $\{F_n^{(2)}\}$. It is defined for all $n \geqslant 1$ by the recurrence relation as follows:

$$F_{n+1}^{(2)} = F_n^{(2)} + F_{n-1}^{(2)} \qquad (n \geqslant 1) \qquad (5)$$

where $F_0^{(2)} = \begin{pmatrix} F_1 & F_0 \\ F_0 & F_{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $F_1^{(2)} = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

The rule (5) can be used in extending the sequence backwards, thus

$$F_{-1}^{(2)} = F_1^{(2)} - F_0^{(2)}, \quad F_{-2}^{(2)} = F_0^{(2)} - F_{-1}^{(2)}, \ \ldots\ldots$$

and so that

$$F_{-(n+1)}^{(2)} = F_{-(n-1)}^{(2)} - F_{-n}^{(2)} \qquad (n \geqslant 0) \qquad (6)$$

This produces

$$F_0^{(2)} = \begin{pmatrix} F_1 & F_0 \\ F_0 & F_{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_{-1}^{(2)} = \begin{pmatrix} F_0 & F_{-1} \\ F_{-1} & F_{-2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix},$$

$$F_{-2}^{(2)} = \begin{pmatrix} F_{-1} & F_{-2} \\ F_{-2} & F_{-3} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \quad F_{-3}^{(2)} = \begin{pmatrix} F_{-2} & F_{-3} \\ F_{-3} & F_{-4} \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}, \ldots\ldots$$

and generally

$$F_{-n}^{(2)} = \begin{pmatrix} F_{-(n-1)} & F_{-n} \\ F_{-n} & F_{-(n+1)} \end{pmatrix} \qquad (n \geqslant 0) \qquad (7)$$

Again, let the Fibonacci matrix of order 4 $F_n^{(4)}$ be equal to a partitioned matrix:

$$F_n^{(4)} = \begin{pmatrix} F_{(n+1)}^{(2)} & F_n^{(2)} \\ F_n^{(2)} & F_{n-1}^{(2)} \end{pmatrix} \qquad (n \geqslant 0) \qquad (8)$$

Then

$$F_n^{(4)}+F_{n-1}^{(4)}=\begin{pmatrix}F_{n+1}^{(2)} & F_n^{(2)}\\ F_n^{(2)} & F_{n-1}^{(2)}\end{pmatrix}+\begin{pmatrix}F_n^{(2)} & F_{n-1}^{(2)}\\ F_{n-1}^{(2)} & F_{n-2}^{(2)}\end{pmatrix}=\begin{pmatrix}F_{n+2}^{(2)} & F_{n+1}^{(2)}\\ F_{n+1}^{(2)} & F_n^{(2)}\end{pmatrix}=F_{n+1}^{(4)}$$

Hence, we obtain the Fibonacci matrix of order 4 sequence $\{F_n^{(4)}\}$. It is defined for all $n\geqslant 1$ by the recurrence relation as follows:

$$F_{n+1}^{(4)}=F_n^{(4)}+F_{n-1}^{(4)} \qquad (n\geqslant 1) \qquad (9)$$

where $F_0^{(4)}=\begin{pmatrix}F_1^{(2)} & F_0^{(2)}\\ F_0^{(2)} & F_{-1}^{(2)}\end{pmatrix}$, $F_1^{(4)}=\begin{pmatrix}F_2^{(2)} & F_1^{(2)}\\ F_1^{(2)} & F_0^{(2)}\end{pmatrix}$

The rule (9) can be used in extending the sequence backwards, thus

$$F_{-1}^{(4)}=F_1^{(4)}-F_0^{(4)} \quad , \quad F_{-2}^{(4)}=F_0^{(4)}-F_{-1}^{(4)} \quad , \ \ldots\ldots$$

and so that

$$F_{-(n+1)}^{(4)}=F_{-(n-1)}^{(4)}-F_{-n}^{(4)} \qquad (n\geqslant 0) \qquad (10)$$

This produces

$$F_0^{(4)}=\begin{pmatrix}F_1^{(2)} & F_0^{(2)}\\ F_0^{(2)} & F_{-1}^{(2)}\end{pmatrix} , \quad F_{-1}^{(4)}=\begin{pmatrix}F_0^{(2)} & F_{-1}^{(2)}\\ F_{-1}^{(2)} & F_{-2}^{(2)}\end{pmatrix} , \quad F_{-2}^{(4)}=\begin{pmatrix}F_{-1}^{(2)} & F_{-2}^{(2)}\\ F_{-2}^{(2)} & F_{-3}^{(2)}\end{pmatrix} , \ \ldots\ldots$$

and generally

$$F_{-n}^{(4)}=\begin{pmatrix}F_{-(n-1)}^{(2)} & F_{-n}^{(2)}\\ F_{-n}^{(2)} & F_{-(n+1)}^{(2)}\end{pmatrix} \qquad (n\geqslant 0) \qquad (11)$$

Thus, and so on and so forth ,we let the Fibonacci matrix of order 2^k $F_n^{(2^k)}$ be equal to a partitioned matrix:

$$F_n^{(2^k)}=\begin{pmatrix}F_{n+1}^{(2^{k-1})} & F_n^{(2^{k-1})}\\ F_n^{(2^{k-1})} & F_{n-1}^{(2^{k-1})}\end{pmatrix} \qquad (n\geqslant 0 , k\geqslant 1) \qquad (12)$$

