Recursive Constructions of Balanced Incomplete Block Designs with Block Size of 7, 8 or 9

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Abstract

This paper gives constructions of balanced incomplete block designs and group divisible designs with $k = 7$, 8, or 9, and $\lambda = 1$. The first objective is to give constructions for all possible cases with the exception of 40, 78, and 157 values of v. Many of these initial exceptions have now been removed by Abel. In an update section, more are removed; group divisble designs with groups of size $k(k-1)$ are constructed for $k = 7$ and 8 with 124 and 87 exceptions; it is also established that $v \geq 294469$ and $v \equiv 7 \bmod 42$ suffices for the existence of a resolvable balanced incomplete block design with $k = 7$. Group divisible designs with group size k and resolvable designs are constructed.

Key words and phrases: BIBD, RBIBD, Group Divisible Design.
AMS subject classification: 05B05.

1 Introduction

This is the third paper in a series on BIBD constructions. In [25, 26], a number of direct constructions were given for BIBD's with $\lambda = 1$, including cases where $7 \leq k \leq 9$. The object of this paper is to use those direct constructions, together with recursive techniques, to produce constructions for the remaining values of v as far as possible. If k is a prime power, then the necessary conditions for a BIBD with $\lambda = 1$ are $v = 1$ or $k \bmod k(k-1)$.

These conditions are not, in general, sufficient, but the only known non-existence result is $43 \notin B(7, 1)$.

A first draft of this paper was informally circulated some time ago, and since then, the status of the existence problem that is the main topic of this paper has been improved upon, primarily by Abel [1, 2], but also in collaborative work. (For a recent published status of the problem, see [7].) However, this paper does form the basis for the later work, and is being published in something close to its original form. One objective is to provide some designs so that this paper, in conjunction with [2], establishes the relevant results in [7]. The main changes made to the earlier draft are in two of the secondary appendices, which detailed the smaller members of $RB(k, 1)$ and $GD(k, 1, k(k-1))$. Also, a new update section (Section 12) has been added, wherein we give a sufficient bound for the existence of $RC(7)$, as well as establishing small possible exception lists for $D(7)$ and $D(8)$. Some improvements to [7] that appear in [8] are also included here. Also, some improvements resulting from some recent small constructions given in [3, 19, 20, 33] are noted or established.

The basic organization of the paper is to give the theoretical constructions in the body of the paper, together with a letter code and the parameters, typically m and n, used in the derivation. The actual constructions are usually performed in the tables in the appendices, and there numerical values are substituted for m and n. This enables a complete presentation of a rather large number of constructions to be given compactly, yet allows the reader interested in any particular value to readily ascertain its construction.

2 Notation

The notation is largely taken from Hanani [30], but in order to compress the tables, and in view of the fact there are two classes of v modulo $k(k-1)$, some additional notation is employed. We introduce the class $A(k)$ through $E(k)$, and the pseudo class $F(k)$, by

$$
\begin{aligned}
t \in A(k) &\iff k(k-1)t + 1 \in B(k, 1), \\
&\iff k(k-1)t \in GD(k, 1, k-1), \\
t \in B(k) &\iff k(k-1)t + k \in B(k.1). \\
&\iff k(k-1)t + k - 1 \in GD(k.1.k-1). \\
t \in C(k) &\iff k(k-1)t + k \in GD(k.1.k), \\
t \in RC(k) &\iff k(k-1)t + k \in RGD(k, 1.k), \\
&\iff k(k-1)t + k \in RB(k.1), \\
t \in D(k) &\iff k(k-1)t \in GD(k, 1.k(k-1)),
\end{aligned}
$$

$$t \in E(k) \iff t \in A(k) \cap B(k).$$

The pseudo-class $F(k)$ is used in some theorems as a substitute for any arbitary one of the above classes (except $RC(k)$). All occurrences of $F(k)$ in the theorem, both in the hypotheses and the conclusions, may be replaced by any one of the above classes (except $RC(k)$), and the theorem will still be valid.

The above definitions also incorporate [30, Lemma 2.12]. In a similar vein, we have from [30, Lemma 2.10]:

Theorem 2.1 $RC(k) \subset C(k) \subset B(k)$.

3 The Basic Constructions

Our main recursive tool is the following special case of Wilson's Fundamental Construction (see e.g., [13, IX.3.2]).

Theorem 3.1 (Link Theorem) *If $v \in GD(K, \lambda, M)$ and $mK \in GD(K', \lambda', m)$, then $mv \in GD(K', \lambda\lambda', mM)$.*

Proof: [17, Lemma 2.27]. ∎

Remark 3.2 This theorem is invoked so often in the rest of the paper that it warrants a shorthand for its use, and so in the remainder of this paper I will implicitly invoke this theorem merely by giving the first design on v points and refering to the second design(s) on mK points as the *link* design(s).

The next two theorems (the *Fill Theorems*) enable the groups of a design to be filled in to some extent.

Theorem 3.3 *If $M' \subset GD(K, \lambda, M)$, then $GD(K, \lambda, M') \subset GD(K, \lambda, M)$.*

Proof: [17, Lemma 2.28]. ∎

Theorem 3.4 *If $v \in GD(K, \lambda, M)$ and $M + m \subset GD(K, \lambda, m)$, then $v + m \in GD(K, \lambda, m)$.*

Proof: [17, Lemma 2.29]. ∎

Remark 3.5 In the rest of this paper these two theorems are used heavily, usually with $m = 1$, $m = k - 1$, or $m = k$. Note that a $GD(k, \lambda, 1)$ is, by definition, a $B(k, \lambda)$.

4 Further Constructions

One obvious set of constructions is to take a known $B(k,1)$, in the GD form, (the m parameter below), and use a transversal design as the link, and fill appropriately using Theorems 3.3 and 3.4. As an illustration, consider code b in Table 4.1 below.

Lemma 4.1 *If $m \in A(k)$, $n \in F(k)$, and $kn \in T(k,1)$, then $kmn \in F(k)$.*

Proof: Use the transversal design as the link on $k(k-1)m \in GD(k,1,k-1)$ to give $k(k-1)kmn \in GD(k,1,k(k-1)n)$. If $F = A$, apply Theorem 3.3 with $M = \{k-1\}$; if $F = B$, apply Theorem 3.4 with $m = k-1$; if $F = C$, apply Theorem 3.4 with $m = k$; if $F = D$, apply Theorem 3.3 with $M = \{k(k-1)\}$; if $F = E$, then the result follows from the proof for $F = A$ and $F = B$. ∎

Table 4.1

Code	$m \in$	$n \in$	Link	Result
a	$A(k)$	$A(k)$	$kn-1 \in T(k,1)$	$(kn-1)m \in B(k)$
b	$A(k)$	$F(k)$	$kn \in T(k,1)$	$knm \in F(k)$
c	$A(k)$	$B(k)$	$kn+1 \in T(k,1)$	$(kn+1)m \in A(k)$
d	$B(k)$	$A(k)$	$kn-1 \in T(k,1)$	$(kn-1)m+n \in A(k)$
e	$B(k)$	$F(k)$	$kn \in T(k,1)$	$knm+n \in F(k)$
f	$B(k)$	$B(k)$	$kn+1 \in T(k,1)$	$(kn+1)m+n \in B(k)$
g	$C(k)$	$F(k)$	$(k-1)n \in T(k,1)$	$(k-1)nm+n \in F(k)$
h	$C(k)$	$B(k)$	$(k-1)n+1 \in T(k,1)$	$((k-1)n+1)m+n \in B(k)$
h	$C(k)$	$C(k)$	$(k-1)n+1 \in T(k,1)$	$((k-1)n+1)m+n \in C(k)$

Remark 4.2 For case h, with $n \in B(k)$, use Theorem 3.3 with $M = \{1\}$.

Simpler constructions result from using the fill theorems alone.

Table 4.2

Code	$m \in$	Condition	Result
HA	$A(k)$	$k(k-1)m-(k-1) \in T(k,1)$	$km-1 \in B(k)$
HB	$B(k)$	$k(k-1)m+k-1 \in T(k,1)$	$km+1 \in A(k)$
Hc	$B(k)$	$k(k-1)m+k \in T(k,1)$	$km+1 \in B(k)$
HC	$C(k)$	$k(k-1)m+k \in T(k,1)$	$km+1 \in C(k)$
HD	$A(k)$	$k(k-1)m+1 \in T(k,1)$	$km \in B(k)$
HF	$F(k)$	$k(k-1)m \in T(k,1)$	$km \in F(k)$

Remark 4.3 For cases Hc and HD, use Theorem 3.3 with $M = \{1\}$. For $7 \le k \le 9$, the relevant transversal designs all exist, now the former possible $T(7,1)$ exceptions of 36, 42, and 48 are known.

Since $\{1\} \in B(k)$ for $7 \le k \le 9$, we can start with designs on larger blocks.

