Copnumber of graphs with strong isometric dimension two

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Abstract. The problem is to determine the number of 'cops' needed to capture a 'robber' where the game is played with perfect information with the cops and the robber alternating moves. The 'cops' capture the 'robber' if one of them occupies the same vertex as the robber at any time in the game. Here we show that a graph with strong isometric dimension two requires no more than two cops.

Key words: isometric, embedding, strong product, paths.

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Abbreviated title: Strong Isometric Dimension.

1. Introduction and preliminaries.

The game of 'Cops and Robber' was introduced by Nowakowski & Winkler [6] and, independently, by Quilliot [7]. The game rules were: given a connected graph G, the cop chooses a vertex of G, then the robber chooses a vertex. Afterward, they move alternately — each can move to an adjacent vertex or pass. The cop wins if he ever occupies the same vertex as the robber; the robber wins if this situation never occurs. In [6] and [7],

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the authors characterize those graphs in which the cop has a winning strategy. Aigner and Fromme [1] pose the question: ***given a graph G, determine the least number of cops required to capture a robber on G.*** In this case, a set of cops choose vertices of G and then alternate moves with the robber. The cops' move consists of some (possibly empty) subset of the cops each moving to an adjacent vertex. The minimum number of cops needed to guarentee a winning strategy is called the **copnumber** of G and denoted $c(G)$. In [1], they show that if G is a finite, planar graph, then $c(G) \leq 3$. Quilliot [8] extended the analysis and showed that on a graph of genus g no more than $2g+3$ cops are required to catch a robber. Andrea [2,3] pursued this further showing that excluded minors play a part in determining the copnumber. A comprehensive list of references concerning this game can be found in [5].

The **strong product** of a set of graphs $\{G_i : i = 1, \ldots, k\}$ is the graph $\boxtimes_{i=1}^{k} G_i$ whose vertex set is the Cartesian product of the sets $\{V(G_i) : i = 1, \ldots, k\}$, and there is an edge between $\bar{a} = (a_1, a_2, \ldots, a_k)$ and $\bar{b} = (b_1, b_2, \ldots, b_k)$ if and only if a_i is adjacent or equal to b_i for $i = 1, \ldots, k$. See [5] for an analysis of copnumbers for several different products of paths. A graph G is an **isometric subgraph** of a graph H if the distances of the two coincide; i.e. $d_G(x, y) = d_H(x, y)$ for all $x, y \in V(G)$. The **strong isometric dimension** of a graph G is defined to be the least number k such that there is a set of k paths $\{P_1, P_2, \ldots, P_k\}$ with G an isometric subgraph of $\boxtimes_{i=1}^{k} P_i$ (see [4]). This is denoted $idim(G) = k$. Let $\pi_j : \boxtimes_{i=1}^{k} P_i \to P_j$ be the projection map onto the path P_j.

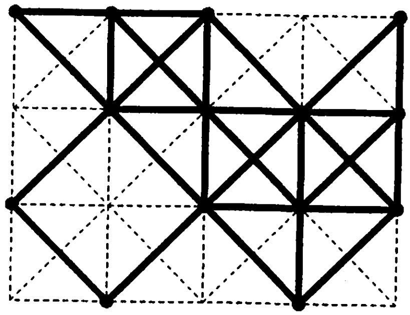

Figure 1: An example of a graph with $idim = 2$

2. Results.

In this section, we show that any graph with strong isometric dimension two has copnumber at most two. The strong product of two paths is known to have copnumber one while the cycle on four vertices has both strong isometric dimension two and copnumber two. Hence, this bound is sharp. In proving this bound, we employ the following lemma :

Lemma 2.1 (Aigner & Fromme [1]). *Let P be an isometric path in a graph G. A single cop moving on P can guarantee that after a finite number of moves the robber will be immediately caught if he moves onto P.*

Since this is a central result, we include the following proof. Let $P = \{a_0, a_1, \dots, a_n\}$ be an isometric path in G and let the map $f : G \to P$ be defined by

$$f(v) = \begin{cases} a_k & \text{if } k = d(a_0, v) \text{ and } k \leq n \\ a_n & \text{otherwise} \end{cases}$$

This map is edge preserving, so the image or "shadow" of the robber will move along the path according to the rules of the game. The copnumber of a path is one, so a single cop is able to catch this shadow and stay with it. Since every vertex on the path is its own shadow, if the robber now moves onto the path he will be immediately captured. The maximum number of moves before the robber is captured is the length of the path.