Then

$$F_n^{(2^k)}+F_{n-1}^{(2^k)}=\begin{pmatrix}F_{n+1}^{(2^{k-1})} & F_n^{(2^{k-1})}\\ F_n^{(2^{k-1})} & F_{n-1}^{(2^{k-1})}\end{pmatrix}+\begin{pmatrix}F_n^{(2^{k-1})} & F_{n-1}^{(2^{k-1})}\\ F_{n-1}^{(2^{k-1})} & F_{n-2}^{(2^{k-1})}\end{pmatrix}=\begin{pmatrix}F_{n+2}^{(2^{k-1})} & F_{n+1}^{(2^{k-1})}\\ F_{n+1}^{(2^{k-1})} & F_n^{(2^{k-1})}\end{pmatrix}=F_{n+1}^{(2^k)}$$

Hence we obtain the Fibonacci matrix of order 2^k sequence $\{F_n^{(2^k)}\}$. It is defined for all $n \geq 1$ by the recurrence relation as follows:

$$F_{n+1}^{(2^k)} = F_n^{(2^k)} + F_{n-1}^{(2^k)} \qquad (n \geq 1, k \geq 1) \qquad (13)$$

where

$$F_0^{(2^k)} = \begin{pmatrix} F_1^{(2^{k-1})} & F_0^{(2^{k-1})} \\ F_0^{(2^{k-1})} & F_{-1}^{(2^{k-1})} \end{pmatrix}, \qquad F_1^{(2^k)} = \begin{pmatrix} F_2^{(2^{k-1})} & F_1^{(2^{k-1})} \\ F_1^{(2^{k-1})} & F_0^{(2^{k-1})} \end{pmatrix}$$

The rule (13) can be used in extending the sequence backwards, thus

$$F_{-1}^{(2^k)} = F_1^{(2^k)} - F_0^{(2^k)}, \quad F_{-2}^{(2^k)} = F_0^{(2^k)} - F_{-1}^{(2^k)}, \quad \ldots\ldots$$

and so that

$$F_{-(n+1)}^{(2^k)} = F_{-(n-1)}^{(2^k)} - F_{-n}^{(2^k)} \qquad (n \geq 0, k \geq 1) \qquad (14)$$

This produces

$$F_0^{(2^k)} = \begin{pmatrix} F_1^{(2^{k-1})} & F_0^{(2^{k-1})} \\ F_0^{(2^{k-1})} & F_{-1}^{(2^{k-1})} \end{pmatrix}, \quad F_{-1}^{(2^k)} = \begin{pmatrix} F_0^{(2^{k-1})} & F_{-1}^{(2^{k-1})} \\ F_{-1}^{(2^{k-1})} & F_{-2}^{(2^{k-1})} \end{pmatrix}, \quad F_{-2}^{(2^k)} = \begin{pmatrix} F_{-1}^{(2^{k-1})} & F_{-2}^{(2^{k-1})} \\ F_{-2}^{(2^{k-1})} & F_{-3}^{(2^{k-1})} \end{pmatrix}$$

...... and generally

$$F_{-n}^{(2^k)} = \begin{pmatrix} F_{-(n-1)}^{(2^{k-1})} & F_{-n}^{(2^{k-1})} \\ F_{-n}^{(2^{k-1})} & F_{-(n+1)}^{(2^{k-1})} \end{pmatrix} \qquad (n \geq 0, k \geq 1) \qquad (15)$$

Now we obtain a basic property of the Fibonacci matrix of order 2^k sequence $\{F_n^{(2^k)}\}$ by the equation (15)

Theorem1: The Fibonacci matrix of order 2^k sequence $\{F_n^{(2^k)}\}$ is satisfied with

$$F_{-n}^{(2^k)} = (-1)^{n+1} E_{2^k} F_n^{(2^k)} E_{2^k} \qquad (n \geq 1, k \geq 1) \qquad (16)$$

where E_{2^k} is equal to a partitioned matrix

$$E_{2^k} = \begin{pmatrix} O_{2^{k-1}} & E_{2^{k-1}} \\ -E_{2^{k-1}} & O_{2^{k-1}} \end{pmatrix}, \quad \text{when } k=1 \ E_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$O_{2^{k-1}}$ is a zero matrix of order 2^{k-1}.

Proof : This is easily proved by induction. When k=1, we have

$$F_{-n}^{(2)} = \begin{pmatrix} F_{-(n-1)} & F_{-n} \\ F_{-n} & F_{-(n+1)} \end{pmatrix} = \begin{pmatrix} (-1)^{n}F_{n-1} & (-1)^{n+1}F_{n} \\ (-1)^{n+1}F_{n} & (-1)^{n+2}F_{n+1} \end{pmatrix} = (-1)^{n+1}\begin{pmatrix} -F_{n-1} & F_{n} \\ F_{n} & -F_{n-1} \end{pmatrix}$$

$$= (-1)^{n+1}\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} F_{n+1} & F_{n} \\ F_{n} & F_{n-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] = (-1)^{n+1} E_2 F_n^{(2)} E_2$$

Then , when k=1, the formula (16) is true. When k=2 , we have

$$F_{-n}^{(4)} = \begin{pmatrix} F_{-(n-1)}^{(2)} & F_{-n}^{(2)} \\ F_{-n}^{(2)} & F_{-(n+1)}^{(2)} \end{pmatrix} = (-1)^{n+1}\begin{pmatrix} -E_2F_{n-1}^{(2)}E_2 & E_2F_{n}^{(2)}E_2 \\ E_2F_{n}^{(2)}E_2 & -E_2F_{n+1}^{(2)}E_2 \end{pmatrix}$$

$$= (-1)^{n+1}\left[\begin{pmatrix} 0_2 & E_2 \\ -E_2 & 0_2 \end{pmatrix}\begin{pmatrix} F_{n+1}^{(2)} & F_{n}^{(2)} \\ F_{n}^{(2)} & F_{n-1}^{(2)} \end{pmatrix}\begin{pmatrix} 0_2 & E_2 \\ -E_2 & 0_2 \end{pmatrix}\right]$$

$$= (-1)^{n+1} E_4 F_n^{(4)} E_4 \ , \qquad \text{where } E_4 = \begin{pmatrix} 0_2 & E_2 \\ -E_2 & 0_2 \end{pmatrix} \ .$$

Then, when k=2, the formula (16) is true. Assuming the formula (16) to be true for k=m-1, in similar manner, we can prove that the formula (16) is true for k=m.