Table 4.3

Code	$m \in$	Link	Fill	Result
i	$A(k+1)$	$\{1\} \in B(k)$	$\{1\} \in F(k)$	$(k+1)m \in F(k)$
j	$B(k+1)$	$\{1\} \in B(k)$	$\{1\} \in F(k)$	$(k+1)m+1 \in F(k)$

Up to now, we have used input designs with a uniform block size, but this is too restrictive. The following provides constructions for all k, but is especially useful for $k = 7$.

Theorem 4.4 *If $v \in GD(K, 1, M)$, $K \subset D(k)$, and $M \subset F(k)$, then $v \in F(k)$.*

Proof: Use $K \subset D(k)$ as links on the v point design. ∎

Remark 4.5 If $7 \le k \le 9$, then $D(k)$ contains at least a pair of adjacent values from $\{k, k+1, k+2\}$. Some constructions of the classes $D(k)$ are given in Appendix B and in Section 12.

There are four general basic methods we use to exploit this, and similar, theorems. These are truncating a group of a GD design, especially a transversal design; adding points to a resolvable design; the last spike construction, which deletes all points from several groups of a design, with the exception of those contained in the last block; and the complete or partial deletion of a block in a GD design.

Lemma 4.6 (Code T) *If $\{d, d+1\} \subset D(k)$, $m \in T(d+1, 1)$, $n \le m$, and $\{m, n\} \subset F(k)$, then $dm + n \in F(k)$.*

Lemma 4.7 (Code R) *Let $\{d, d+1\} \subset D(k)$ and $m \in RB(d, 1)$:*

a) *if $n \le (m-1)/(d-1)$, and $\{1, n\} \subset F(k)$, then $m + n \in F(k)$;*

b) *if $n < (m-1)/(d-1)$, and $\{d, n\} \subset F(k)$, then $m + n \in F(k)$.*

Proof: For the second part, we add points to the resolution classes of the $RGD[d, 1, d; m]$. ∎

Lemma 4.8 (Code ZR) *If $\{d, d+1\} \subset D(k)$, $m \in RC(d)$, $n \le dm + 1$, and $\{1, d-1, n\} \subset F(k)$, then $d(d-1)m + d + n - 1 \in F(k)$.*

Proof: This variant of the first part of Lemma 3.4 removes one original point of the $RB(d, 1)$ design, together with the unaugmented blocks in which it occurred, thus producing groups of size $d - 1$. ∎

Lemma 4.9 (Code L) *If $\{d, d+1, m\} \subset D(k)$, $m \in T(n,1)$, $n \geq d$, and $\{1, n\} \subset F(k)$, then $d(m-1)+n \in F(k)$.*

Proof: Since $m \in T(n,1)$, we can perform a last spike construction to give $dm + n - d \in GD(\{d, d+1, n^*\}, 1, \{1, m\})$. We now fill in the groups of size m and remove the single block of size n, and apply Theorem 4.4. ∎

Lemma 4.10 (Code BD) *Let $\{d, d+1\} \subset D(k)$ and $m \in T(d+1, 1)$;*

a) *if $m-1 \in F(k)$, then $(d+1)(m-1) \in F(k)$ (with $n = d+1$);*

b) *if $\{m-1, m\} \subset F(k)$, then $(d+1)(m-1)+1 \in F(k)$ (with $n = d$).*

Proof: Delete n points from a single block of the transversal design to give $(d+1)m - n \in GD(\{d, d+1, d+1-n^*\}, 1, \{m-1, m\})$. Note that, if $n = d+1$, then there will be no groups of size m. ∎

This has covered most of the constructions for the $k = 7$ case. For $8 \leq k \leq 9$, we have $\{1\} \in E(k)$, and this yields more powerful constructions. In general, if $\{x\} \in E(k)$, then $(k-1)\{kx, kx+1\} \subset GD(k, 1, k-1)$, and this allows a lot of additional flexibility.

Lemma 4.11 *If $km + u \in T(kx+1, 1)$, $kn + v \leq km + u$, and $\{x\} \in E(k)$, then $k(k-1)(x(km+u)+n) + v(k-1) \in GD(k, 1, \{k(k-1)m + u(k-1), k(k-1)n + v(k-1)\})$.*

Proof: This is a truncated transversal design construction, using $\{x\} \in E(k)$ as the links. ∎

Remark 4.12 The interesting applications follow by taking $-1 \leq (u, v) \leq 1$, with $|u - v| \leq 1$. Taking $x = 1$ gives us Table 4.4.

Table 4.4

Code	$m \in$	$n \in$	Conditions are $\{1\} \in E(k)$ and:		Result
k	$A(k)$	$A(k)$	$km - 1 \in T(k+1, 1)$	$1 \leq n \leq m$	$km - 1 + n \in A(k)$
l	$A(k)$	$B(k)$	$km - 1 \in T(k+1, 1)$	$n < m$	$km - 1 + n \in B(k)$
m	$B(k)$	$A(k)$	$km \in T(k+1, 1)$	$1 \leq n \leq m$	$km + n \in A(k)$
n	$F(k)$	$F(k)$	$km \in T(k+1, 1)$	$n \leq m$	$km + n \in F(k)$
o	$A(k)$	$B(k)$	$km \in T(k+1, 1)$	$n < m$	$km + n \in B(k)$
p	$B(k)$	$A(k)$	$km + 1 \in T(k+1, 1)$	$n \leq m$	$km + 1 + n \in A(k)$
q	$B(k)$	$B(k)$	$km + 1 \in T(k+1, 1)$	$n \leq m$	$km + 1 + n \in B(k)$

If we delete a group in the constructions of Table 4.3, we get:

Table 4.5

Code	$m \in$	Link	Fill	Result
Zi	$A(k+1)$	$\{1\} \in E(k)$	$\{1\} \in F(k)$	$(k+1)m-1 \in F(k)$
Zj	$B(k+1)$	$\{1\} \in E(k)$	$\{1\} \in F(k)$	$(k+1)m \in F(k)$

We can also delete n points from the same block of a transversal design.

Table 4.6

Code	$m \in$	n	Transversal	Link	Result
Ba	$A(k)$	$k+1$	$km \in T(k+1,1)$	$\{1\} \in E(k)$	$(k+1)m-1 \in A(k)$
BA	$E(k)$	k	$km \in T(k+1,1)$	$\{1\} \in E(k)$	$(k+1)m-1 \in B(k)$
BB	$F(k)$	$k+1$	$km+1 \in T(k+1,1)$	$\{1\} \in E(k)$	$(k+1)m \in F(k)$
BC	$B(k)$	$k+1$	$km+2 \in T(k+1,1)$	$\{1\} \in E(k)$	$(k+1)m+1 \in B(k)$

Remark 4.13 For the application of construction BC, the transversal designs of possible future interest require $26 \in T(9,1)$, and $\{29, 38, 56, 173, 191\} \subset T(10,1)$. Of these values, only the primes 29, 173, and 191 are known, and only 191 (and now 173) can be used currently.

We can also add points to a resolvable design.

Lemma 4.14 (Code r) *If $m \in RC(k)$, $n \leq m$, $\{1\} \in E(k)$, and $\{1,n\} \subset F(k)$, then $(k-1)m+1+n \in F(k)$.*

Proof: Add kn points to the $RGD(k,1,k)$, and use $\{1\} \in E(k)$ as the links. ∎

5 Standard Designs (Code S)

Having given most of the constructions we will be using, in the next two sections we look at what designs are available as input for those constructions. The next set of constructions are based on projective geometries, or on configurations that are found in them.

Lemma 5.1 *If q is a prime or prime power, then*

1. *$1 \in A(q+1)$, and*
2. *$1 \in RC(q)$, and*
3. *$q-1 \in RC(q+1)$, and*
4. *$q \in RC(q+1)$.*

Proof: The first two are the usual $PG(2,q)$ and $AG(2,q)$ designs. The third is the unital design (see [14, 32]). Lorimer [37] has shown the resolvability in the fourth design. ∎

Lemma 5.2 *If q is a power of 2, then $2^n \in RC(q)$.*

Proof: The designs were constructed by Denniston [22]. The resolvabilty in the case of $n = 1$ was shown by Seiden [44]. She also found it necessary to dualize, but the basic technique in the proof consisted of using the points of a deleted line to exhibit the resolvability and is more generally applicable. ∎

We do not insist on the *incompleteness* of BIBD's, nor do we exclude trivial designs.

Lemma 5.3 *For any k, we have*

1. $0 \in A(k)$, *and*
2. $0 \in RC(k)$, *and*
3. $\{0,1\} \subset D(k)$.

6 Previously Constructed Designs (Code P)

The first set of (cyclic in the case of primes) constructions were given in [25]. The relevant results can be summarized briefly.

Lemma 6.1 *If $42t + 1$ is a prime power, $t \leq 780$, and $t \notin$* {1, 3, 4, 5, 13, 35, 145, 159, 209, 224, 266, 268, 306, 360, 365, 372, 379, 383, 390, 403, 460, 476, 509, 605, 609, 619, 625, 645, 649, 664, 680, 681, 684, 730, 734, 744}, *then $t \in A(7)$.*

Remark 6.2 No construction, whether cyclic or not, is known for $t \subset \{1,3,5\}$, (and $t = 1$ is known as a non-existant design).