Note that the result only holds for *isometric* paths.

Theorem 2.2. *Let G be a finite connected graph. If $idim(G) = 2$, then $c(G) \leq 2$.*

Proof. Suppose $idim(G) = 2$, then G is an isometric subgraph of $P_n \boxtimes P_m$ for some $n, m \geq 1$. We may assume that $n + m$ is minimum. Let $V(P_n) = \{1, \ldots, n\}$ and let $f : V(G) \to \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq m\}$ be an edge preserving map such that $f(G)$ is an isometric subgraph of $P_n \boxtimes P_m$. We identify G with this subgraph.

First, we partition the vertices of G according to their first coordinate. Let $A_i = \{v \in V(G) : \pi_1(v) = i\}$ for all $1 \leq i \leq n$. We claim that for each $1 \leq i \leq n$ there exists an isometric path of G which contains all the vertices of A_i. To see this, let $A = A_i$ for some $1 \leq i \leq n$. Order the vertices of A by their second coordinate, i.e. $A = \{v_1, v_2, \ldots, v_k\}$ where $\pi_2(v_j) < \pi_2(v_{j+1})$ for all $1 \leq j \leq k - 1$. For each $1 \leq j \leq k - 1$, let Q_j be an isometric path in G from v_j to v_{j+1}. Now let $R = Q_1 \cup \cdots \cup Q_{k-1}$. Note that for any pair of vertices u and v in A, $d(u, v) = |\pi_2(u) - \pi_2(v)|$. Therefore, $\sum_{j=1}^{k-1} d(v_j, v_{j+1}) = \sum_{j=1}^{k-1} (\pi_2(v_{j+1}) - \pi_2(v_j)) = \pi_2(v_k) - \pi_2(v_1) = d(v_1, v_k)$, and the path R is isometric.

Hence, there exists a set of isometric paths $R_1, \ldots, R_n$ such that for each $i = 1, \ldots, n$ all the vertices of A_i are on the path R_i. By Lemma 2.1, we know that one cop moving on R_i can, after a finite number of moves, prevent the robber from moving onto R_i. Therefore, one cop can "protect" the vertices in A_i by moving on the larger set R_i. Now suppose that we have two cops. Place one cop on a vertex in A_1 and the second on a vertex in A_2. After a finite number of moves the cops can prevent the robber from moving onto A_1 and A_2. Therefore, once the cops have A_1 and A_2 protected, the robber, if uncaptured, must occupy a vertex in A_i for some $i > 2$.

By induction, assume that for some $k \leq n-1$ the cops have the sets A_{k-1} and A_k protected and the robber occupies a vertex in A_i for some $i > k$. The cop protecting A_{k-1} now leaves R_{k-1} and moves to protect the set A_{k+1}. Meanwhile, the other cop continues to protect the vertices of A_k. The robber can obviously not move from a vertex in A_i for some $i > k$ to a vertex in A_j for $j < k$ without moving through a vertex in A_k. Since a move onto A_k would result in his immediate capture, we can assume that the robber restricts his movement to the vertices in $A_{k+1} \cup \cdots \cup A_n$. Once the set A_{k+1} is protected, the robber's movement is further restricted to $A_{k+2} \cup \cdots \cup A_n$.

If we continue in this manner, the cops will eventually protect the sets A_{n-1} and A_n and the robber will have no vertex which he can safely occupy. Therefore, two cops are sufficient to apprehend the robber. □

We can extend this result to obtain an upper bound for the copnumber of a graph of strong isometric dimension three.