To sum up, the formula (16) is proved .

3. The Sum Formula of $\{ F_n^{(2^k)} \}$

Theorem 2 : The sum formula of $\{ F_n^{(2^k)} \}$ is as follows :

$$\sum_{i=1}^{n} F_i^{(2^k)} = F_{n+2}^{(2^k)} - F_2^{(2^k)} \qquad (n \geqslant 1, k \geqslant 1) \qquad (17)$$

Proof :

$$\sum_{i=1}^{n} F_i^{(2^k)} = \sum_{i=3}^{n+2} F_i^{(2^k)} - \sum_{i=2}^{n+1} F_i^{(2^k)}$$

$$= \left(\sum_{i=1}^{n+2} F_i^{(2^k)} - F_1^{(2^k)} - F_2^{(2^k)}\right) - \left(\sum_{i=1}^{n+2} F_i^{(2^k)} - F_1^{(2^k)} - F_{n+2}^{(2^k)}\right)$$

$$= F_{n+2}^{(2^k)} - F_2^{(2^k)}$$

4. Other properties of $\{ F_n^{(2^k)} \}$

Theorem 3 :

$$F_n^{(2^{k+1})} = F_1^{(2^{k+1})} \begin{pmatrix} I_{2^k} & I_{2^k} \\ I_{2^k} & 0_{2^k} \end{pmatrix}^{n-1} \qquad (n \geqslant 1, k \geqslant 0) \qquad (18)$$

where I_{2^k} is an unit matrix of order 2^k, O_{2^k} is a zero matrix of order 2^k.

Proof :

$$F_n^{(2^{k+1})} = \begin{pmatrix} F_{n+1}^{(2^k)} & F_n^{(2^k)} \\ F_n^{(2^k)} & F_{n-1}^{(2^k)} \end{pmatrix} = \begin{pmatrix} F_n^{(2^k)} & F_{n-1}^{(2^k)} \\ F_{n-1}^{(2^k)} & F_{n-2}^{(2^k)} \end{pmatrix} \begin{pmatrix} I_{2^k} & I_{2^k} \\ I_{2^k} & 0_{2^k} \end{pmatrix} = \cdots\cdots$$

$$= \begin{pmatrix} F_3^{(2^k)} & F_2^{(2^k)} \\ F_2^{(2^k)} & F_1^{(2^k)} \end{pmatrix} \begin{pmatrix} I_{2^k} & I_{2^k} \\ I_{2^k} & 0_{2^k} \end{pmatrix}^{n-2}$$

$$= F_1^{(2^{k+1})} \begin{pmatrix} I_{2^k} & I_{2^k} \\ I_{2^k} & 0_{2^k} \end{pmatrix}^{n-1}$$

Theorem 4 :

$$\sum_{i=1}^{n} F_{2i-1}^{(2^k)} = F_{2n}^{(2^k)} - F_0^{(2^k)} \qquad (n \geqslant 1, k \geqslant 1) \qquad (19)$$

Proof :

$$\sum_{i=1}^{n} F_{2i-1}^{(2^k)} = \sum_{i=1}^{n} F_{2i}^{(2^k)} - \sum_{i=1}^{n} F_{2i-2}^{(2^k)}$$

$$= \sum_{i=1}^{n} F_{2i}^{(2^k)} - \sum_{i=0}^{n-1} F_{2i}^{(2^k)}$$

$$= F_{2n}^{(2^k)} - F_0^{(2^k)}$$

Theorem 5 :

$$\sum_{i=1}^{n} F_{2i}^{(2^k)} = F_{2n+1}^{(2^k)} - F_1^{(2^k)} \qquad (n \geqslant 1, k \geqslant 1) \qquad (20)$$

Proof :

$$\sum_{i=1}^{n} F_{2i}^{(2^k)} = \sum_{i=1}^{n} F_{2i+1}^{(2^k)} - \sum_{i=1}^{n} F_{2i-1}^{(2^k)}$$

$$= \sum_{i=1}^{n} F_{2i+1}^{(2^k)} - \sum_{i=0}^{n-1} F_{2i+1}^{(2^k)}$$

$$= F_{2n+1}^{(2^k)} - F_{1}^{(2^k)}$$

Theorem 6 :

$$\sum_{i=1}^{2n} (-1)^i F_i^{(2^k)} = F_{2n-1}^{(2^k)} + F_0^{(2^k)} - F_1^{(2^k)} \quad (n \geqslant 1, k \geqslant 1) \qquad (21)$$

Proof : Subtraction of (19) from (20) produces (21)

Theorem 7 :

$$\sum_{i=1}^{n} F_{i-1}^{(2^k)} / 2^i = \frac{1}{2}(F_0^{(2^k)} + F_3^{(2^k)}) - F_{n+2}^{(2^k)} / 2^n \quad (n \geqslant 1, k \geqslant 1) \qquad (22)$$

Proof : This is easily proved by induction. Obviously, the formula (22) is true for n=1. Assuming it to be true for 2, 3, ... , n-1, we add $F_{n-1}^{(2^k)} / 2^n$ on both sides. The right-hand side becomes

$$\frac{1}{2}(F_0^{(2^k)} + F_3^{(2^k)}) - (2F_{n+1}^{(2^k)} - F_{n-1}^{(2^k)}) / 2^n$$

$$= \frac{1}{2} (F_0^{(2^k)} + F_3^{(2^k)}) - (F_{n+1}^{(2^k)} + F_n^{(2^k)}) / 2^n$$

$$= \frac{1}{2} (F_0^{(2^k)} + F_3^{(2^k)}) - F_{n+2}^{(2^k)} / 2^n \qquad \text{QED}.$$

Theorem 8 :

$$F_{n+p}^{(2^k)} = \sum_{i=0}^{p} \binom{p}{i} F_{n-i}^{(2^k)} \qquad (n \geqslant 1, k \geqslant 1) \qquad (23)$$

Proof : It is observe that the formula (23) holds for all integer n when p=1. We shall prove, by induction, that it holds for every positive integer p .