Lemma 6.3 *If $t \in$* {8, 18, 56, 66, 111, 113, 147, 173, 242, 260, 278, 348, 356, 446, 467, 555, 573}, *then $t \in A(8)$.*

Remark 6.4 No construction, whether cyclic or not, was known for $t \subset$ {2, 3, 5, 6, 11, 12, 13, 23, 38, 45, 47, 48, 53, 60, 75, 126, 182, 188, 192}. For these values $56t + 1$ is a prime power. Constructions are now known for all but the first four of these, and for $t \in \{6, 8, 14\}$ in the next lemma.

Lemma 6.5 *If $72t + 1$ is a prime power, $t \leq 455$, and $t \notin \{4,5,6,8,14\}$, then $t \in A(9)$.*

Lemma 6.6 *If* $90t+1$ *is a prime power,* $t \leq 364$, *and* $t \notin \{2, 3, 4, 6, 7, 9, 11, 17, 24, 25, 26, 55, 88, 356\}$, *then* $t \in A(10)$.

Corollary 6.7 *If* $t \in \{13, 18, 20, 28, 31, 33, 34, 37, 39, 41, 45, 47, 51, 54, 56, 62, 65, 68, 69, 70, 72, 73, 74, 76, 84, 86, 89, 90, 91, 94, 96, 97, 98\}$, *then* $t \in A(10)$.

Remark 6.8 The only other small (<100) designs I originally knew of were the values $\{1, 81, 82\} \subset A(10)$. These follow from Lemma 5.1. 1 and Table 4.1 (c 1.8, d 9.1).

The next set of constructions were given in [26]. The constructions for k odd are cyclic, (at least for primes), whilst those for k even have a fixed element. The relevant results can again be summarized briefly.

Lemma 6.9 *If* $6t+1$ *is a prime power,* $4 \leq t \leq 832$, $t \neq 6$, *and* t *is even, then* $t \in RC(6)$.

Lemma 6.10 *If* $6t+1$ *is a prime power,* $t \leq 512$, *and* $t \notin \{3, 4, 6\}$, *then* $t \in C(7)$.

Remark 6.11 The exceptions 4 and 6 have now been removed by Janko and Tonchev [33] and Abel [3].

Lemma 6.12 *If* $8t+1$ *is a prime power,* $5 \leq t \leq 512$, *and* $t \neq 11$, *then* $t \in RC(8)$.

Lemma 6.13 *If* $8t+1$ *is a prime power,* $t \leq 729$ *and* $t \notin \{2, 3, 5, 10, 12, 14\}$, *then* $t \in C(9)$.

Lemma 6.14 *If* $10t+1$ *is a prime power,* $8 \leq t \leq 729$, t *is even, and* $t \notin \{10, 12, 18, 24\}$, *then* $t \in RC(10)$.

Corollary 6.15 *If* $t \in \{8, 28, 36, 40, 42, 46, 52, 54, 60, 64, 66, 70, 76, 82, 84, 88, 94, 96\}$, *then* $t \in RC(10)$.

Remark 6.16 The only other small (<100) designs I knew of were $9 \in RC(10)$, (by Lemma 5.1. 4) and $\{80, 81\} \subset C(10)$ by Table 4.2 (HF 8, HC 8).

The next construction is a modification to a design of Mills [38], (also given in [13, p. 313].

Lemma 6.17 $4 \in A(6)$.

Proof: In $Z_{11} \times Z_{11}$, take as base blocks $(0_0, 0_4, 0_3, 1_1, 1_7, 4_6)$, $(0_5, 0_7, 2_{10}, 4_1, 8_5, 6_9)$, $(0_1, 1_6, 2_1, 4_2, 7_3, 6_1)$, $(0_2, 1_2, 4_{10}, 9_7, 3_0, 6_3)$. ∎

The final construction is due to Hanani [31].

Lemma 6.18 $4 \in A(7)$.

7 Some Resolvable Design Constructions

Theorem 7.1 (Code RW) *If v is an odd prime power, and there is a $(v,k,1)$ difference family $\{B_1, B_2, \ldots, B_t\}$ in $GF(v)$ such that the base blocks are mutually disjoint, then $kv \in RB(k,1)$.*

Proof: See Ray-Chaudhuri and Wilson [43]. (Also in [13, p. 356]). ∎

Corollary 7.2 *If $v = 30m+1$ is a prime power, $m \le 67$, and $m \ne 2$, then $6m \in RC(6)$.*

Proof: For $m = 4$, we take the modified Mills' design given in Lemma 6.17. For the other values, use the constructions given by Wilson [45, p. 45], (and Mills [39] for the omitted $v = 841$); finally, use Theorem 7.1. ∎

Corollary 7.3 $\{56, 70, 77\} \subset RC(7)$, *and* $9 \in RC(9)$.

Proof: For $k = 7$, use the constructions given by Wilson [45], and for $k = 9$, use the Singer difference set. ∎

Ray-Chaudhuri and Wilson [43] give some constructions that parallel some of those given in Table 4.1. Taking $v_2 = k$ and $v_2 = 1$ in their Theorem 4, and using their Theorem 3 gives:

Table 7.1

Code	$m \in$	$n \in$	Link	Value $\in RC(k)$
E	$RC(k)$	$RC(k)$	$kn \in T(k+1.1)$	$knm + n$
F	$RC(k)$	$RC(k)$	$kn + 1 \in T(k+1,1)$	$(kn+1)m + n$
H	$RC(k)$	$RC(k)$	$(k-1)n + 1 \in T(k+1,1)$	$((k-1)n+1)m + n$

The PBD closure result for $RB(k,1)$ designs [43, Theorem 1] has consequences that parallel the constructions of Table 4.3. Notice that $1 \in RC(k)$ ensures that $k+1$ is the replication number of a $RB(k,1)$ design.

Table 7.2

Code	$m \in$	Condition	Value $\in RC(k)$
I	$A(k+1)$	$\{1\} \in RC(k)$	$(k+1)m$
J	$B(k+1)$	$\{1\} \in RC(k)$	$(k+1)m + 1$

Lemma 7.4 (Code RT) *If $\{1, m, n\} \subset RC(k)$, $\{1\} \in E(k+1)$, $n \le m$, and $m \in T(k+2,1)$, then $(k+1)m + n \in RC(k)$.*

Proof: The values given by this construction are already covered by construction T, but with the exception of the resolvability. Use $\{1\} \in RC(k+1)$ as the link on a truncated transversal construction; since $\{1\} \in RC(k)$, the value $k+1$ is the replication count of a $RB(k,1)$ design, and PBD closure gives the result. ∎

Appendix C gives constructions of the smaller resolvable designs for $k = 6$, 7, and 9. The smaller constructions for $k = 8$ are incorporated into Table A.6.

8 Transversal Designs

In this section, I collect together previously constructed designs for later use. The first four lemmas are in [30, Lemmas 3.1, 3.4 3.6].

Lemma 8.1 *If $s' \leq s$, then $T(s, \lambda) \subset T(s', \lambda)$.*

Lemma 8.2 *If $r \in T(s, \lambda)$ and $r' \in T(s, \lambda')$, then $rr' \in T(s, \lambda\lambda')$.*

Lemma 8.3 *If q is a prime or a prime power, then $q \in T(q + 1, 1)$.*

Lemma 8.4 $RT(s, 1) = T(s + 1, 1)$.

The existence of the transversal designs given above and in the following lemmas is taken from [4], with improvements from [21]. Their weaker original versions, which were used to generate the designs in this paper were mostly taken from [15], with amendments from [11, 16, 18, 47].

Lemma 8.5 *If n_k is such that $v > n_k$ implies $v \in T(k+2, 1)$, then $n_5 \leq 62$, $n_6 \leq 75$, $n_7 \leq 780$, and $n_8 \leq 2774$.*

Lemma 8.6 *We have $v \in T(7, 1)$ for $v \notin \{2, 3, 4, 5, 6, 10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60, 62\}$.*

Lemma 8.7 *If $v \in \{7, 40, 50, 63, 69, 70\}$, then $v \in T(8, 1)$.*

Lemma 8.8 *If $v \in \{8, 56, 57, 65, 72, 80, 82, 88, 91, 96, 99, 100\}$, then it follows that $v \in T(9, 1)$.*

Lemma 8.9 *If $v \notin \{111, 119, 123, 159, 175, 183, 291, 295, 303, 335\}$ and v is odd and $v > 100$, then $v \in T(9, 1)$.*

Remark 8.10 A construction of Brouwer [16] partitions $PG(2, 23)$ into 7 disjoint $B[\{0, 3, 5\}, 1; 79]$ designs, and 4 of these give a $B[\{9, 11, 16\}, 1; 316]$ design; a consequence of Lemma 5.1.4 is $585 \in T(10, 1)$. The only transversal designs we use that are not covered by the above lemmas are $\{435, 519, 792, 892\} \subset T(10, 1)$; note that $435 = 16 \cdot 27 + 3$ (and $16 + 3 = 19$), $519 = 16 \cdot 31 + 23$, $792 = 31 \cdot 25 + 17$, and $892 = 81 \cdot 11 + 1$.

