On some characterization of strongly regular graphs

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1 Introduction

A graph Γ is said to be strongly regular if Γ is a connected regular graph such that for any two distinct vertices x and y, the number of vertices which are adjacent to both of them only depends on whether x and y are adjacent or not. In this paper, we look at some geometric condition of a representation of some strongly regular graphs. In the representation of a strongly regular graph on a sphere, we will consider the following question: Can the vertices of the graph be partitioned into two sets such that each set is on a hyperplane? This will give one characterization of a strongly regular graph. This problem arose from the geometric condition of some coherent configuration, as proposed by Prof. Bannai in AMS conference, 1998 [1]. Some coherent configuration has the property that it can be embedded on two concentric spheres, such that the representations of the vertices on one sphere can be partitioned into two hyperplanes with respect to one of the vertices lying on the other sphere. This characterization may be of some help in the classification of such coherent configuration. In this paper, we will give a classification of the partition of the graph of Johnson graph of class 2 and Hamming graph of class 2, and some results on a few other graphs.

*This is a research for the master thesis of the author in Kyushu University, which includes the complete proofs for this discussion.

2 Basic concepts

For a graph Γ, let A be an adjacency matrix of Γ, which is a matrix indexed by the vertices of Γ and whose (x,y) entry is 1 if x and y are adjacent and 0 otherwise. The adjacency matrix of strongly regular graph has 3 distinct eigenvalues, one of which is the valency of the graph. Let p_1 and p_2 be the other eigenvalues and let E_1 and E_2 be projection matrices to the eigenspaces with respect to p_1 and p_2, respectively. These projections map the vertices of the graph on a sphere S^{m_i-1} (by rescaling) where m_i is the multiplicity of the eigenvalue. The distance of the vertices on the sphere is determined by its adjacency. For a subset Y of vertices in Γ, Y is said to be a 1-design if, for $\boldsymbol{Y} = \sum_{x\in Y} x$, $E_1\boldsymbol{Y} = 0$, and $E_2\boldsymbol{Y} \neq 0$, while Y is a 1-antidesign if $E_1\boldsymbol{Y} \neq 0$ and $E_2\boldsymbol{Y} = 0$.

3 The Johnson graph

Definition 3.1 [Johnson graph] Let $S = \{1,\dots v\}$ and $X = \{T \subset S \mid |T| = k\}$ $(k \leq v/2)$. Johnson graph $J(v,k)$ has X as a vertex set with $T_1, T_2 \in X$ adjacent if and only if $|T_1 \cap T_2| = k-1$.

Now we consider $J(v,2)$, which is a strongly regular graph. The number of vertices of the graph is $n = \frac{v(v-1)}{2}$. Let E_0, E_1, E_2 be the projection to the eigenspaces of $J(v,2)$, then the rank of these matrices are 1, $v-1$ and $\frac{(v-2)(v-1)}{2}$, respectively. Identify a vertex x with a vector in $\mathbf{R}^{\mathbf{n}}$ which is indexed by the elements of X having 1 in the x-th place, 0 otherwise. Consider the set $\bar{X} = \{E_1 x | x \in X\}$ as a representation of the graph which are on the surface of a sphere in $\mathbf{R}^{\mathbf{v}-1}$ vector space. Our purpose is to divide the points of $\bar{X}$ into two in order that they are on two parallel hyperplanes, which is equivalent to finding a vector which is normal to the planes.

The inner product between two points are defined by the adjacency between the points, because E_1 can be written as a linear combination of the adjacency matrix. Observe that $(E_1 x, E_1 y)$ is equal to $\dfrac{2}{v}$ if $x = y$; $\dfrac{v-4}{v(v-2)}$ if x, y are adjacent; and $\dfrac{-4}{v(v-2)}$ otherwise.

As a solution to our problem, we give a complete classification of the division of the vertices of $J(v,2)$.