Let the formula (23) be true from 1 up to some value of p . Then

$$F^{(2^k)}_{m+p} = \sum_{i=0}^{p}\binom{p}{i} F^{(2^k)}_{m-i} = \sum_{i=0}^{p}\binom{p}{i}\left(F^{(2^k)}_{m-i-1} + F^{(2^k)}_{m-i-2} \right)$$

$$= F^{(2^k)}_{m-1} + \sum_{i=1}^{p}\binom{p}{i} F^{(2^k)}_{m-i-1} + \sum_{i=0}^{p-1}\binom{p}{i} F^{(2^k)}_{m-i-2} + F^{(2^k)}_{m-p-2}$$

$$= F^{(2^k)}_{m-1} + \sum_{i=1}^{p}\binom{p}{i} F^{(2^k)}_{m-i-1} + \sum_{i=1}^{p}\binom{p}{i-1} F^{(2^k)}_{m-i-1} + F^{(2^k)}_{m-p-2}$$

$$= F^{(2^k)}_{m-1} + \sum_{i=1}^{p}\binom{p+1}{i} F^{(2^k)}_{m-i-1} + F^{(2^k)}_{m-p-2}$$

$$= \sum_{i=0}^{p+1}\binom{p+1}{i} F^{(2^k)}_{m-i-1}$$

Let m=n+1, then

$$F^{(2^k)}_{n+p+1} = \sum_{i=0}^{p+1}\binom{p+1}{i} F^{(2^k)}_{n-i}$$

This is the formula (23) again , nothing but p is replaced by p+1. Thus the formula (23) holds for all p.

Theorem 9 :

$$F^{(2^k)}_{2n} = \sum_{i=0}^{n}\binom{n}{i} F^{(2^k)}_{i} \qquad (n\geqslant 1, k\geqslant 1) \qquad (24)$$

proof : Because of $\binom{p}{i} = \binom{p}{p-i}$, the formula (23) can also be written

$$F^{(2^k)}_{n+p} = \sum_{i=0}^{p}\binom{p}{i} F^{(2^k)}_{n-p+i} \qquad (n\geqslant 1, k\geqslant 1) \qquad (25)$$

When n=p, then

$$F^{(2^k)}_{2n} = \sum_{i=0}^{n}\binom{n}{i} F^{(2^k)}_{i}$$

Corollary 1: When n=m+(t-1)p , the formula (25) can be written

$$F^{(2^k)}_{m+tp} = \sum_{i=0}^{p}\binom{p}{i} F^{(2^k)}_{m+(t-2)p+i} \qquad (k\geqslant 1) \qquad (26)$$

Corollary 2: When t=2, the formula (26) can be written

$$F^{(2^k)}_{m+2p} = \sum_{i=0}^{p}\binom{p}{i} F^{(2^k)}_{m+i} \qquad (k\geqslant 1) \qquad (27)$$

Theorem 10 : For m=1, p=n, the formula (27) can be transformed into

$$F^{(2^k)}_{2n+1} = F^{(2^k)}_{-1} + \sum_{i=0}^{n}\binom{n+1}{i+1} F^{(2^k)}_{i} \qquad (n\geqslant 1, k\geqslant 1) \qquad (28)$$

Proof :

$$F^{(2^k)}_{2n+1} = \sum_{i=0}^{n} \binom{n}{i} F^{(2^k)}_{i+1}$$

$$= F^{(2^k)}_{1} + \sum_{i=1}^{n} \binom{n}{i} (F^{(2^k)}_{i} + F^{(2^k)}_{i-1})$$

$$= F^{(2^k)}_{1} + \sum_{i=1}^{n} \binom{n}{i} F^{(2^k)}_{i} + \sum_{i=0}^{n-1} \binom{n}{i+1} F^{(2^k)}_{i}$$

$$= F^{(2^k)}_{1} + F^{(2^k)}_{n} + \sum_{i=1}^{n-1} \binom{n}{i} F^{(2^k)}_{i} + \binom{n}{1} F^{(2^k)}_{0} + \sum_{i=1}^{n-1} \binom{n}{i+1} F^{(2^k)}_{i}$$

$$= F^{(2^k)}_{1} + F^{(2^k)}_{n} + \sum_{i=0}^{n-1} \binom{n+1}{i+1} F^{(2^k)}_{i} - F^{(2^k)}_{0}$$

$$= F^{(2^k)}_{-1} + \sum_{i=0}^{n} \binom{n+1}{i+1} F^{(2^k)}_{i}$$

Theorem 11 :

$$F^{(2^k)}_{n-p} = \sum_{i=0}^{p} (-1)^i \binom{p}{i} F^{(2^k)}_{n+p-i} \qquad (n \geqslant 1, k \geqslant 1) \qquad (29)$$