Theorem 2.3. *Let G be a finite connected graph. If $idim(G) = 3$ and G is an isometric subgraph of $P_n \boxtimes P_m \boxtimes P_l$ where $n \leq m \leq l$, then $c(G) \leq n+2$.*

Proof. Let $f : V(G) \to \{(i,j,k) : 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq k \leq l\}$ be an edge preserving map such that $f(G)$ is an isometric subgraph of $P_n \boxtimes P_m \boxtimes P_l$. Identify G with this subgraph. We wish to partition the vertices of G according to their first and second coordinates. First, let $B_j = \{v \in V(G) : \pi_2(v) = j\}$ and then let $A_{i,j} = \{v \in B_j : \pi_1(v) = i\}$. Hence, $A_{i,j}$ is the set of all vertices in $V(G)$ with i as their first coordinate and j as their second coordinate. As in Theorem 2.2, there is an isometric path of G, $R_{i,j}$, which contains all the vertices of $A_{i,j}$. If $A_{i,j}$ is empty for some i, j, then let $R_{i,j}$ be a path with zero vertices. Any cop assigned to an empty set may occupy any vertex in G and still be considered to be protecting that set. Hence, the set of paths $S_j = \{R_{1,j}, R_{2,j}, \ldots, R_{n,j}\}$ covers all the vertices of B_j, and a set of n cops can prevent a robber from moving onto B_j.

Now suppose we have a set of $n+2$ cops, $\{c_1, \ldots, c_n, c_{n+1}, c_{n+2}\}$. Let the cop c_i protect the set $A_{i,1}$ for $i = 1, \ldots, n$. Let c_{n+1} protect $A_{1,2}$ and c_{n+2} protect $A_{2,2}$. The cops now force the robber to occupy some vertex in $B_2 \cup B_3 \cdots \cup B_m$. Suppose, by induction, that for some j, $n+1$ cops protect the sets in $\mathcal{A}_p = \{A_{1,j+1}, A_{2,j+1}, \ldots, A_{p,j+1}, A_{p,j}, A_{p+1,j}, \ldots, A_{n,j}\}$. The $(n+2)^{nd}$ cop may be protecting a set or may be in transition. If the robber is on some vertex, x, in B_{j+1}, then he must be in $A_{i,j+1}$ for some $i \geq p+1$. Hence, any vertex in B_j adjacent to x is protected by one of the cops. Therefore, if the robber occupies a vertex in the set $B_{j+1} \cup \cdots \cup B_m$, then it is impossible for him to move out of that set. The $(n+2)^{nd}$ cop can now move to protect the set $A_{p+1,j+1}$ and thus, $\mathcal{A}_p$ can be replaced with $\mathcal{A}_{p+1}$. Eventually the entire set B_{j+1} is protected by cops in B_{j+1} and the robber will be confined to vertices in $B_{j+2} \cup \cdots \cup B_m$. Note that this requires only n cops. Move the other two cops to protect $A_{1,j+2}$ and $A_{2,j+2}$ and repeat this procedure.

By induction, the cops will eventually protect B_m and the robber will have no vertex which he can safely occupy. Hence, the robber will be apprehended by one of the cops and $c(G) \leq n+2$. □

Corollary 2.4. *If $idim(G) = 3$, then $c(G) \leq diam(G) + 3$.*

Proof. Suppose G is an isometric subgraph of $P_n \boxtimes P_m \boxtimes P_l$ where $n \leq m \leq l$ and $n + m + l$ is minimized. Then $l - 1 = diam(G)$ and, by Theorem 2.3, $c(G) \leq n + 2 \leq l + 2 = diam(G) + 3$.

□

We believe that the results in Theorem 2.3 and Corollary 2.4 are not the best possible.