Theorem 3.1 *The Johnson graph $J(v,2)$, when it is represented in a $v-1$ dimensional space, can be partitioned in this way: the vertices are placed on two parallel hyperplanes such that one hyperplane contains $\{i,1\}$, $\{i,2\}$, $\dots$, $\{i,v\}$ where $i \in S$, while the other one has the other vertices. Moreover, this is the only partition of vertices into two parallel hyperplanes.*

Proof. Since E_1 has rank $v-1$, there are $v-1$ vertices which are linearly independent. One can easily check that $X_1 = \{x_1 = \{1,2\}, \dots, x_{v-1} = \{1, v\}\}$ are such a set of vertices. Restate our problem using the above vertices as follows: For $\boldsymbol{u} = \sum_{i=1}^{v-1} a_i E_1 x_i$ and for every $x \in X$, Determine $a_1, \dots, a_{v-1} \in \boldsymbol{R}$ such that $(\boldsymbol{u}, E_1 x)$ has one of two values.

First, let us consider the innerproduct of $\boldsymbol{u}$ and the vertices of X_1. By permutation of $\{2, \dots, v\}$, we may assume that $(\boldsymbol{u}, E_1 x_i) = p$ for $i = 1, \dots, s$ and $(\boldsymbol{u}, E_1 x_j) = q$ for $j = s+1, \dots, v-1$. It is straightforward that $a = a_i$ for $i = 1, \dots, s$ and $b = a_j$ for $j = s+1, \dots, v-1$.

Next, let us calculate the innerproduct of $\boldsymbol{u}$ and $x \in X \setminus X_1$. If $s \geq 2$, each of them take one of the following values:

$$
\begin{aligned}
r_1 &= 2\frac{v-4}{v(v-2)}a + (s-2)\frac{-4}{v(v-2)}a + (v-1-s)\frac{-4}{v(v-2)}b, \\
r_2 &= \frac{-4}{v(v-2)}sa + 2\frac{v-4}{v(v-2)}b + (v-1-s-2)\frac{-4}{v(v-2)}b, \\
r_3 &= \frac{-4}{v(v-2)}a + \frac{-4}{v(v-2)}b + (s-1)\frac{-4}{v(v-2)}a \\
&\quad +(v-1-s-1)\frac{-4}{v(v-2)}b.
\end{aligned}
$$

Since they take one or two values, $r = r_i$, and so $p = q$. Therefore $a = b$.

Assume that $s = 1$. Then the innerproducts p, q are given by $p = \frac{1}{v}(2a + (v-4)b)$ and $q = \frac{1}{v}(2b + \frac{v-4}{v-2}a + \frac{v-4}{v-2}(v-3)b)$. The innerproduct of $\boldsymbol{u}$ and $x \in X \setminus X_1$ are $\frac{1}{v(v-2)}\{(v-4)a - (3v-8)b\} = r_1$, if $x = \{2, k\}$ for some k, and $-\frac{1}{v(v-2)}\{2(v-4)b + 4a\} = r_2$, otherwise. Let $p = r_2$ and $q = r_1$ then immediately we know this case does not fit our hypothesis. Let $p = r_1$ and $q = r_2$, then we have $a = -(v-3)b$.

Let $\boldsymbol{u} = (-(v-3), \overbrace{1, \dots, 1}^{v-2 \text{ times}})$. Let $X_2 \subset X$ with $X_2 = \{\{2,1\}, \{2,3\}, \dots, \{2, v\}\}$. and $\boldsymbol{v} = \sum_{x \in X_2} E_1 x$. Then we have $\boldsymbol{v} = -\boldsymbol{u}$. Remembering that we permuted $\{2, \dots, v\}$ for calculation, we have the following result: Let X' be a subset of X with cardinality $v-1$, such that for all $x, y \in X'$, $|x \cap y| = 1$, and $\boldsymbol{u} = \sum_{x \in X'} E_1 x$. Then this $\boldsymbol{u}$ divides the points of the graph on two hyperplanes so that one hyperplane has X' and the other one has the other vertices, and these are the only vectors which divide the vertices. ■

Let us see briefly the case $J(v,d)$, $d \geq 2$. Let X' be the set of d subset of S, which includes a given letter i. In this case, by the same discussion, $\sum_{x\in X'} E_1 x$ divides the points of X into two hyperplanes, X' and the other vertices.