Proof : Obviously, the formula (29) holds for p=1. Write

$$F^{(2^k)}_{m-p} = \sum_{i=0}^{p} (-1)^i \binom{p}{i} (F^{(2^k)}_{m+p+2-i} - F^{(2^k)}_{m+p+1-i})$$

$$= F^{(2^k)}_{m+p+2} + \sum_{i=1}^{p} [(-1)^i \binom{p}{i} F^{(2^k)}_{m+p+2-i} + (-1)^i \binom{p}{i-1} F^{(2^k)}_{m+p+2-i}] + (-1)^{p+1} F^{(2^k)}_{m+1}$$

$$= \sum_{i=0}^{p+1} (-1)^i \binom{p+1}{i} F^{(2^k)}_{m+p+2-i}$$

Let m=n-1, then

$$F^{(2^k)}_{n-p-1} = \sum_{i=0}^{p+1} (-1)^i \binom{p+1}{i} F^{(2^k)}_{n+p+1-i}$$

which is the same as the formula (29), with p+1 replacing p. Hence the formula (29) is proved.

Corollary 3 : The formula (29) can be written

$$F^{(2^k)}_{n-p} = \sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} F^{(2^k)}_{n+i} \qquad (k \geqslant 1) \qquad (30)$$

Corollary 4 : In the formula (30), let n=m-(t-1)p, then

$$F^{(2^k)}_{m-tp} = \sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} F^{(2^k)}_{m-(t-1)p+i} \qquad (k \geqslant 1) \qquad (31)$$

Corollary 5 : In the formula (31), let t=2, then

$$F^{(2^k)}_{m-2p} = \sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} F^{(2^k)}_{m-p+i}$$
$$= \sum_{i=0}^{p} (-1)^{i} \binom{p}{i} F^{(2^k)}_{m-i} \qquad (k \geqslant 1) \qquad (32)$$

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AMS Classification Number: 11B39, 11B83, 11C20.

Combinatorics of topmost discs of multi-peg Tower of Hanoi problem

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Abstract

Combinatorial properties of the multi-peg Tower of Hanoi problem on n discs and p pegs are studied. Top-maps are introduced as maps which reflect topmost discs of regular states. We study these maps from several points of view. We also count the number of edges in graphs of the multi-peg Tower of Hanoi problem and in this way obtain some combinatorial identities.

1 Introduction

The Tower of Hanoi problem posed in 1884 [1] is by now very well understood. The classical problem consists of finding the minimum number of moves necessary to transfer a tower of n discs from one peg to another. Several variants and generalizations of the problem have been proposed, cf., for instance, [5, 7]. Papers [2, 3, 9] nicely survey the topic and/or give the corresponding large bibliography. When the classical problem with three pegs is generalized to more pegs, the problem becomes notoriously difficult although Hinz [4] suspects that the problem might be solvable by closer examining the graphs associated to the problem.

*Supported by the Ministry of Science and Technology of Slovenia under the grant J1-0498-0101.

In this paper we introduce a partial description of regular states of the problem by considering only the topmost disc of every peg. Formally this is done via so-called top-maps which assign to every regular state a vector whose components are labels of topmost discs. For a fixed number of pegs this description is polynomial in the number of discs, in contrast to the usual description of regular states which is exponential in the number of discs.

In the rest of this section we introduce the concepts and notations needed later. In the next section we describe several properties of top-maps, for instance we compute sizes of their images and classify their unique preimages. In Section 3 we then apply our considerations to graphs of multi-peg Tower of Hanoi problem and to obtain certain combinatorial identities related to the Stirling numbers of the second kind.

The multi-peg Tower of Hanoi problem consists of $p \geq 3$ pegs numbered $0, 1, \ldots p-1$ and $n \geq 1$ discs of different sizes. Discs will be numbered $1, 2, \ldots, n$ and we assume that they are ordered by size, disc 1 being the smallest one. Initially all discs lie on peg 0 in small-on-large ordering. The objective is to transfer all the discs to peg $p-1$ in the minimum number of legal moves. A *legal move* is a transfer of the topmost disc from one peg to another peg such that no disc can be moved onto a smaller one.

As usual, a state is *regular* if no larger disc is placed on a smaller one. A regular state can be uniquely described with an n-tuple $r \in (\mathbb{Z}_p)^n = \{0, 1, \ldots, p-1\}^n$. More precisely, we set

$$r = (r_1, r_2, \ldots, r_n),$$

where r_d denotes the peg on which the disc d is placed. An example of a regular state on $p = 5$ pegs with $n = 6$ discs is shown in Figure 1. The corresponding 6-tuple r is also given (as well as a 5-tuple s to be defined in the next section).

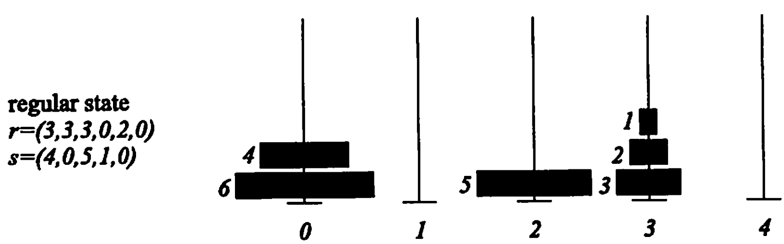

Figure 1: A regular state

A regular state is *perfect* if all the discs lie on the same peg. In Figure 2 an example of a perfect state is shown with all discs on peg 3.