Some results for v of special forms are given in Tables 8.1 and 8.2.

Table 8.1

Table of unknown $T[k,1;v]$

v	k	Parity	Value of m
$6m$	7	all m	1, 3, 5, 9, 10
$6m+1$	7	all m	none
$7m-1$	7	all m	1, 3, 5, 9
$7m$	7	all m	2
	8	all m	2, 3, 4, 5, 6
$7m+1$	7	all m	2, 3
	8	all m	2, 3, 5
$8m-1$	8	all m	2, 5
	9	all m	1, 2, 5, 7, 8, 11, 12, 14, 15, 20, 22, 23, 37, 38, 42
$8m$	8	all m	none
	9	all m	3
$8m+1$	8 or 9	all m	4
$9m-1$	9	m even	4
	10	m even	4, 18, 24, 32, 44, 54
$9m$	9	m odd	5, 7
	10	m odd	5, 7, 15, 45, 55, 63, 215
$9m+1$	9	m even	6
	10	m even	6, 10, 16, 26

Table 8.2

Table of known $T[k,1;v]$ with $v \leq 780$

v	k	Parity	Value of m
$9m-1$	9	m odd	1, 9, 17, 25, 33, 39, 41, 45, 49, 57, 65, 67, 69, 73, 75, 77, 79, 81, 83
	10	m odd	9, 25, 39, 41, 45, 57, 67, 73
$9m$	9	m even	8, 16, 24, 32, 38, 40, 46, 48, 56, 58, 60, 64, 68, 72, 74, 76, 80, 82, 84
	10	m even	16, 32, 38, 46, 48, 58, 60, 64, 80, 82
$9m+1$	9	m odd	7, 9, 11, 15, 17, 23, 29, 31, 33, 35, 39, 47, 55, 57, 63, 65, 67, 71, 73, 75, 77, 79, 81, 83, 85
	10	m odd	7, 9, 11, 17, 23, 29, 33, 39, 55, 71, 81

9 The Incomplete Transversal Design Construction

In this section, we deal with a construction given by Mullin et. al. [40, Theorem 1.1]. This construction requires an incomplete transversal design as a component, and establishing this makes the construction too lengthy to be given in main tables in Appendix A; the full construction is given in Appendix D, and only a brief reference is given in the main Appendix A tables. A more up-to-date list is given in [5], with ammendments in [21].

Theorem 9.1 *Suppose there exists a* $T[k,1;v-f+a]-T[k,1;a]$, *and* $v \in B(\{k,f^*\},1)$, *and that* $f \geq a$, *and* $f+(k-1)a \in B(k,1)$. *Then* $kv-(k-1)(f-a) \in B(k,1)$.

Proof: Take the blocks of the given incomplete transversal design, and use the $B[\{k,f^*\},1;v]$ to start filling in the groups with $f-a$ points at infinity; label the points so that the flat of size f covers the infinite points and the missing subgroup. Finally, use the $B[k,1;f-a+ka]$ to fill in the flats. ∎

For the incomplete transversal designs we have:

Lemma 9.2 *If* $0 \leq a \leq m$ *and* $0 \leq t \leq m$, *and* $\{s,s+1,s+2,t\} \subset T(k,1)$, *and* $m \in T(k+2,1)$, *then there exists a* $T[k,1;sm+t+a]-T[k,1;a]$.

Proof: See Wilson [46, Theorem 2.4]. ∎

Remark 9.3 This is the lemma that Mullin et al. use, but further lemmas are also useful.

Lemma 9.4 *If* $0 \leq t \leq m$ *and* $\{s,s+1\} \subset T(k,1)$, *and* $m \in T(k+1,1)$, *then there exists a* $T[k,1;sm+t]-T[k,1;t]$. *If, in addition,* $t \in T(k,1)$, *then there exists a* $T[k,1;sm+t]-T[k,1;m]$.

Proof: See Wilson [46, Theorem 2.3]. ∎

We even have occassion to use the more basic MacNeish construction.

Lemma 9.5 *If* $\{m,n\} \subset T(k,1)$, *then there exists a* $T[k,1;mn]-T[k,1;n]$.

Lemma 9.6 *If there exists* $GD[K,1,\{M,a^*\};v+a]$, *and* $K \subset T(k+1,1)$, *and* $M \subset T(k,1)$, *then there exists a* $T[k,1;v+a]-T[k,1;a]$.

Proof: See Hanani [30, Theorem 3.2]. ∎

Corollary 9.7 *The design* $T[7,1;179]-T[7,1;23]$ *exists.*

Proof: Consider $T[9,1;23]$; truncate two groups to size 9, and fill in all groups except for one of size 23, to give $179 \in GD(\{7,8,9,23\},1,\{1,23^*\})$, and apply the lemma. ∎

Lemma 9.8 *If* $m \in T(t,1)$, *and* $s \leq t$ *and* $\{s, s+1, m\} \subset T(k+1,1)$, *then there exists a* $T[k,1; sm+t-s] - T[k,1;t]$. *If further,* $t \in T(k+1,1)$, *then there exists a* $T[k,1; sm+t-s] - T[k,1;m]$.

Proof: This is really a corollary to Lemma 9.6. We may perform the last spike construction to show $sm+t-s \in GD(\{s,s+1,t^*\},1,\{1,m\})$. Lemma 9.6 establishes the further part of this lemma. To establish the main part we fill in the groups and delete the spike to give $sm+t-s \in GD(\{s,s+1,m\},1,\{1,t^*\})$, and then apply Lemma 9.6. ∎

For clarification, we state:

Lemma 9.9 *If* $m \in T(k,1)$, *and* $r \in \{0,1\}$, *then there exists a* $T[k,1;m] - T[k,1;r]$.

Proof: This follows by removing r blocks from the original transversal design. ∎

Having dealt with some methods for constructing incomplete transversal designs, we now turn to some methods for constructing designs with flats. These constructions are summarized in Table 9.1. The constructions IA through IF parallel those of Table 4.2, and we partially fill in a transversal design, possibly with the aid of some infinite points; for IG, we use $k \in T(k,1)$ as a link on the $D(k)$ design; for the Ir construction, we add points to every resolution set of a resolvable design; for IR, we use $k \in T(k,1)$ as a link on this. Note in constructions Ic and IF, that if $m \in C(k)$, rather than $m \in B(k)$, then $n \in C(k)$ yields a result in $C(k)$. Similarly, in constructions IG and IR, since $T[k,1;k]$ was used as the link in constructing the v point design, it is easy to see that $n \in C(k)$ yields a result in $C(k)$.

Table 9.1

Code	$m \in$	f	v	Conditions
IA	$A(k)$	$k(k-1)m+1$	$k(f-k)+k$	$f-k \in T(k,1)$
IB	$B(k)$	$k(k-1)m+k$	$k(f-1)+1$	$f-1 \in T(k,1)$
Ic	$B(k)$	$k(k-1)m+k$	kf	$f \in T(k,1)$
ID	$A(k)$	$k(k-1)m+1$	kf	$f \in T(k,1)$
If	$A(k)$	$k(k-1)m+1$	$k(f-1)+1$	$f-1 \in T(k,1)$
IF	$B(k)$	$k(k-1)m+k$	$k(f-k)+k$	$f-k \in T(k,1)$
IG	$D(k)$	k^2	$k(k-1)m+k$	$k \in T(k,1)$
Ir	$RC(k-1)$	$(k-1)m+1$	$(k-1)^2m+k$	
IR	$RC(k-1)$	$k(k-1)m+k$	$(k-1)f+k$	$k \in T(k,1)$

In the tables in Appendix A, the parameter m is as given in Table 9.1; the parameter n is the integer part of $(f+(k-1)a)/(k(k-1))$, and application of Theorem 9.1 will show $(v-f)/(k-1)+n$ is in $A(k)$ or $B(k)$ or $C(k)$.

10 The Special Constructions (Code ZZ)

In this section, some miscellaneous constructions for special values are given.

Lemma 10.1 $45 \in GD(7,1,3)$.

Proof: See Baker [10]. ∎

Theorem 10.2 *If* $t \in \{7, 121, 122, 133, 134, 145, 152, 231, 347, 517, 795\}$, *then* $t \in A(7)$.