4 The Hamming graph

In this section we see the same problem on the Hamming graph.

Definition 4.1 [Hamming graph] Let $F = \{1,\ldots,q\}$ and X be an n direct product of F. Hamming graph has X as a vertex set and adjacency on X is defined that for $x,y \in X$, where $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, x,y are adjacent if and only if $\#\{j | x_j \neq y_j\} = 1$. We call this graph Hamming graph $H(n,q)$.

Now we consider the graph $H(2,q)$, which is a strongly regular graph. The number of vertices of the graph is $n = q^2$. Let E_1 be a projection to the eigenspace of $H(2,q)$ with respect to the second largest eigenvalue with the rank $2(q-1)$. The question is: Can the vertices be partitioned into two parallel hyperplanes? How they are partitioned?

Theorem 4.1 *For the graph $H(2,q)$, in the representation on $2(q-1)$ dimensional Euclidean space, there is a vector $\boldsymbol{u}$ such that any embedded point of the graph lies on one of two hyperplanes with respect to the vector $\boldsymbol{u}$. The expression of $\boldsymbol{u}$ is as follows: Let $\boldsymbol{v}_t$ be the sum of the embeddings of (j,t), $j = 1..q$, $\boldsymbol{v}^t$ be the sum of the embeddings of (t,j), $j = 1..q$, and S be a proper subset of $\{1,\ldots,q\}$. Then $\boldsymbol{u}_S = \sum_{t\in S} \boldsymbol{v}_t$ or $\boldsymbol{u}^S = \sum_{t\in S} \boldsymbol{v}^t$ gives the description for $\boldsymbol{u}$.*

Proof. For the vertices of $H(2,q)$, by abuse of notation, identify the embedded points with (i,j) $(i = 1,\ldots,q,\ j = 1,\ldots,q)$. They are embedded on a sphere with its radius $(2q-2)^{1/2}$ and the innerproduct of two vertices is $q-2$ for the adjacent pair and -2 for the non-adjacent pair.

Let $X_1 := \{(1,2),(1,3),\ldots,(1,q),(2,1),(3,1),\ldots,(q,1)\}$, (the set of vertices which contains 1 as either coordinate) then if $q > 2$, X_1 is the basis of representation space. For the case $q = 2$, it is easy to see that the assertion holds, so we may assume that $q > 2$.

Let $\boldsymbol{u} = \sum_{i=2}^{q} a_i(1,i) + \sum_{j=2}^{q} b_j(j,1)$. Let us determine $\{a_i, b_j\}$ so that for every vertex $x \in X$, the innerproduct of $\boldsymbol{u}$ and x has one of two values. Denote $\alpha = 2(q-1), \beta = q-2, \gamma = -2$.

First, let us see the value $(\boldsymbol{u},(1,i))$ or $(\boldsymbol{u},(i,1))$.

$$(\boldsymbol{u},x)=\begin{cases}\alpha a_i+\sum_{j\neq i}\beta a_j+\sum_{j=2}^{q}\gamma b_j & \text{if } x=(1,i)\ (i=2,\dots q)\\ \alpha b_i+\sum_{j\neq i}\beta b_j+\sum_{j=2}^{q}\gamma a_j & \text{if } x=(i,1)\ (i=2,\dots q).\end{cases}$$