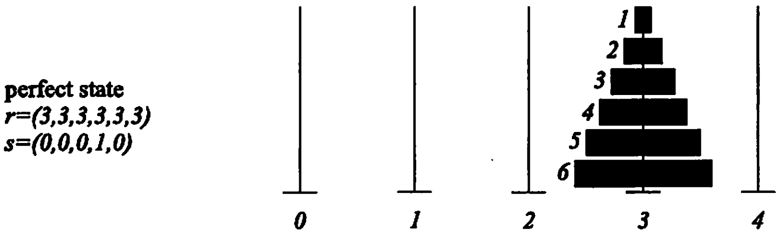

Figure 2: A perfect state

We call a regular state a *spread state*, if all the discs are on different pegs, see Figure 3. Clearly, for $n > p$ there are no spread states.

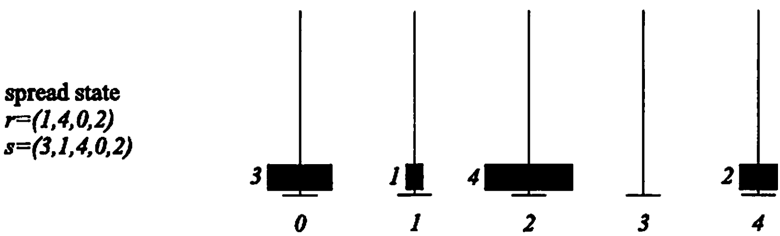

Figure 3: A spread state

A regular state is an *almost spread state*, if for some $k \geq 1$, discs $1, \ldots, k$ are on a common peg, while discs $k+1, \ldots, n$ are each on a private peg, cf. Figure 4.

Note that a perfect state is an almost spread state with $k = n$ and a spread state is an almost spread state with $k = 1$.

Let $r = (r_1, r_2, \ldots, r_{n-1}, r_n)$ be a regular state. Then r is a perfect state if and only if all the components r_i are equal. The state r is a spread state if and only if $r_i \neq r_j$ for any $i \neq j$. Finally, r is an almost spread state if and only if there exists $k \geq 1$ such that all r_i are equal for $i \leq k$ and $r_i \neq r_j \neq r_1$ for $i, j > k, i \neq j$.

Finally, as usual, let $(n)_m = n(n-1)\cdots(n-m+1)$ and let $S(n,k)$ denote the Stirling numbers of the second kind.

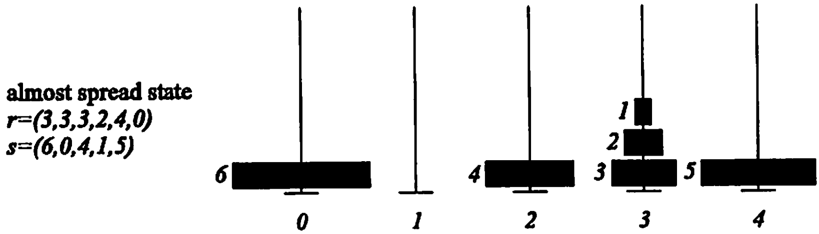

Figure 4: An almost spread state

2 Top-maps

Information about topmost discs suffices to know all the possible legal moves from a given regular state. From this reason we introduce mappings

$$\mathcal{T}_{n,p} : (\mathbb{Z}_p)^n \to (\mathbb{Z}_{n+1})^p$$

as $\mathcal{T}_{n,p}(r_1, r_2, \ldots, r_n) = (s_0, s_1, \ldots, s_{p-1})$, where

$$s_i = \begin{cases} 0; & r_k \neq i \text{ for } 1 \leq k \leq n, \\ \min\limits_{1 \leq k \leq n} \{k \; ; \; r_k = i\}; & \text{otherwise.} \end{cases}$$

Hence, the component s_i of s is the topmost disc on peg i, if peg i is nonempty; otherwise $s_i = 0$. We will briefly refer to these mappings as *top-maps*. As usual, $\mathcal{R}(\mathcal{T}_{n,p})$ denotes the image of $\mathcal{T}_{n,p}$ and $|\mathcal{R}(\mathcal{T}_{n,p})|$ its size.

Observe that in general a top-map need not be surjective. For instance, the 4-tuples $(2, 3, 4, 5)$ and $(2, 1, 4, 4)$ are not in the image of some top-map. In the first case the smallest disc is not present and in the second case the disc 4 is supposed to be simultaneously on two discs.

We begin with the lemma which is useful for computing $|\mathcal{R}(\mathcal{T}_{n,p})|$.

Lemma 2.1 *Let $s = (s_0, s_1, \ldots, s_{p-1}) \in (\mathbb{Z}_{n+1})^p$. Then $s \in \mathcal{R}(\mathcal{T}_{n,p})$ if and only if the following two conditions are fulfilled:*

(i) $\exists i \in \{0, \ldots, p-1\} : s_i = 1,$
(ii) $\forall i, j \in \{0, \ldots, p-1\}$: $s_i = s_j \Rightarrow (i = j \vee s_i = 0).$

Proof. The conditions are clearly necessary. Indeed, the smallest disc must be one of the top discs and no disc can be topmost simultaneously on two pegs.

Assume now that $s = (s_0, s_1, \ldots, s_{p-1})$ fulfills the two conditions. We need to show that there exists $r = (r_1, r_2, \ldots, r_n) \in (\mathbb{Z}_p)^n$ with $\mathcal{T}_{n,p}(r) = s$.