Proof: For 7; note that $8 \in RC(6)$. Add a point at infinity to each resolution set of the $RB(6,1)$ to give $42 * 7 + 1 \in GD(7,1,\{1,49\})$, and use the $AG(2,7)$ design to fill in the flat. For $\{121,122\}$; consider a Denniston arc in $PG(2,16)$. Delete all the off-arc points except for those in the last secant; delete some of these only. Thus $120 + t \in B(\{8,9,(8+t)^*\},1)$ for $8 + t \leq 17$. Now delete an off-arc point on the big block, and all its blocks to give $120 + t - 1 \in GD(\{8,9\},1,\{8,(8+t-1)^*\})$ for $t \leq 9$. We can take $t = 2$ or 3, and apply Theorem 4.4. For 133; use $124 \in T(7,1)$ as a link on Lemma 10.1, and fill with the aid of 7 infinite points. For 134; a variant of Lemma 4.8 is obtained by adding points to 15 of the resolution sets in the Seiden design, and then deleting one of the original 120 points to give $134 \in GD(\{8,9,15^*\},1,\{7,8\})$. Finally, apply Theorem 4.4. For 145; note $17 \in RT(9,1)$. Deleting 8 points from a group gives $145 \in RGD(\{8,9\},1,\{17,9\})$. Now fill in groups and delete one resolution set to give $145 \in GD(\{8,9,17\},1,\{8,9\})$. Finally, apply Theorem 4.4. For 152; note $17 \in T(9,1)$. Now fill in groups and delete one point to give $152 \in GD(\{9,17\},1,\{8,16^*\})$. Finally, apply Theorem 4.4. For 231; note that $4 \in B(8)$, hence $231 \in GD(8,1,7)$; apply Theorem 4.4. For 347; use Lemma 4.11 with $x = 7$, $m = 7$, $n = 4$, $u = v = 0$. For 517; delete 29 points from a single block of $T[57,1;64]$, to give $517 \in GD(\{28^*,56,57\},1,\{63,64\})$. Use $4 \in A(7)$ with $8 \in E(7)$ as links, and $9 \in E(7)$ for the fill. For 795; use Lemma 4.11 with $x = 7$, $m = 16$, $n = 4$, $u = 1$, $v = 0$. ∎

Theorem 10.3 *If* $t = 24$, *then* $t \in B(7)$.

Proof: Note that $28 \in RC(6)$, so adding a point at infinity to each resolution set (of the $RB(6,1)$) gives $42 \cdot 24 + 7 \in GD(7,1,\{1,169^*\})$. Use Lemma 6.18 to fill the flat. ∎

Theorem 10.4 *If* $t \in \{31,75,157\}$, *then* $t \in C(7)$.

Proof: For $\{31, 157\}$; note that $\{5, 26\} \subset RC(6)$. If $m \in RC(6)$, then adding a point at infinity to each resolution set (of the $RGD(6, 1, 6)$) gives $6(6m + 1) \in GD(7, 1, \{6, 6m^*\})$. Use $AG(2, 7)$ for the link, and fill with 7 points at infinity. For 75; use $70 \in T(7, 1)$ as a link on Lemma 10.1, and fill with the aid of 7 infinite points. ∎

Lemma 10.5 *If q is a prime power, and $0 \le t \le q+1$, then $q^2 + q + 1 - t \in B(\{q - 1, q, q + 1\}, 1)$.*

Proof: Form an oval in $PG(2, q)$, and delete t points of the oval. ∎

Corollary 10.6 $\{83, 84, 87\} \subset B(\{8, 9, 10\}, 1)$.

Lemma 10.7 $\{92, 98\} \subset B(\{8, 9, 10\}, 1)$.

Proof: Delete one block of $T[10, 1; 11]$ to give a $GD[\{9, 10\}, 1, 10; 100]$. For 92, delete 8 points from a block of size 9; and for 98, delete a single point. In either case, the result is a $GD(\{8, 9, 10\}, 1, \{9, 10\})$, so the result follows. ∎

Lemma 10.8 *If q is a prime power, and $0 < s \le q$, and $0 \le t \le q + 1 - s$, then $sq + t \in GD(\{s, s + 1, s + 2\}, 1, \{q, 1\})$.*

Proof: Form an oval in $PG(2, q)$, pick a point on the oval and remove it and use its blocks to define groups in $GD[q + 1, 1, q; q(q + 1)]$. The remaining q oval points fall in q distinct groups of this design, and there is (at most) one group (arising from the tangent through the chosen point when q is odd), that has no oval points. Retain this group, and $s - 1$ others, and the oval points of t of the remaining groups. ∎

Corollary 10.9 *If $m \in \{132, 133, 140, 141, 236, 237, 238, 244, 245, 308\}$, we have $m \in GD(\{8, 9, 10\}, 1, \{1, 16, 17, 29, 37\})$.*

Proof: With $q \in \{16, 17\}$, take $t \in \{4, 5\}$. With $q = 29$, take $t \in \{4, 5, 6, 12, 13\}$. With $q = 37$, take $t = 12$. ∎

Lemma 10.10 *If $t \in \{124, 125\}$, then $t \in B(\{8, 9, 10\}, 1)$.*

Proof: Consider a Seiden's design embedded in the (dual) plane PG(2,16). This plane contains a block oval, with the blocks spanning 153 points, and each of those points falls on two blocks of the block oval. The remaining 120 points form an $RB(8, 1)$ design, and every line of the plane is either a block of the block oval, or a secant to this 120 point arc. We wish to adjoin

n points to this arc, giving a $B[\{8,9,10\},1;120+n]$ design. By picking two off-arc points on a secant, we clearly could add two points to the design. Suppose we have added x points, with $x \leq 4$. We wish to show a further point can be added. We already have $x(x-1)/2$ blocks of size 10, so there are at most $7x(x-1)/2$ further points on these blocks which are forbidden to us. Our x points fall on $2x$ blocks of the oval. These $2x$ blocks have $x(2x-1)$ intersections amongst themselves, and so these blocks cover $34x - x(2x-1)$ points, including our set of x points, and these are forbidden to us. The number of forbidden points is at most $34x - x(2x-1) + 7x(x-1)/2$, which evaluates to 69, 108, and 150 for $x = 2, 3$ and 4. Since these values are less than the number of points available (153), we may increment x from 2 to 3, 4 or 5. ∎

Theorem 10.11 *If* $t \in \{83, 84, 92, 124, 125, 132, 133, 140, 141, 172, 236, 237, 238, 244, 245, 308\}$, *then* $t \in A(8)$.

Proof: For 172; delete one block of $T[10,1;19]$, then delete 8 more points from one group, to give a $GD[\{8,9,10\},1,\{10^*,18\};172]$. Thus, noting Corollaries 10.6 and 10.9, and Lemmas 10.7 and 10.10, we have shown, for all the t of concern, that $t \in GD(\{8,9,10\},1,M)$, where $M \subset A(8)$, so our result follows from Theorem 4.4. ∎

Theorem 10.12 *If* $t = 147$, *then* $t \in C(8)$.

Proof: Delete one block of $T[10,1;16]$, then delete 3 more points from one group to give a $GD[\{8,9,10\},1,\{12^*,15\};147]$; apply Theorem 4.4. ∎

Theorem 10.13 *If* $t = 69$, *then* $t \in B(9)$.

Proof: A $GD[\{9,10,55\},1,\{1,64\};622]$ may be constructed by using the last spike construction on $T[55,1;64]$. Note $8 \cdot \{9,10,55\} \subset GD(9,1,8)$; use these as links to give $622 = 8 \cdot (9 \cdot 64 + 55 - 9) \in GD(9,1,\{8,(72 \cdot 7 + 8)^*\})$, and fill with a point at infinity. ∎

11 The Larger Designs

In this section, we give recursive proofs that the designs larger than the range covered by the tables in Appendix A all exist. Note that this is not established here, (nor was it known to be true), for the class $C(9)$. The results for this class were more sporadic, and so we will deal with $B(9)$ instead, judging that the importance of the results that we could obtain for $C(9)$ did not warrant the increase in length to the paper they would cause.

Theorem 11.1 *If $t > 1095$, then $t \in A(7)$.*

Proof: If $x \geq 15$, then $8x \in T(9,1)$ and $8x \in A(7)$, so for $r \leq 120$, we can construct $64x + r \in GD(\{8,9\}, 1, \{8x, r\})$, and apply Theorem 4.4. Noting that $64 * 15 + 120 = 1080$, and that the Appendix Table A.1 covers values up to 1095, we see we only need concern ourselves with the mod(64) residues that do not occur in A(7) before 120. These missing residues are 2, 3, 19, 43, 59, 61, and 62. Note that if $x \geq 15$, then $8x + 1 \in T(9,1)$ and $8x+1 \in A(7)$. This provides an alternative construction, using the residues 58, ?, 11, 35, 51, 53, and 54. For the residue 3 mod(64), we will use the value 131 with $8x \in T(9,1)$, and note that $1155 = 8 \cdot 143 + 11$ which fills the gap between the end of the table and the first valid construction for this residue. ∎

Theorem 11.2 *If $t > 591$, then $t \in C(7)$ and $t \in B(7)$.*

Proof: If $x \geq 7$, then $\{8x, 8x+1\} \subset T(9,1)$, and if $x \neq 3$, then $\{8x, 8x+1\} \subset C(7)$. Now, if $0 \leq r \leq 63$ and $r \neq 34$, then either r or $r + 8$ are in $C(7)$, and, if $t \geq 584$, then we can write t as $t = 64x + r + 8$ with $0 \leq r \leq 63$ and $x \geq 9$, so that $8x + 1 > 8x > r + 8$. Using the two transversal designs mentioned above, we can show that $t \in GD(\{8,9\}, 1, \{8x + 1, r\})$ and $t \in GD(\{8,9\}, 1, \{8x, r+8\})$. Now, if $t \neq 42 \bmod 64$ and $t \geq 584$, then for at least one of these forms the group sizes are good, so we can apply Theorem 4.4 to yield a design for t. If $t = 42 \bmod 64$, then $t = 64x + 42 = 64(x-1) + 106$ and for $x \geq 15$ we have $8(x-1) > 106$, and noting that $106 \in C(7)$, we only have to worry about $t = 64x + 42$ for $9 \leq x < 15$. These missing values of t are $8 \cdot 71 + 50$, $8 \cdot 79 + 50$, $8 \cdot 91 + 18$, $8 \cdot 101 + 2$, $8 \cdot 109 + 2$, and $8 \cdot 117 + 2$, so these values are also constructable. The result for $B(7)$ follows from Theorem 2.1. ∎

