A Table of Lower Bounds for the Number of Mutually Orthogonal Frequency Squares

Charles F. Laywine * and Gary L. Mullen †

May 4, 2000

Abstract

We construct a small table of lower bounds for the maximum number of mutually orthogonal frequency squares of types $F(n;\lambda)$ with $n \leq 100$.

1 Introduction

Let $n = \lambda m$. An $F(n;\lambda)$ *frequency square* is an $n \times n$ array in which each of m distinct symbols appears exactly λ times in each row and column. Two such squares are said to be *orthogonal* if upon superposition, each of the m^2 distinct ordered pairs occurs exactly λ^2 times. Finally a set of $t \geq 2$ such squares is said to be *orthogonal* if any two distinct squares are orthogonal. We refer to Laywine [11] for a brief survey of sets of mutually orthogonal frequency squares.

Thus $F(n;1)$ frequency squares are simply latin squares of order n. A table of lower bounds for the maximum number of mutually orthog-

*Department of Mathematics, Brock University, St. Catharines, Ontario, L2S 3A1, Canada, Email: claywine@spartan.ac.brocku.ca

†Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA, Email: mullen@math.psu.edu

onal latin squares (MOLS) of order $n \leq 10,000$ is provided in [1]. The purpose of this note is to provide a small table of lower bounds for the maximum number of mutually orthogonal frequency squares (MOFS) of type $F(n; \lambda)$. This new table has been, to a large extent, inspired and influenced by the older table which directly provided the $F(n; 1)$ entries, and, indirectly, many others.

An $F(n; \lambda_1, \ldots, \lambda_m)$ *frequency square* based upon the symbols $1, \ldots, m$ is an $n \times n$ array in which each symbol i occurs exactly λ_i times in each row and column. Two such squares are *orthogonal* if upon superpositioning of them, each of the ordered pairs (i, j) occurs exactly $\lambda_i \lambda_j$ times. A set of $t \geq 2$ such squares is *orthogonal* if each pair of distinct squares is orthogonal. While this note focusses only on the case of sets of MOFS with constant frequency vectors, sets of at least two MOFS exist for any frequency vector $(\lambda_1, \ldots, \lambda_m)$ which partitions a given order $n > 2$ except for the partition $\lambda_1 = \cdots = \lambda_6 = 1$ in the case where $n = 6$. Such sets can always be found by the method of substitution of symbols outlined in (h) below and discussed by Laywine and Mullen in [12].

For $n = m\lambda$, an upper bound on the maximum number of MOFS of type $F(n; \lambda)$ was given in [8]. This upper bound is

$$(n-1)^2/(m-1),$$

and a set of MOFS attaining this bound is said to be *complete.*

We now list the major constructions for sets of MOFS with constant frequency vectors.

(a) Let q denote a prime power and let $i \geq 1$ be an integer. There are several methods of constructing complete sets of MOFS of type

$F(q^i; q^{i-1})$ containing $(q^i - 1)^2/(q - 1)$ squares. These include statistical methods as in [8], linear polynomials over the finite field F_q as in [15], substitutions to the symbols of a complete set of MOLS in which the symbols of the MOLS are grouped together to form the blocks of a related affine design as in [13], and using certain properties of subsquares as in [10]. These methods all yield complete sets of $F(q^i; q^{i-1})$ MOFS. See also [17] for a considerable extension of [15] to sets of very general hyperrectangles.

(b) In [6] complete sets of MOFS of type $F(4t; 2t)$ are constructed whenever there is a Hadamard matrix of order 4t (which for $n \leq 100$, is always the case).

(c) In [7] Finney constructs 8 $F(6; 2)$ MOFS and 7 $F(6; 3)$ MOFS.

(d) In [14] the authors show that if there is a Hadamard matrix of order $4t$, then there are $4t - 2$ MOFS of type $F(4t; t)$.

(e) The Kronecker product is an effective technique for constructing sets of MOLS of order $n_1 n_2$ from sets of MOLS of orders n_1 and n_2. A similar technique was extended to sets of MOFS by Hedayat, Raghavarao and Seiden [8]. In particular, if one has k MOFS of type $F(n_1; \lambda_1)$ and k MOFS of type $F(n_2; \lambda_2)$, the Kronecker product can be used to construct k MOFS of type $F(n_1 n_2; \lambda_1 \lambda_2)$. This technique is especially effective for non-prime powers where few other techniques are currently available.