We define the components r_i as follows. For any i with $s_i \neq 0$ we set $r_{s_i} = i$. By condition (ii) such i is uniquely determined. For all the other discs i (i.e. for all those which do not appear as the topmost discs) we set $r_i = r_1$. (Note that by condition (i) the component r_1 has already been defined.) It is now easy to verify that $r \in (\mathbb{Z}_p)^n$ represents a regular state and that $\mathcal{T}_{n,p}(r) = s$. □

Proposition 2.2 *For any $n \geq 1$ and $p \geq 3$ we have*

$$|\mathcal{R}(\mathcal{T}_{n,p})| = p \sum_{k=0}^{p-1} \binom{p-1}{k} (n-1)_{p-k-1} .$$

Proof. Let $s \in \mathcal{R}(\mathcal{T}_{n,p})$. Then, by condition (i) of Lemma 2.1 we have $s_i = 1$ for some i. There are p possibilities for that. After fixing this i, there may be k empty pegs for any k with $0 \leq k \leq p-1$. For a fixed k there are $\binom{p-1}{k}$ possible selections of k empty pegs. The remaining $p-1-k$ pegs are nonempty, and by the condition (ii) of Lemma 2.1, which asserts that the topmost discs must be different, we infer that there are

$$\Big(n-1\Big)\Big((n-1)-1\Big)\cdots\Big((n-1)-(p-1-k-1)\Big) = (n-1)_{p-k-1}$$

possibilities to select the $p-1-k$ top discs. □

Let p be fixed. Then, as $\mathcal{R}(\mathcal{T}_{n,p}) \subsetneq (\mathbb{Z}_{n+1})^p$, the size of $\mathcal{R}(\mathcal{T}_{n,p})$ is a polynomial function in n. This observation can be made more precise as follows:

Corollary 2.3 *Let $p \geq 3$. Then $|\mathcal{R}(\mathcal{T}_{n,p})| = pn^{p-1} + O(n^{p-2})$.*

Proof. Result follows by noting that the leading term from Proposition 2.2 is obtained for $k = 0$. □

Recall that the number of regular states is p^n, i.e. the number of regular states is exponential in the number of discs. Therefore Corollary 2.3 seems to be interesting from the algorithmic point of view, because the exponential number of regular states is a source of difficulties in studying the graphs of the Tower of Hanoi problem.

Let $s \in \mathcal{R}(\mathcal{T}_{n,p})$. In order to determine $|\mathcal{T}_{n,p}{}^{-1}(s)|$ we introduce the following function. For a positive integer d and a p-tuple $s = (s_0, s_1, \ldots, s_{p-1})$ let

$$h(d, s) = \big|\{i \in \{0, \ldots, p-1\} \; ; \; 0 < s_i < d\}\big| .$$

Now we have:

Lemma 2.4 *Let $s \in \mathcal{R}(\mathcal{T}_{n,p})$. Then*

$$|\mathcal{T}_{n,p}{}^{-1}(s)| = \begin{cases} 1; & h(n+1,s) = n, \\ \prod\limits_{\substack{i=1 \\ i \neq s_0,\ldots,s_{p-1}}}^{n} h(i,s); & \text{otherwise.} \end{cases}$$

Proof. Suppose that $h(n+1,s) = n$. Then every disc in (any element of) $\mathcal{T}_{n,p}{}^{-1}(s)$ is topmost. It follows that the preimage of s is unique, i.e. $|\mathcal{T}_{n,p}{}^{-1}(s)| = 1$.

Assume now that at least one disc in $\mathcal{T}_{n,p}{}^{-1}(s)$, say i, is not topmost. Then $h(i,s)$ is nonzero and represents the number of pegs on which we can place the disc i in a regular state corresponding to s. Therefore $|\mathcal{T}_{n,p}{}^{-1}(s)|$ is at least $\prod_i h(i,s)$, where i runs over all discs that are not topmost. On the other hand, a disc i can only be placed on a peg j if $i < s_j$. If there are several discs that can be placed on the same peg, then they must be on this peg sorted by their sizes, i.e. there is only one possibility to do that. Therefore, $|\mathcal{T}_{n,p}{}^{-1}(s)|$ is at most $\prod_i h(i,s)$. □

The case when the preimage of s is unique is characterized in the next theorem.

Theorem 2.5 *Let $\mathcal{T}_{n,p}(r) = s$. Then $\mathcal{T}_{n,p}{}^{-1}(s) = \{r\}$ if and only if r is an almost spread state.*

Proof. Suppose first that r is a perfect state, a spread state, or an almost spread state. Then, using Lemma 2.4, it is easy to verify that $|\mathcal{T}_{n,p}{}^{-1}(s)| = 1$.

Conversely, let $|\mathcal{T}_{n,p}{}^{-1}(s)| = 1$. By Lemma 2.4 we then either have $h(n+1,s) = n$ or $\prod_i h(i,s) = 1$. In the first case r is a spread state and thus an almost spread state.

Assume now that $\prod_i h(i,s) = 1$. It follows that the index set of this product is nonempty and for any such i we have $h(i,s) = 1$. Let i be such number with $h(i,s) = 1$. Clearly, $i > 1$. Moreover, Lemma 2.1, implies that there exists a peg j such that $s_j = 1$. Therefore, $i < s_j$ and as $h(i,s) = 1$ disc i must lie on peg s_j. Hence, all the discs i with $h(i,s) = 1$ must be on the same disc (i.e. on the disc s_j). Now, if the number of such discs i is $n-1$, then r is a perfect state which is an almost spread state. Otherwise, for any other disc k we have $k = s_t$ for some t. In other words, all the other discs are topmost. They clearly lie on pegs different from s_j, and we can conclude that we have an almost spread state. □

3 Top-maps and combinatorial identities

We have seen in the previous section that the image of a top-map is of polynomial size in the number of discs. Therefore it is natural to ask which information can be deduced from the image of $\mathcal{T}_{n,p}$ itself. Here we show how to compute the number of legal moves from a regular state r using only information of $\mathcal{T}_{n,p}(r)$.