By the reordering of the indices, let

$$\begin{aligned}(\boldsymbol{u},(1,2)),\dots,(\boldsymbol{u},(1,r)) &= p_1,\\ (\boldsymbol{u},(1,r+1)),\dots,(\boldsymbol{u},(1,q)) &= p_2,\\ (\boldsymbol{u},(1,2)),\dots,(\boldsymbol{u},(1,s)) &= p_1',\\ (\boldsymbol{u},(1,s+1)),\dots,(\boldsymbol{u},(1,q)) &= p_2'.\end{aligned}$$

Hence, we have the following for the a_i's and b_i's.

$$\begin{aligned}a &= a_2=\cdots=a_{r+1}\\ &= \alpha p_1+(r-1)\beta p_1+(q-1-r)\beta p_2+\gamma(sp_1'+(q-1-s)p_2'),\\ a' &= a_{r+2}=\cdots=a_q\\ &= \alpha p_2+r\beta p_1+(q-2-r)\beta p_2+\gamma(sp_1'+(q-1-s)p_2'),\\ b &= b_2=\cdots=b_{s+1}\\ &= \gamma(rp_1+(q-1-r)p_2)+\alpha p_1'+(s-1)\beta p_1'+(q-1-s)\beta p_2',\\ b' &= b_{s+2}=\cdots=b_q\\ &= \gamma(rp_1+(q-1-r)p_2)+\alpha p_2'+s\beta p_1'+(q-2-s)\beta p_2'.\end{aligned}$$

Conversely, this means that p_i, q_i are described as follows.

$$\begin{aligned}p_1 &= 2(q-1)a+(r-1)(q-2)a+(q-1-r)(q-2)a'\\ &\quad -2(sb+(q-1-s)b'),\\ p_2 &= 2(q-1)a'+r(q-2)a+(q-2-r)(q-2)a'\\ &\quad -2(sb+(q-1-s)b'),\\ p_1' &= -2(ra+(q-1-r)a')+2(q-1)b+(s-1)(q-2)b\\ &\quad +(q-1-s)(q-2)b',\\ p_2' &= -2(ra+(q-1-r)a')+2(q-1)b'+s(q-2)b\\ &\quad +(q-2-s)(q-2)b'.\end{aligned}$$

Then, here we have $p_1-p_2=q(a-a')$, $p_1'-p_2'=q(b-b')$. Note that we have at most two distinct values for p_i's and p_i''s. Let $p=(\boldsymbol{u},(1,1))=(q-2)(ra+(q-1-r)a'+sb+(q-1-s)b')$. This is equal to either of

p_1, p_2, p'_1, p'_2 or it is equal to none of them and p_1, p_2, p'_1, p'_2 have the same value.

Let us see case by case.

Suppose $p_1 = p_2$ *and* $p'_1 = p'_2$. Then $p = (q-2)(q-1)(a+b)$, and $p_1 = (q^2 - 2q + 2)a - 2(q-1)b$, $p'_1 = -2(q-1)a + (q^2 - 2q + 2)b$. If $p = p_1$, then we have $a = (q-1)b$, and conversely if $p = p'_1$, then we have $b = (q-1)a$.

For these two cases, $p = p_1$ or $p = p'_1$, then it is easy to see that $u = (q-2)\sum_{i=1}^{q}(1,i)$ or $u = (q-2)\sum_{i=1}^{q}(i,1)$.

Suppose $p_1 = p_2$ *and* $p'_1 \neq p'_2$. Let $p_1 = p'_1$. Then we have $qa = (s+1)b + (q-1-s)b'$. If $p = p_1$ then $(q-1)a = b$ and if $p = p'_2$ then $(q-1)a = b'$. If $p = p_1$, $a = sb + (q-1-s)b'$ therefore $(q-1)a = b$. and if $p = p'_2$, $(q-1)a = b$.

Let $p' = (u, (i,j)) = -(q-2)a + (q-2-2(s-1))b - 2(q-1-s)b'$ for $(i,j) \in F \times F$ $2 \le i \le r+1$, $2 \le j \le q$. When $p' = p_1$, $(q-1)a = b$ and when $p' = p'_2$, $a = (s-1)b + (q-s)b'$.