(f) In [5] Dillon, Ferragut, and Gealy give a construction for $q^4 - 1$ MOFS of type $F(q^3; q)$ when q is a prime power. Using a more general technique, for any prime power q and positive integers r and s with $(r, s) = 1$, they are able to construct a set of $q^{2s+r} - 1$ MOFS of type

$F(q^{r+s}; q^s)$. For $r = 2$ and odd $s \geq 1$, they obtain

$$\frac{q^{2s+4}-1}{q^2-1} - \frac{q^{s+2}-1}{q-1} + (q-1)(q^s+1)$$

MOFS of type $F(q^{s+2}; q^s)$. In addition, when $t+2$ divides s, the last factor of $q^s + 1$ can be improved to $(q^{s+t+2}-1)/(q^{t+2}-1)$.

(g) In a private communication [16] to the authors, Stufken describes various sets of MOFS of small orders. Further details for these constructions are given in [9].

(h) By making a substitution on the symbols of a set of δ MOLS, one can easily obtain a set of δ MOFS with any prescribed frequency. Specifically to obtain $F(n; \lambda_1, \ldots, \lambda_m)$ MOFS from a set of MOLS of order n, replace λ_i of the symbols in the MOLS by i for $i = 1, \ldots, m$. More generally, one can make similar substitutions on a set of MOFS under suitable circumstances. Suppose $A = \{\mu_1, \ldots, \mu_k\}$ and $B = \{\lambda_1, \ldots, \lambda_\ell\}$ are partitions of n for $\ell \leq k$. Then A is a *refinement* of B if the set A can be partitioned into ℓ subsets $A_1, \ldots, A_\ell$ such that if $A_i = \{\mu_{i_1}, \ldots, \mu_{i_m}\}$ then $\lambda_i = \mu_{i_1} + \cdots + \mu_{i_m}$ for $i = 1, \ldots, \ell$. Then a set of δ MOFS of type $F(n; \lambda_1, \ldots, \lambda_\ell)$ can be obtained from a set of δ MOFS of type $F(n; \mu_1, \ldots, \mu_k)$ by replacing the symbols in A_i by i for $i = 1, \ldots, \ell$. Complete details are given by Laywine and Mullen in [12].

We now include our table. We use the notation $f(n; \lambda)$ to denote the number of MOFS that can currently be constructed of type $F(n; \lambda)$ where $n = \lambda m$. Hence $f(n; \lambda) \leq (n-1)^2/(m-1)$. In this notation, m denotes the number of distinct symbols, and λ denotes the frequency with which each symbol occurs. Underlined values indicate that the given set is complete; i.e. that the number of MOFS equals the upper bound given in [8]. All entries in the latin cases, i.e., all entries of the form $f(n;1)$ are taken from [1]. In order to save space we have omitted all entries of the form $f(p;1) = p-1$, where p is a prime. The letter that occurs as an exponent on each value refers to the method of construction, and should not be interpreted as an exponent. In those cases where more than one construction provide the table entry for a particular value of $f(n;\lambda)$ we have identified, somewhat subjectively, the method we consider the most efficient.

$f(4;1) = \underline{3}; f(4;2) = \underline{9}^a$

$f(6;1) = 1; f(6;2) = 8^c; f(6;3) = 7^c$

$f(8;1) = \underline{7}; f(8;2) = 15^f; f(8;4) = \underline{49}^a$

$f(9;1) = \underline{8}; f(9;3) = \underline{32}^a$

$f(10;1) = 2; f(10;2) = f(10;5) = 2^h$

$f(12;1) = 5; f(12;2) = 5^h; f(12;3) = 10^d; f(12;4) = 5^h; f(12;6) = \underline{121}^b$

$f(14;1) = 3; f(14;2) = f(14;7) = 3^h$

$f(15;1) = 4; f(15;3) = f(15;5) = 4^h$

$f(16;1) = \underline{15}; f(16;2) = 31^f; f(16;4) = \underline{75}^a; f(16;8) = \underline{225}^a$

$f(18;1) = 3; f(18;2) = f(18;3) = 3^h; f(18;6) = 119^g; f(18;9) = 3^h$

$f(20;1) = 4; f(20;2) = f(20;4) = 4^h; f(20;5) = 18^d; f(20;10) = \underline{361}^b$