3. Genus.

We saw in the previous section that if $idim(G) = 2$, then $c(G) \leq 2$. We can contrast this result with the genus result of Quilliot, $c(G) \leq 2g + 3$ [8]. It is the case that for arbitrarily high genus, g, there is a graph of that genus with strong isometric dimension two, and, thus, restricted clique size. To demonstrate this we use the following theorem, where $\gamma(G)$ denotes the genus of a graph G.

Theorem 3.1 (Battle, Harary, Kodama & Youngs [9]). *If two graphs G and H are joined at a vertex v (called $G *_v H$), then $\gamma(G *_v H) = \gamma(G) + \gamma(H)$.*

Now consider the graph $P_5 \boxtimes P_4$ in Figure 3. It has a vertex set of size 20 and an edge set of size 55. For any planar graph, G, $|E(G)| \leq 3|V(G)| - 6$. Hence, this graph has genus at least 1. If we take n copies of this graph and associate vertices as in Figure 4, then, by Theorem 3.1, the resulting graph, F, has genus at least n. Now, any isometric subgraph of $P_{5n} \boxtimes P_{4n}$ that has F as a subgraph (in particular, $P_{5n} \boxtimes P_{4n}$ and F itself) has genus at least n and copnumber at most 2.

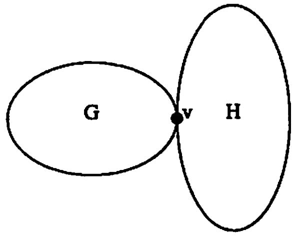

Figure 2: $G *_v H$

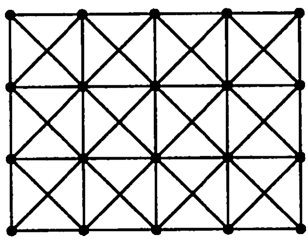

Figure 3: $P_5 \boxtimes P_4$

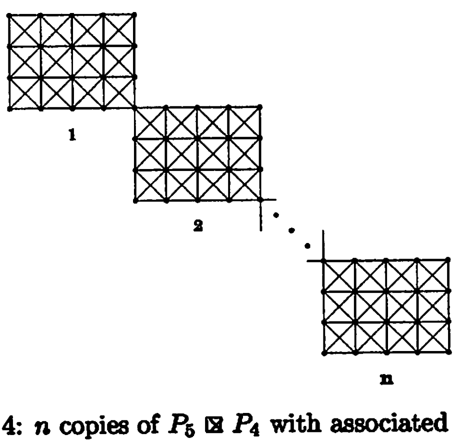

Figure 4: n copies of $P_5 \boxtimes P_4$ with associated vertices

REFERENCES

[1] M. Aigner & M. Fromme, *A game of cops and robbers*, Math. **8** (1984), 1–12.

[2] T. Andreae, *Note on a pursuit game played on graphs*, Discrete Appl. Math. **9** (1984), 111–115.

[3] T. Andreae, *On a pursuit game played on graphs for which a minor is excluded*, J. Combin. Theory Ser. B **41** (1986), 37–47.

[4] S. L. Fitzpatrick & R. J. Nowakowski, *The strong isometric dimension of finite reflexive graphs*, to appear in Discussiones Mathematicae Graph Theory.

[5] S. Neufeld & R. J. Nowakowski, *Copnumbers of Cartesian, strong and categorical products of graphs*, Discrete Math. **186** (1998), 253–268.

[6] R. J. Nowakowski &, P. Winkler, *Vertex to vertex pursuit in a graph*, Discrete Math. **43** (1983), 235–239.

[7] A. Quilliot, *Thèse d'Etat*, Université de Paris VI, 1983.

[8] A. Quilliot, *A short note about pursuit games played on a graph with a given genus*, J. Combin. Theory (Ser. B) **38** (1985), 89–92.

[9] J. Battle, F. Harary, Y. Kodama &, J. W. T. Youngs, *Additivity of the genus of a graph*, Bull. Amer. Math. Soc. **68** (1962), 565–568.