Theorem 11.3 *If $t > 319$, then $t \in A(8)$.*

Proof: If $m \neq 4$, then $8m + 1 \in T(9,1)$. If $m \geq 29$, and $m \neq 191$, then $m \in B(8)$. Let $S = \{0, 1, 10, 43, 36, 29, 30, 15\}$ and $n \in S$. Then $\{n, n + 120\} \subset A(8)$, and $120 + n \leq 163$. Now, if $t \neq 4 \bmod 8$ and $t \geq 320$, then we can write t as $t = (8m + 1) + n$ with $m \geq 36$, and so $m \geq n$. Construction p from Table 4.4 now yields a construction for most of the values of t. If $t = 4 \bmod 8$ and $m \geq 43$, then the above construction works, so we only have to worry about $324 \leq t \leq 380$ with $t = 4 \bmod 8$, and $m = 191$. For the first set of cases we can write $t = 8m + 36$, with $36 \leq m \leq 43$, and use construction m to give a successful construction. For $m = 191$, we replace m by 176 and n by $n + 120$ and apply construction p. ∎

Theorem 11.4 *If $t > 263$, then $t \in C(8)$ and $t \in B(8)$.*

Proof: If $x \geq 7$, then $8x \in T(9,1)$. If $x \geq 29$, then either x or $x+1$ is in $C(8)$. Let $S = \{0,1,2,35,4,21,6,7\}$ and $r \in S$. Then $\{r, r+8\} \subset C(8)$. Now, if $t \neq 3 \bmod 8$, and $t \geq 9 * 29 = 261$, then we can write t as either $t = 8(x+1)+r$ or $t = 8x+(r+8)$. If $x \geq 29$, then $x \geq r+8$, so for at least one of these forms we can apply construction n from Table 4.4 to yield a design for t. If $t = 3 \bmod 8$ and $x \geq 43$, then the above construction works, so we only have to worry about $267 \leq t \leq 379$ with $t = 3 \bmod 8$, but in these cases we can write $t = 8x+19$, with $31 \leq x \leq 45$, and construction n again yields a successful construction. The result for $B(8)$ follows from Theorem 2.1. ∎

For the $k = 9$ cases, we intend to apply the constructions of Table 4.4. Since the needed $T(10,1)$ are more readily available with odd group sizes, the first step is to look at the residues mod 18. We want a pair of consecutive residues that are in the respective sets $A(9)$, $B(9)$, $C(9)$.

Table 11.1

res mod 18	$A(9)$		$B(9)$		$C(9)$	
0	18	36	36	54	36	54
1	1	19	55	73	55	73
2	146	164	56	74	56	74
3	39	57	21	39	21	39
4	112	130	112	130	490	508
5	23	41	77	95	77	95
6	240	258	6	24	6	24
7	169	187	79	97	295	313
8	26	44	80	98	152	170
9	81	99	45	63	45	63
10	10	28	64	82	64	82
11	65	83	11	29	11	29
12	30	48	66	84	66	84
13	85	103	175	193	463	481
14	104	122	32	50	32	50
15	33	51	51	69	87	105
16	70	88	70	88	70	88
17	71	89	17	35	17	35

The residues for $C(9)$ are distressingly large in some cases, so we will no longer deal with $C(9)$, and will concentrate on $B(9)$ instead. We now state some preliminary lemmas that summarize the existence of transversal designs and features of the Appendix Tables A.7 and A.8.

Lemma 11.5 *If $x \geq 70$ is even, then $9x - 1 \in T(10, 1)$.*

Lemma 11.6 *If $x \geq 70$ is odd, and $x \neq 215$, then $9x \in T(10, 1)$.*

Lemma 11.7 *If $x \geq 70$ is even, then $9x + 1 \in T(10, 1)$.*

We now look for values m and $m + 2$ that are "not suitable", either because they are not in the appropriate set, or because the appropriate transversal design is unknown.

Lemma 11.8 *If $m \geq 70$ is odd, and $m \notin \{75, 113, 195, 201, 245\}$, then either $m \in A(9)$ and $9m \in T(10, 1)$, or $m + 2 \in A(9)$ and $9(m + 2) \in T(10, 1)$.*

Lemma 11.9 *If $m \geq 70$ is odd, and $m \neq 203$, then either $m \in B(9)$ and $9m \in T(10, 1)$, or $m + 2 \in B(9)$ and $9(m + 2) \in T(10, 1)$.*

Lemma 11.10 *If $m \geq 70$ is even, and $m \notin \{76, 84, 92, 94, 174, 192, 194, 204\}$, then either $m \in A(9)$ and $9m - 1 \in T(10, 1)$, or $m + 2 \in A(9)$ and $9(m + 2) - 1 \in T(10, 1)$.*

Lemma 11.11 *If $m \geq 70$ is even, and $m \notin \{92, 146\}$, then either $m \in B(9)$ and $9m + 1 \in T(10, 1)$, or $m + 2 \in B(9)$ and $9(m + 2) + 1 \in T(10, 1)$.*

In the remaining discussion of $k = 9$ constructions, it is sometimes convenient to divide numbers by 9. When this is done, we will write the result in *mixed radix* notation, with the integer part in the usual base 10 and the fractional part in base 9. The maximum value in the appendix tables is 728, which when divided by 9 yields 80.8.

The next two tables outline the approach we will adopt. We can deal with most cases using the constructions given in the following tables using one of the constructions of Table 4.4 and with m and $n + 18$ or with $m + 2$ and n. For the $A(9)$ residues of 0 and 7, use construction k for the smaller values, and then use construction n for $t/9 \geq 110.0$ and 98.7.

Theorem 11.12 *If $t > 728$, then $t \in A(9)$ and $t \in B(9)$.*

Proof: The main part of the proof of the theorem follows from the constructions outlined in Tables 11.2 and 11.3. There are some messy details of providing constructions for the exceptional cases of Lemmas 11.9, 11.10 and 11.11, and also of ensuring that the gaps between the end of the appendix tables and the first valid construction can be filled. Gaps between the end of Tables A.7 and A.8, (at 9×80.8), and the first valid construction appear for the $A(9)$ residues of 5, 8, and 15, and for the $B(9)$ residues of 6, 11, and 14. These messy details are given in Appendix E. ∎

Table 11.2

Constructions for $A(9)$.

Res mod 18	Constr.	n	$n+18$	$n+18$ $\div 9$	First good m	First valid $t \div 9$
0	k	1	19	2.1	70	72.0
	n	81	99	11.0	99	110.0
1	n	10	28	3.1	71	74.1
2	p	1	19	2.1	70	72.2
3	n	30	48	5.3	71	76.3
4	p	39	57	6.3	70	76.4
5	p	112	130	14.4	130	144.5
6	n	33	51	5.6	71	76.6
7	k	26	44	4.8	70	74.7
	n	70	88	9.7	89	98.7
8	n	71	89	9.8	89	98.8
9	n	18	36	4.0	71	75.0
10	n	1	19	2.1	71	73.1
11	p	10	28	3.1	70	73.2
12	n	39	57	6.3	71	77.3
13	p	30	48	5.3	70	75.4
14	n	23	41	4.5	71	75.5
15	k	70	88	9.7	88	97.6
16	p	33	51	5.6	70	75.7
17	n	26	44	4.8	71	75.8

Table 11.3

Constructions for $B(9)$.

Res mod 18	Constr.	n	$n+18$	$n+18$ $\div 9$	First good m	First valid $t \div 9$
0	n	45	63	7.0	71	78.0
1	q	36	54	6.0	70	76.1
2	n	11	29	3.2	71	74.2
3	q	56	74	8.2	74	82.3
4	q	21	39	4.3	70	74.4
5	n	32	50	5.5	71	76.5
6	q	77	95	10.5	96	106.6
7	q	6	24	2.6	70	72.7
8	n	17	35	3.8	71	74.8
9	n	36	54	6.0	71	77.0
10	n	55	73	8.1	73	81.1
11	n	56	74	8.2	75	83.2
12	n	21	39	4.3	71	75.3
13	l	32	50	5.5	70	75.4
14	n	77	95	10.5	95	105.5
15	n	6	24	2.6	71	73.6
16	l	17	35	3.8	70	73.7
17	l	36	54	6.0	70	75.8

12 Update

The objective of this section is to provide some designs so that this paper, in conjunction with [2], establishes the relevant results in [7]. We also either note, or establish further results, so that this section, in conjunction with [7] gives the current status of the existence problem for $B[k, 1; v]$ for $k \in \{7, 8, 9\}$. Also, we give a sufficient bound for the existence of $RC(7)$, as well as establishing small possible exception lists for $D(7)$ and $D(8)$.