$f(21;1) = 5; f(21;3) = f(21;7) = 5^h$

$f(22;1) = 3; f(22;2) = f(22;11) = 3^h$

$f(24;1) = 6; f(24;2) = f(24;3) = 6^h; f(24;4) = 8^e; f(24;6) = 22^d; f(24;8) = 8^h; f(24;12) = \underline{529}^b$

$f(25;1) = \underline{24}; f(25;5) = \underline{144}^a$

$f(26;1) = 4; f(26;2) = f(26;13) = 4^h$

$f(27;1) = \underline{26}; f(27;3) = 80^f; f(27;9) = \underline{338}^a$

$f(28;1) = 5; f(28;2) = f(28;4) = 5^h; f(28;7) = 26^d; f(28;14) = \underline{729}^b$

$f(30;1) = 4; f(30;2) = f(30;3) = f(30;5) = f(30;6) = f(30;10) = f(30;15) = 4^h$

$f(32;1) = \underline{31}; f(32;2) = 63^f; f(32;4) = 127^f; f(32;8) = 319^f; f(32;16) = \underline{961}^c$

$f(33;1) = 5; f(33;3) = f(33;11) = 5^h$

$f(34;1) = 4; f(34;2) = f(34;17) = 4^h$

$f(35;1) = 5; f(35;5) = f(35;7) = 5^h$

$f(36;1) = 6; f(36;2) = 8^e; f(36;3) = 6^h; f(36;4) = 8^h; f(36;6) = 9^e; f(36;9) = 34^d; f(36;12) = 9^h; f(36;18) = \underline{1225}^b$

$f(38;1) = 4; f(38;2) = f(38;19) = 4^h$

$f(39;1) = 5; f(39;3) = f(39;13) = 5^h$

$f(40;1) = 7; f(40;2) = f(40;4) = f(40;5) = f(40;8) = 7^h; f(40;10) = 38^d; f(40;20) = \underline{1521}^b$

$f(42;1) = 5; f(42;2) = 6^e; f(42;3) = 6^e; f(42;6) = 6^h; f(42;7) = 5^h; f(42;14) = 6^h; f(42;21) = 6^h$

$f(44;1) = 5; f(44;2) = f(44;4) = 5^h; f(44;11) = 42^d; f(44;22) = \underline{1681}^b$

$f(45;1) = 6; f(45;3) = f(45;5) = f(45;9) = f(45;15) = 6^h$

$f(46;1) = 4; f(46;2) = f(46;23) = 4^h$

$f(48;1) = 7; f(48;2) = f(48;3) = 7^h; f(48;4) = 8^e; f(48;6) = 7^h; f(48;8) = 8^h; f(48;12) = 46^d; f(48;16) = 8^h; f(48;24) = \underline{2209}^b$

$f(49;1) = \underline{48}; f(49;7) = \underline{384}$[a]

$f(50;1) = 6; f(50;2) = f(50;5) = 6$[h]; $f(50;10) = 539$[g]; $f(50;25) = 6$[h]

$f(51;1) = 5; f(51;3) = f(51;17) = 5$[h]

$f(52;1) = 5; f(52;2) = f(52;4) = 5$[h]; $f(52;13) = 50$[d]; $f(52;26) = \underline{2601}$[b]

$f(54;1) = 5; f(54;2) = 8$[e]; $f(54;3) = 7$[e]; $f(54;6) = 8$[e]; $f(54;9) = 7$[h]; $f(54;18) = 1325$[g]; $f(54;27) = 7$[h]

$f(55;1) = 6; f(55;5) = f(55;11) = 6$[h]

$f(56;1) = 7; f(56;2) = f(56;4) = f(56;7) = f(56;8) = 7$[h]; $f(56;14) = 54$[d]; $f(56;28) = \underline{3025}$[b]

$f(57;1) = 7; f(57;3) = f(57;19) = 7$[h]

$f(58;1) = 5; f(58;2) = f(58;29) = 5$[h]

$f(60;1) = 4; f(60;2) = f(60;3) = f(60;4) = f(60;5) = f(60;6) = f(60;10) = f(60;12) = 4$[h]; $f(60;15) = 58$[d]; $f(60;20) = 4$[h]; $f(60;30) = \underline{3481}$[b]

$f(62;1) = 4; f(62;2) = f(62;31) = 4$[h]