Let $p'' = (u, (i,j)) = -(q-2)a - 2sb + (2s - q + 2)b'$ for $(i,j) \in F \times F$ $r+2 \le i \le q$, $2 \le j \le q$. If $p'' = p_1$ then $(q-1)a = b'$ and if $p'' = p'_2$ then $a = sb + (q-1-r)b'$.

Checking each of the cases above, we finally get one case that $p_1 = p = p'$ and $b = (q-1)a$, $(r(1-q)+1)b = (q-1-r)(q-1)b'$. Now let v be a sum of the following vertices: $(1,1), (1,2), \ldots (1,q)$, and (i,j) for $i \le r+1, j = 1, \ldots, q$, $v = \sum_{i=1}^{r+1}\sum_{j=1}^{q}(i.j)$. Then by direct calculation, we see that v is scalar product of u.

Let $p_1 = p'_2$. Then we can get similar result, by the same step, that u is the scalar product of $v = \sum_{i=r+2}^{q}\sum_{j=1}^{q}(i.j)$.

Suppose $p_1 \neq p_2$ *and* $p'_1 = p'_2$. By the same discussion, u is the scalar product of $v = \sum_{i=1}^{q}\sum_{j=1}^{r+1}(i.j)$ or the scalar product of $v = \sum_{i=1}^{q}\sum_{j=r+2}^{q}(i.j)$.

Now suppose $p_1 \neq p_2$ *and* $p'_1 \neq p'_2$. In this case, it is easy to see that $a' - a = b' - b = 0$, and this leads the same discussion as above.

Now, remember that we reordered the indices for easy calculation, so we have the desired result.

■

Now let us see the case $H(n,q)$, $n \ge 2$. In this case, as the same manner we saw in the above, there is a vector u such that the embedded vertices of the graph are partitioned into the two hyperplanes with respect to the vector u. The expression of u is as follows: Let $v_{t,i}$ be the sum of the vertices which has t in ith place, and S be a proper subset of $\{1, \ldots, q\}$ then for some number i, $u_S = \sum_{t \in S} v_{t,i}$ gives the description for u.

5 General results and other topics

Do all strongly regular graphs have such a property that when it is represented on a sphere, the vertices can be placed on two hyperplanes? The author found a few graphs which do not satisfy this property. They are three graphs which are Chang graphs with the same parameters as $J(8,2)$. For each of them, we cannot give a division such that they can be placed on two hyperplanes. The proof is in [10]. From this fact, we can say that the property depends not on the parameters but on the property of association scheme. For the second smallest strongly regular graph, a pentagon, we can say that there is no such a division of the vertex set. On the other hand, we also have some other strongly regular graphs which can divided on two hyperplanes. For the complete bipartite graph $K_{n,n}$, we can classify the division of the graph as follows: let X_1 and X_2 be the two maximal cocliques of $K_{n,n}$ and for $a = 1, \ldots, n-1$, Y_1 and Y_2 be any subsets of X_1 and X_2, with cardinality a. Then $Y_1 \cup Y_2$ are on one hyperplane while the other vertices are placed on the other hyperplane, and these are only division of the graph $K_{n,n}$. For the Grassmann graph over $\boldsymbol{F}_q$, q the prime power, the set of 2 dimensional subspaces which contains a given one dimensional subspace is on one hyperplane, while the other 2 dimensional subspaces are on the other hyperplane. For any Steiner triple system $(\mathcal{P}, \mathcal{B})$, the set of blocks which includes a given point $\alpha \in \mathcal{P}$ is on one hyperplane, while the other blocks are on the other plane. For the Shrikhande graph, we can give a complete classification for divisions into two hyperplanes, with 8 vertices lying each hyperplane. For the Paley graph of 9 vertices, the graph can be divided into two sets of vertices, one of them has 3 vertices. But the Paley graph of 13 vertices can not be divided into two hyperplanes.