12.1 Designs with Block Size 7

The values in Theorem 12.2 and 12.3 were constructed in [8], and are included here for completeness. The key design is the $T[7, 1; 42]$ which is given in [4]; it is actually given as a $T[7, 1; 42] - T[7, 1; 7]$, which fact we use later.

Lemma 12.1 $\{61, 66, 67, 83\} \subset B(\{7, 8, 9\}, 1, \{4, 7, 8, 9, 10\})$.

Proof: For 61: take two lines in $PG(2, 8)$, and remove their point of intersection, plus 7 other points from the first line and 4 other points from the second line; use lines through one of these last 4 points as groups. For 66, 67: delete 7 or 6 oval points from $PG(2, 8)$ and use lines through one of these deleted points as groups. For 83: delete 8 points each from two parallel lines in $T[9, 1; 11]$, with every group having at least one point deleted. ∎

Theorem 12.2 $\{61, 66, 67, 83\} \subset A(7)$

Proof: Essentially this is a corollary of Lemma 12.1, and follows by Theorem 4.4. ∎

Theorem 12.3 $44 \in C(7)$

Proof: Constuction IG 7.2. ∎

Remark 12.4 In [3], Abel showed that $\{29, 43\} \subset A(7)$ and $\{4, 6, 29\} \subset C(7)$ as a consequence of the designs noted in Remark 6.11.

12.2 Designs with Block Size 9

Our first aim is to show that $A(9)$ contains 17, 176, 215; that $B(9)$ contains 166, 191; that $C(9)$ contains 19, 173; and that $RC(9)$ contains 37. This will validate [7].

The result for 17 and 19 follows by using the appropriate fill theorem on Theorem 12.17; we can then get 173, 176, and 191 by constructions L 19.11, L 19.14, and BC 19.10. For 215, a variant of the code L construction removes all but one point from 8 groups of a $T[17, 1; 23]$, with the 8 singleton points being collinear, giving a $GD[\{9, 10, 17^*\}, 1, \{1, 23\}; 215]$; the result now follows from Theorem 4.4. For 166, start with a $TD[10, 1; 17]$ and fill in 6 groups; take a block, and delete its four points from the remaining 17-groups, then remove the 6-block to give a group; this shows $166 \in GD(\{9, 10, 17\}, 1, \{1, 6, 16\})$; the result now follows from Theorem 4.4. For the last result, we take Mathon's $RGD[9, 1, 3; 99]$ (see [34, Lemma 3.5]), and use a $RT[9, 1; 27]$ as the link design, then use a $RT[9, 1; 9]$ as the fill design.

Theorem 12.5 $8 \in A(9)$

Proof: Buratti gives a difference family in $GF(577)$ in [19]. We take the cube roots (1, 363, 213), multiply each by 1, 8, and 208 to give a base block; form a second by multiplying by 46 (= 5^6 and 5 is a primitive element); multiply this pair by 513^i for $0 \leq i \leq 3$ (note $513 = 5^{24}$) to get the required 8 base blocks. Since $8 = 5^{444}$ and $208 = 5^{483}$, we also see that these base blocks are disjoint. ∎

We have several designs as a consequence of Theorem 12.5; these are given in Table 12.1, with further details in Appendix D. The $TD[9, 1; 740] - TD[9, 1; 60]$ needed to show $93 \in A(9)$ follows from a construction of Abel [2, Lemma 5.7] and relies on the existence of $TD[9, 1; 56 + h_1 + h_2] - TD[9, 1; h_1] - TD[9, 1; h_2]$ for $h_1 \in \{0, 8\}$ and $h_2 \in \{0, 1, 8\}$.

Table 12.1

Constructions for $t \in A(9)$.

t	constr.	t	constr.	t	constr.	t	constr.
72	b 8.1	93	Ir 12.8	97	k 10.8	107	Ir 14.8
128	Ir 17.8	197	m 21.8	212	Ir 29.8	219	Ir 30.8

Theorem 12.6 $\{20, 48, 60\} \subset B(9)$.

Proof: For 48, use constuction a 6.1; the other two values are given in [20]. ∎

Remark 12.7 Hanani's purported $GD[9, 1, 9; 153]$, given in Hall [29], is flawed. I can see no way to correct it, although one can adapt it to give a $GD[9, 2, 9; 153]$.

This flawed design affects [23, 24], and no further valid designs are to be found there. However, (distinguishing methods from results), I did learn from [24] the power of non-constant weightings in Wilson's Fundamental Construction, and Lemma 12.11 is modelled after that paper.

12.3 Resolvable Designs

The improvements that follow from recent work are the result of having new small BIBDs to use recursively, from a result of Abel's given below, and the following use of frames. (For the definition of a frame, see [6, I.6.1]).

Theorem 12.8 $57 \in RC(6)$

Proof: Removing a sub-$T[7,1;7]$ from a $T[7,1;42]$ gives $GD[6,1,35;245]$; this GDD is also a frame (by [42, III.6.10]); now use this design as the link design on a $GD[7,1,7;49]$ to produce a $GD[6,1,35*7=245;245*7=1715]$ which is also a frame by [6, I.6.10]; fill the groups with the aid of an infinite point, using [6, I.6.12], to get $1715+1 \in RB(6,1)$, (i.e., $57 \in RC(6)$). Replacing the 49 point design by other designs in $C(7)$ shows that $t \in C(7)$ implies $49t+8 \in RC(6)$. ∎

One problem with using Theorem 7.1 with a difference family is exhibiting a set of suitable shifts so that the shifted base blocks of the family are disjoint; many difference families are formed by considering multiples of a "master" base block, and Abel exploited this structure rather nicely to exhibit a systematic set of shifts; the following is a slight expansion of Abel's construction [2, Theorem 2.2].

Theorem 12.9 *Let $q = k(k-1)t+1$ be a prime power. Given we have a difference family over $GF(q)$ for $B[k,1;q]$, with blocks $B_0, B_1, \ldots, B_{t-1}$, of the form $B_n = a^n B_0$. If $k \geq 2$, then there is a disjoint difference family.*

Proof: Shift the n-th base block by $a^n y$. Let $B_0 = \{b_1, b_2, \ldots, b_k\}$. Now if $a^m(b_i + y) = a^n(b_j + y)$ for $m < n$, then $(b_i + y) = a^{n-m}(b_j + y)$; (so if y is bad, it is really bad). The systematic shift, and this consequence, are the key elements of the proof; we now show a suitable y exists; note we need only look at disjointedness of B_0 with the other blocks. Since $\lambda = 1$, we may note that $a^n \neq 1$ (except for $n = 0$ of course). If we fix $i \neq j$ and $0 < n < t$, then $b_i + y = a^n(b_j + y)$ has a unique solution in y; this eliminates at most $k(k-1)(t-1)$ values of y; if we fix $i = j$ and $0 < n < t$, then $b_i + y = a^n(b_i + y)$ implies $b_i + y = 0$ (independent of $n > 0$); this eliminates at most k more values of y; there remain at least $q - k(k-1)(t-1) - k = k(k-2) + 1$ possible choices of y. ∎

Remark 12.10 For $k = 7$ and $q \notin \{43, 127, 211\}$, suitable difference families exist for $q \leq 6007$ (and in particular, for $q = 1597$), and for $k = 9$ and $q \notin \{289, 361\}$ suitable difference families exist for $q \leq 32767$ [25, 2, 19] (noting the disjointness exhibited in Theorem 12.5).

We now look at an application of Wilson's Fundamental Construction, of which our link Theorem 3.1 is a special case. (See [13, Theorem IX.3.2] or [41, III.2.7]).

Lemma 12.11 *If a* $T[10,1;m]$ *exists,* $0 \leq u \leq m$ *and* $0 \leq x+y \leq m$*, then* $t \in GD(\{8,9\},1,\{8m,8u,(7x+8y)\})$ *and* $7t+1 \in B(\{8,56m+1,56u+1,7(7x+8y)+1\})$*, where* $t = 64m+8u+7x+8y$*.*

Proof: We require $GD(\{8,9\},1,\{7,8\})$ designs, with group types of 8^8, 8^9, 8^{10}, $8^8 7^1$, $8^9 7^1$. The first two of these are TDs, the next is a punctured $AG(2,9)$, and the last two are the previous two with a point removed. We next weight a $T[10,1;m]$, giving all m points of 8 groups a weight of 8; we give u points a weight of 8 in the next group, and give x points a weight of 7 and y points a weight of 8 in the last group. Using $7*\{8,9\} \subset GD(8,1,7)$ as links on the t point design just constructed gives the second result. ∎

Theorem 12.12 *If* $t \geq 7011$*, then* $t \in RC(7)$*; equivalently, if* $v \equiv 7 \bmod 42$ *and* $v \geq 294469$*, then* $v \in RB(7,1)$*.*

Proof: First, note that if $n \in RC(7)$, then that design has a replication number of $7n+1$. The basic aim of the proof is to use Lemma 12.11, express t as $t = 64m+8u+7x+8y$ for suitable m, u, x and y; then from the second part of Lemma 12.11, we have $7t+1 \in B(7 \cdot RC(7)+1,1)$, so the result will follow by PBD closure.