$f(63;1) = 6; f(63;3) = f(63;7) = f(63;9) = f(63;21) = 6$[b]

$f(64;1) = \underline{63}; f(64;2) = 127$[f]; $f(64;4) = 255$[f]; $f(64;8) = \underline{567}$[a]; $f(64;16) = \underline{1323}$[a]; $f(64;32) = \underline{3969}$[a]

$f(65;1) = 7; f(65;5) = f(65;13) = 7$[h]

$f(66;1) = 5; f(66;2) = 8$[e]; $f(66;3) = 7$[e]; $f(66;6) = 8$[h]; $f(66;11) = 5$[h]; $f(66;22) = 8$[h]; $f(66;33) = 7$[h]

$f(68;1) = 5; f(68;2) = 9$[e]; $f(68;4) = 9$[h]; $f(68;17) = 66$[d]; $f(68;34) = \underline{4489}$[b]

$f(69;1) = 6; f(69;3) = f(69;23) = 6$[h]

$f(70;1) = 6; f(70;2) = f(70;5) = f(70;7) = f(70;10) = f(70;14) = f(70;35) = 6^h$

$f(72;1) = 7; f(72;2) = 8^e; f(72;3) = 7^h; f(72;4) = 8^h; f(72;6) = 15^e; f(72;8) = 8^h; f(72;9) = 7^h; f(72;12) = 32^e; f(72;18) = 70^d; f(72;24) = 32^h; f(72;36) = \underline{5041}^b$

$f(74;1) = 5; f(74;2) = f(74;37) = 5^h$

$f(75;1) = 5; f(75;3) = f(75;5) = f(75;15) = f(75;25) = 5^h$

$f(76;1) = 6; f(76;2) = 9^e; f(76;4) = 9^h; f(76;19) = 74^d; f(76;38) = \underline{5625}^b$

$f(77;1) = 6; f(77;7) = f(77;11) = 6^h$

$f(78;1) = 6; f(78;2) = 8^e; f(78;3) = 7^e; f(78;6) = 8^h; f(78;13) = 6^h; f(78;26) = 8^h; f(78;39) = 7^h$

$f(80;1) = 9; f(80;2) = f(80;4) = f(80;5) = f(80;8) = f(80;10) = f(80;16) = 9^h; f(80;20) = 78^d; f(80;40) = \underline{6241}^b$

$f(81;1) = \underline{80}; f(81;3) = 242^f; f(81;9) = \underline{800}^a; f(81;27) = \underline{3200}^a$

$f(82;1) = 8; f(82;2) = f(82;41) = 8^h$

$f(84;1) = 6; f(84;2) = f(84;3) = f(84;4) = f(84;6) = f(84;7) = f(84;12) = f(84;14) = 6^h; f(84;21) = 82^d; f(84;28) = 6^h; f(84;42) = \underline{6889}^b$

$f(85;1) = 6; f(85;5) = f(85;17) = 6^h$

$f(86;1) = 6; f(86;2) = f(86;43) = 6^h$

$f(87;1) = 6; f(87;3) = f(87;29) = 6^h$

$f(88;1) = 7; f(88;2) = 10^e; f(88;4) = 10^h; f(8;8) = f(88;11) = 10^h; f(88;22) = 80^d; f(88;44) = \underline{7569}^b$

$f(90;1) = 6; f(90;2) = f(90;3) = f(90;5) = f(90;6) = f(90;9) = f(90;10) = f(90;15) = f(90;18) = f(90;30) = f(90;45) = 6^h$

$f(91;1) = 7; f(91;7) = f(91;13) = 7^h$

$f(92;1) = 6; f(92;2) = 8^e; f(92;4) = 8^h; f(92;23) = 90^d; f(92;46) = \underline{8281}^b$

$f(93;1) = 6 : f(93;3) = f(93;31) = 6^h$

$f(94;1) = 6; f(94;2) = f(94;47) = 6^h$

$f(95;1) = 6; f(95;5) = f(95;19) = 6^h$

$f(96;1) = 7; f(96;2) = 8^e; f(96;3) = 7^h; f(96;4) = 8^h; f(96;6) = 10^e; f(96;8) = 8^h; f(96;12) = 15^e; f(96;16) = 8^h; f(96;24) = 94^d; f(96;32) = 8^h; f(96;48) = \underline{9025}^b$

$f(98;1) = 6; f(98;2) = f(98;7) = 6^h; f(98;14) = 1455^g; f(98;49) = 6^h$

$f(99;1) = 8; f(99;3) = 10^e; f(99;9) = f(99;11) = f(99;33) = 10^h$

$f(100;1) = 8; f(100;2) = 9^e; f(100;4) = 9^h; f(100;5) = 8^h; f(100;10) = f(100;20) = 9^h; f(100;25) = 98^d; f(100;50) = \underline{9801}^b$

We close with two conjectures. Let $\mathcal{F}(n;\lambda)$ denote the maximum number of MOFS of type $F(n;\lambda)$ with $n = \lambda m$.