Proposition 3.1 *Let $\mathcal{T}_{n,p}(r) = s$. Then the number of legal moves from r is*

$$h(n+1,s)\left(p-\frac{1}{2}\right)-\frac{1}{2}h(n+1,s)^2\,.$$

Proof. Observe first that the number of nonempty pegs is $h(n+1,s)$, and so the number of empty pegs is $p-h(n+1,s)$. Any topmost disc of a peg can be moved to any empty peg. Therefore, there are $h(n+1,s)(p-h(n+1,s))$ legal moves of this kind.

It remains to consider the legal moves in which a topmost disc is moved to a nonempty peg. Recall that the $h(n+1,s)$ topmost discs are different. Thus, the smallest one can be moved to any other of $h(n+1,s)-1$ nonempty pegs, the second smallest can be moved to $h(n+1,s)-2$, and so on. Therefore, there are

$$\sum_{i=1}^{h(n+1,s)-1} i = \frac{1}{2}(h(n+1,s)-1)h(n+1,s)$$

different moves from a nonempty onto a nonempty peg.

Summing the above expressions we get the result. □

The number of topmost discs that can be moved to a peg i is equal to $h(s_i,s)$. Moreover, if $s_j=0$, i.e. if peg j is empty, then by definition we have $h(s_j,s)=0$. Therefore, the number of legal moves in which a topmost disc is moved to an nonempty peg is equal to $\sum_i h(s_i,s)$ and so we also have (cf. the above proof)

$$\sum_{i=0}^{p-1} h(s_i,s) = \frac{1}{2}(h(n+1,s)-1)h(n+1,s)\,.$$

By Theorem 3.1 we thus only need to know the number of nonempty pegs $h(n+1,s)$ in order to compute the number of legal moves.

In order to obtain some additional results, we shall consider graphs of the multi-peg tower of Hanoi problem. They are defined as follows. The graph $G_p^n=(V_p^n,E_p^n)$ of n discs and p pegs has regular states as vertices, two vertices being adjacent if one state is obtained from the other by a legal move.

Theorem 3.2 *For any $n \geq 1$ and $p \geq 3$ we have:*

$$\sum_{k=1}^{p} \left(k(p - \frac{1}{2}) - \frac{1}{2}k^2 \right) S(n,k)(p)_k = (p-1) \sum_{i=0}^{n-1} p^{i+1}(p-2)^{n-i-1} .$$

Proof. We will prove the theorem by counting the number of edges in G_p^n in two ways.

Note first that the number of regular states where exactly k pegs are nonempty is equal to $S(n,k)(p)_k$. Therefore, using Proposition 3.1, we have

$$|E_p^n| = \frac{1}{2} \sum_{k=1}^{p} \left(k(p - \frac{1}{2}) - \frac{1}{2}k^2 \right) S(n,k)(p)_k .$$

Consider now the set of states in which the largest disc is fixed on some peg. We infer that the corresponding vertices induce a subgraph of G_p^n isomorphic to G_p^{n-1}. There are p such subgraphs and they form a partition of V_n^p. Two vertices belonging to two such subgraphs are adjacent if and only if they differ exactly in position of the largest disc. Since all the remaining discs except the largest one lie on $p-2$ discs (i.e. on the pegs that are not involved in the move of the largest disc), there are $|V_{p-2}^{n-1}|$ edges connecting two such subgraphs. It follows that the number of edges of G_p^n can be expressed recursively as

$$|E_p^n| = p|E_p^{n-1}| + \binom{p}{2} |V_{p-2}^{n-1}| .$$

Thus, $|E_p^n|$ is equal to:

$$p|E_p^{n-1}| + \binom{p}{2}|V_{p-2}^{n-1}| =$$

$$p\left(p|E_p^{n-2}| + \binom{p}{2}|V_{p-2}^{n-2}|\right) + \binom{p}{2}|V_{p-2}^{n-1}| =$$

$$p\left(p\left(p|E_p^{n-3}| + \binom{p}{2}|V_{p-2}^{n-3}|\right) + \binom{p}{2}|V_{p-2}^{n-2}|\right) + \binom{p}{2}|V_{p-2}^{n-1}| =$$

$$p\left(p\left(\cdots\left(p|E_p^{1}| + \binom{p}{2}|V_{p-2}^{1}|\right) + \binom{p}{2}|V_{p-2}^{2}|\right) + \cdots\right) + \binom{p}{2}|V_{p-2}^{n-1}| =$$

$$\underbrace{p(p(\ldots(p}_{n-1}\binom{p}{2} + \underbrace{\binom{p}{2}(p-2)^1) + \binom{p}{2}(p-2)^2) + \ldots) + \binom{p}{2}(p-2)^{n-1}}_{n-1} =$$

$$\sum_{i=0}^{n-1} \binom{p}{2} p^i (p-2)^{(n-1)-i} .$$

Combining the above expressions the result follows. □

Using standard methods the right-hand side expression of Theorem 3.2 can be summed up. In this way we obtain:

Corollary 3.3 *For any $n \geq 1$ and $p \geq 3$ we have:*

$$\sum_{k=1}^{p} \left(k(p - \frac{1}{2}) - \frac{1}{2}k^2 \right) S(n,k)(p)_k = \binom{p}{2} [p^n - (p-2)^n] .$$

In the proof of Theorem 3.2 we observed that the number of regular states where exactly k pegs are nonempty is $S(n,k)(p)_k$. On the other hand the number of all regular states is p^n. Thus in passing we get the following well-known identity, cf. [6, 8]:

Corollary 3.4

$$p^n = \sum_{k=1}^{p} S(n,k)(p)_k = \sum_{k=1}^{n} S(n,k)(p)_k .$$

In this formula the choice of the upper bound for summation makes no difference, since $S(n,k) = 0$, if $n < k$, and $(p)_k = 0$, if $p < k$.

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