In general, when can a strongly regular graph be divided into two hyperplanes? The author considered one condition such that the vertices of the graph is divided by a vector which is described as a sum of a set of vertices.

In the following discussion, Γ is a SRG, X is its vertex set, and E_i is a primitive idempotent of Γ. Let $\boldsymbol{u}$ be a vector of Euclidean space V. We say that H is a hyperplane with respect to $\boldsymbol{u}$ if for any $x, y \in H$, $1/|x|(\boldsymbol{u}, x) = 1/|y|(\boldsymbol{u}, y)$. Note that if V_1 is an eigenspace of Γ with respect to the eigenvalue not equal to the valency, then the sum of all the representative vectors of X is zero.

Let $\boldsymbol{u}$ be a vector in V_1, and H_1, H_2 be hyperplanes with respect to $\boldsymbol{u}$ in V_1. Let all the points of X be on either H_1 or H_2. Let $X_1 = X \cap H_1, X_2 = X \cap H_2$, with cardinalities x_1 and x_2, respectively. Since $(\boldsymbol{u}, E_1 \sum_{x \in X_1} x) = -(\boldsymbol{u}, E_1 \sum_{x \in X_2} x)$, for any $x \in X_1, y \in X_2$, $x_1(\boldsymbol{u}, E_1 x) = -x_2(\boldsymbol{u}, E_1 y)$.

Lemma 5.1 *Let* $\sum_{x \in X_1} E_1 x = \boldsymbol{u}$ *and* $(\boldsymbol{u}, E_1 x) = \alpha$ *for any* $x \in X_1$, $(\boldsymbol{u}, E_1 y) = \beta$ *for any* $y \in X_2$, *where* $X_2 = X \setminus X_1$. *Then for any* $x \in X_1$, $a = \#\{y \in X_1 | y$ *is adjacent to* $x\}$ *is constant and does not depend on the choice of* $x \in X_1$.

Proof. Let x(resp. y) $\in X_1$ and a_1(resp. a_1') be the number of the vertices which are adjacent to x(resp. y) in X_1, and a_2(resp. a_2') be the number of the points which is not adjacent to x(resp. y) in X_1. Let q_0, q_1, q_2 be the values of innerproduct of a pair of vertices which are equal, adjacent, and non-adjacent, respectively. Then

$$\begin{aligned} \alpha &= (q_0 + a_1 q_1 + a_2 q_2) \\ &= (q_0 + a_1' q_1 + a_2' q_2). \end{aligned}$$

Suppose $a_1 \neq a_1'$. Since $a_2 = x_1 - a_1$ and $a_2' = x_1 - a_1'$, $a_2 \neq a_2'$. Thus $(a_1 - a_1')q_1 + (a_2 - a_2')q_2 = 0$, and $q_1 = -\frac{a_2 - a_2'}{a_1 - a_1'} q_1 = q_2$, which is a contradiction. Therefore, $a_1 = a_1'$. ■

Definition 5.1 A vertex set X_1 in the SRG is said to be a regular subset if for any vertex x in X_1, the number of adjacent vertex within X_1 is d, and for any vertex y in $X \setminus X_1$, the number of adjacent vertices in X_1 is e, and it does not depend on the choice of x and y. If $d > e$, then the regular set is said to be positive and if $d < e$, it is said to be negative.

In the following discussion, we will see that for any strongly regular graph, if there is a regular subset, it gives a division of the graph on two hyperplanes. Now we prepare a lemma on a SRG with a regular subset, which proved in [7]. For a strongly regular graph Γ, let n be the cardinality of X, k be the valency of Γ, λ be the number of points which are adjacent to a pair of adjacent vertices, while μ be the number of points which are adjacent to a pair of non-adjacent vertices, and r, s be eigenvalues of Γ where $r > s$.