Conjecture 1: If $\lambda_1 < \lambda_2$, then $\mathcal{F}(n;\lambda_1) < \mathcal{F}(n;\lambda_2)$.

This conjecture reflects the fact that the expression $(n-1)^2/(m-1)$ increases as m, the number of symbols, decreases. For the most part the values in the table seem to reflect this. If true those values in the table which are not consistent with the conjecture, such as $F(6;2)$ and $F(6;3)$, would be obvious candidates for further investigation.

Conjecture 2: There exists a complete set of $F(m^h;m^{h-1})$ MOFS if and only if m is a prime power.

When $h = 1$, Conjecture 2 reduces to the well known prime power conjecture for sets of MOLS which postulates that there is a complete set of MOLS of order n if and only if n is a prime power.

Acknowledgment: We would like to thank J. Stufken for a number of helpful comments related to the table.

References

[1] R.J. Abel, A.E. Brouwer, C.J. Colbourn, J.H. Dinitz, *Mutually orthogonal latin squares*, In: Handbook of Combinatorial Designs, C.J. Colbourn and J.H. Dinitz, Eds., CRC Press, Boca Raton, FL 1996, 111-142.

[2] R.C. Bose, *On the application of the properties of Galois fields to the construction of hyper-Graeco-Latin squares*, Sankhya 3(1938), 323-338.

[3] J. Dénes and A.D. Keedwell, Latin Squares and Their Applications, Academic Press, New York, 1974.

[4] J. Dénes and A.D. Keedwell, Latin Squares, Annals of Disc. Math., Vol. 46(1991).

[5] J.F. Dillon, E. Ferragut, and M. Gealy, private communication.

[6] W.T. Federer, *On the existence and construction of a complete set of orthogonal $F(4t; 2t, 2t)$-squares design*, Ann. Statist. 5(1977), 561-564.

[7] D.J. Finney, *Some enumerations for the* 6×6 *latin squares*, Utilitas Math. 21(1982), 137-153.

[8] A. HEDAYAT, D. RAGHAVARAO, AND E. SEIDEN, *Further contributions to the theory of F-squares design*, Ann. Statist. 3(1975), 712-716.

[9] A.S. HEDAYAT, N.J.A. SLOANE, AND J. STUFKEN, Orthogonal Arrays: Theory and Applications, Springer Verlag, New York, 1999.

[10] C.F. LAYWINE, *Complete sets of frequency squares with subsquares*, Utilitas Math. 40(1991), 87-96.

[11] C.F. LAYWINE, *Frequency squares*, in Handbook of Combinatorial Designs, C.J. Colbourn and J.H. Dinitz, Eds., CRC Press, Boca Raton, FL, 1996, 354-356.

[12] C.F. LAYWINE AND G.L. MULLEN, *Mutually orthogonal frequency squares with non-constant frequency vectors*, Ars Combinatoria 29(1990), 259-264.

[13] C.F. LAYWINE AND G.L. MULLEN, *Generalizations of Bose's equivalence between complete sets of mutually orthogonal latin squares and affine planes*, J. Combin. Theory ser. A 61(1992), 13-35.

[14] C.F. LAYWINE, G.L. MULLEN, AND S.J. SUCHOWER, *Orthogonal frequency squares of type $F(4t;t)$*, Utilitas Math. 37(1990), 207-214.

[15] G.L. MULLEN, *Polynomial representation of complete sets of mutually orthogonal frequency squares of prime power order*, Disc. Math. 69(1988), 79-84.

[16] J. STUFKEN, private communication.

[17] S.J. SUCHOWER, *Polynomial representation of complete sets of frequency hyperrectangles with prime power dimensions*, J. Combin. Theory, Ser. A 62(1993), 46-65.