Lemma 5.2 (Neumaier) *Let* Γ *be a strongly regular graph having a regular subset* X_1. *If* X_1 *is positive*, $|X_1| = \frac{en}{k-r}$ *and* $|X_1| = \frac{en}{k-s}$ *otherwise. Moreover, the eigenvalue of* Γ *is* $r = d - e$ *if* X_1 *is positive*, $s = d - e$ *otherwise.*

Proposition 5.3 *Let* X_1 *be a subset of* X, *and* $X_2 = X \setminus X_1$. *Then the following conditions are equivalent;*

(i) $\sum_{x \in X_1} E_1 x$ *makes two hyperplanes so that each point on* X *is on either of two hyperplanes.*

(ii) X_1 *is a regular subset, i.e.,* X_1 *satisfies the following conditions:*

(a) For any point x in X_1, the number of points which are adjacent to x within X_1 is constant.

(b) For any point y in X_2, the number of points which are adjacent to y within X_2 is constant.

(iii) X_1 is a 1-antidesign or 1-design.

Proof. By abuse of notation, identify x as $E_1 x$. When $\sum_{x\in X_1} x$ makes two hyperplanes which have every point of X on either of them, then so does $\sum_{x\in X_2} x = -\sum_{x\in X_1} x$. Therefore the statement (ii) holds from Lemma 5.1. Conversely, if X_1 satisfies the conditions in (ii), for $\boldsymbol{u} := \sum_{z\in X_1} z$, $x \in X_1$, $y \in X_2$,

$$\begin{aligned}(\boldsymbol{u}, x) &= (q_0 + a_1 q_1 + a_2 q_2),\\ (\boldsymbol{u}, y) &= (q_0 + b_1 q_1 + b_2 q_2),\end{aligned}$$

where a_1, b_1 are the number of the elements in X_1 or X_2 which are adjacent to x and y. Since $1 + a_1 + a_2 = x_1, 1 + b_1 + b_2 = x_2$, $(\boldsymbol{u}, x) = \alpha$ for $^\forall x \in X_1$, $(\boldsymbol{u}, y) = \beta$ for $^\forall y \in X_2$.

Suppose (ii) holds. Then from Lemma 5.2, we know the cardinality of X_1 and the eigenvalues of Γ. Let E_1 be a projective map to the eigenspace corresponds to r, and E_2 be one corresponds to s. When X_1 is positive, then we get $|E_1 \sum_{x\in X_1} x|^2 = (\sum_{x\in X_1} x)^t E_1 \sum_{x\in X_1} x = 0$ by a direct calculation, therefore X_1 is a 1-antidesign. When X_1 is negative, then we have $|E_2 \sum_{x\in X_1} x| = 0$, and it proves that X_1 is a 1-design.

Suppose X_1 to be a 1-antidesign, then for $x \in X$, the number of the adjacent point from x in X_1 is;

$$\begin{aligned}(\sum_{y\in X_1} y)^t A x &= (\sum_{y\in X_1} y)^t (kE_0 + rE_1 + sE_2)x\\ &= (\sum_{y\in X_1} y)^t \{(k-r)E_0 + rI + (s-r)E_2\} x\\ &= (\sum_{y\in X_1} y)^t (k-r)E_0 x + r(\sum_{y\in X_1} y)^t I x\\ &= (k-r)\frac{|X_1|}{n} + r\delta_{x,X_1}\end{aligned}$$

Here,

$$\delta_{x,X_1} = \begin{cases} 1 & \text{if } x \in X_1\\ 0 & \text{otherwise.}\end{cases}$$

The case that X_1 is a 1-design leads the same discussion.

■

Remark. All examples of the divisions previously we saw are regular subset in strongly regular graph. In section 3 and 4, the author gave the classification of the regular subset of $J(v,2)$ or $H(2,q)$, and we know that the Chang graphs has no regular subset.

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