Now, from the Appendix C constructions, and Remark 12.10, note that $R = \{0,49,266,91,28,77,70,63\} \subset RC(7)$; these values are all multiples of 7, and distinct modulo 8. Also, note $\{m,u\} \subset A(8)$ for $m \geq u \geq 68$ (using the updated results), and so $\{8m,8u\} \subset RC(7)$ by construction I. Given $t \geq 7011$, we can set $t \equiv r \bmod 8$, with $r \in R$, express r as $r = 7x+8y$, with $0 \leq x \leq 7$ and $y \in \{0,0,28,7,0,8,8,8\}$, pick m such that $64m \leq t-r-8*68$; we pick the maximum value for m that also satisfies $m \in T(10,1)$; finally, let $u = (t-r-64m)/8$. Note that any 10 consecutive integers contain at least one that is not divisible by any of 2, 3, 5 or 7, and hence is in $T(10,1)$; this means that we will only pick u in the range 68 through 147; clearly the construction works once we have $m = 149 > 147$, but the members of $T(10,1)$ in the range 97 through 149 are close enough together that the range for u of 68 through m suffices. Finally, $7010 = 64*97+8*67+266$ is the largest inapplicable case below our range (due to 67) in the worst (i.e., largest R member) modulo 8 class. ∎

12.4 The Set D(k)

The set $D(k)$ is the number of groups in a $GD(k,1,k(k-1))$ design, and is PBD closed. With Abel's construction of a $TD[7,1;42]$ design, it is easy to show $\{k, k+1, k+2\} \subset D(k)$ for $k \in \{7, 8\}$. Ling, Zhu, Colbourn and Mullin [36] show $B(\{7,8,9\},1)$ contains all numbers with at most 251 exceptions, and Ling and Colbourn [35] establish a similar result for $B(\{8,9,10\},1)$ with 380 possible exceptions, (a minor flaw was corrected in [9]); the detailed results are given in [12, III.3.2], and quoted below. Using these results together with some recursive constructions detailed in the Appendix (which can be exploited through PBD closure), it is possible to give reasonable list of possible exceptions to inclusion in $D(k)$ for $k \in \{7, 8\}$.

The list of possible exceptions for $B[\{7,8,9\},1;v]$ due to Ling et al. [36] is: 2 6, 10 48, 51 55, 59 62, 93 111, 116 118, 132, 138 168, 170 216, 219 223, 228 230, 243 279, 283 286, 298 300, 303 307, 311 335, 339 342.

Abel [3] has since removed 175, 176, 259 and 260 using Remark 6.11.

Lemma 12.13 *If* $v \in \{132, 141\ 145, 149\ 151, 153, 190, 196\ 198, 202\ 206, 210\text{-}214, 220, 222, 223, 243\ 266, 268, 271, 273, 274, 279, 284, 298\ 325, 327\ 333, 335, 340, 341\}$, *then* $v \in B(Q,1)$ *where* $Q = \{7, 8, 9, 15, 17, 28, 33, 36, 41\}$.

Proof: For 287 322, we can use Lemma 10.8 with $s = 7$ and $q = 41$. The $B(8,1)$ Denniston arc designs on 120 points in $PG(2,16)$ and on 232 points in $PG(2,32)$ contain ovals on 18 and 34 points; add an external line on 17 or 33 points, thereby incrementing all blocks by one, and delete some oval points, to get 132 and 243 265. The complement of the 120 point arc in $PG(2,16)$ is a $B[\{9,17\},1;153]$ where the 17-blocks are an oval in the dual plane; delete three non-collinear points whose pairs lie on 17-blocks to get a $B[\{8,9,15,17\},1;150]$. Most of our designs come from truncating up to two groups of a $T[9,1;t]$ design, and filling in the groups, possibly with the aid of an infinite point, for $t \in \{16,17,27,32,40,41\}$. For $274 = 6*32+3*27+1$, we use the Wojtas method of truncating three groups to get $274 \in B(\{7,8,9,28,33\},1)$; (see [13, Theorem X.3.8]). We can use a spike and a group to get $v \in B(\{7,8,9,g^*,h^*,41\},1)$ for $v = 7*41+g+(h-7)$ with $g \le 40$ and $h \le 41$; suitable triples are $(v,g,h) = (325,36,9), (333,36,17), (341,33,28)$. ∎

Remark 12.14 In the Appendix Table B.1, we establish that $Q \subset D(7)$, and so the constructions of Lemma 12.13 can be used to remove those values from Ling et al.'s exception list using PBD closure. Appendix Table B.1 also removes further values from Ling et al.'s exception list. A complete list of t for which $t \in D(7)$ has not been shown is given in Appendix B; note the values 2 through 6 ($= k-1$) are impossible.

The list of possible exceptions for $B[\{8,9,10\},1;v]$ due to Ling and Colbourn [35] is: 2 7, 11 56, 58 63, 66 71, 75 79, 101 109, 111 113, 115 119, 126, 127, 133 135, 155 160, 166, 167, 173 231, 239, 247 287, 290 295, 299 343, 346 351, 355 399, 403 407, 411 423, 426 431, 435 439, 443 448, 452 455, 472 497, 499 503, 507 511, 580 582.

Lemma 12.15 *If* $v \in \{127, 133\ 135, 155\ 160, 166, 167, 239, 247\ 287, 290\ 295, 299\ 343, 346\ 351, 355\ 399, 403\ 407, 411\ 423, 426\ 431, 435\ 439, 443\ 448, 452\ 455, 472\ 497, 499\ 503, 507\ 511, 580\ 582\}$, *then* $v \in B(Q, 1)$ *where* $Q = \{8, 9, 10, 15, 16, 17, 29, 30, 33, 36, 37, 41, 43, 44, 49, 50, 53, 54, 71\}$.

Proof: Applying Lemma 10.8 with $s = 8$, and taking $q \in K = \{16, 29, 37, 41, 43, 49, 53, 71\}$, we can show $v \in B(\{8,9,10\} \cup K, 1)$ for v values of 128 137, 232 254, 296 326, 328 380, 392 470, 568 632. We can also delete oval points from $PG(2,16)$ and $AG(2,17)$ to show $v \in B(\{15,16,17\},1)$ for $255 \leq v \leq 289$. The Denniston arc on 120 points in $PG(2,16)$ has intersections of size 0 or 8 with the lines of the plane; extending one 8-line to size 15 shows $127 \in B(\{8,9,15\},1)$. As in Lemma 10.8, we may construct a $T[10,1;16]$, where each group contains an oval point; remove up to five oval points to show $v \in B(\{8,9,10,15,16\},1)$ for $155 \leq v \leq 160$. For the remaining values, we can use the truncated transversal design construction, truncating two groups of a $TD[10,1;q]$ for $q \in \{17,37,43\} \cup \{29,32,53\}$, filling in the groups with the aid of an infinite point for the latter three values. ∎

Remark 12.16 In the Appendix Table B.2, we establish that $Q \subset D(8)$, and so the constructions of Lemma 12.15 can be used to remove those values from Ling and Colbourn's exception list using PBD closure. Appendix Table B.2 also removes further values from Ling and Colbourn's exception list. A complete list of t for which $t \in D(8)$ has not been shown is given in Appendix B; note the values 2 through 7 ($= k - 1$) are impossible.

Theorem 12.17 $\{17,19,72,82\} \subset D(9)$

Proof: [27] The constructions for the values 72 and 82 are given in the Appendix; they are included here primarily to highlight their absence from [7, I.2.22]. For $q = 17$ and 19: construct an oval in $PG(2,q)$, and note the set of $q(q-1)/2$ points that do not lie on any tangent intersect secants in $(q-1)/2$ points, and external lines in $(q+1)/2$ points; use these points (plus 17 oval points for $q = 17$) to form a design, with the lines through an unused oval point as groups, to get $\{153,171\} \subset GD(\{9,10\},1,9)$; use $1 \in E(9)$ as link designs for the result. ∎

13 Summary

The main results of this paper is the construction of $B(k, 1)$ designs with some exceptions. These results are summarized in the table below. Lists of the exceptions are given in Appendix F.

Table 13.1

Summary of BIBD Exceptions.

	Established in this paper			Current best known		
Class	Number not constructed	Largest t	Largest v	Number not constructed	Largest t	Largest v
$A(7)$	31	117	4915	18	62	2605
$B(7)$	9	82	3451	4	39	1645
$C(7)$	18	203	8533	5	39	1645
$A(8)$	67	231	12937	28	67	3753
$B(8)$	11	191	10704	10	28	1576
$C(8)$	13	195	10928	10	28	1576
$A(9)$	89	372	26785	47	209	15049
$B(9)$	68	454	32697	44	229	16497
$C(9)$	213	1726	124281	55	229	16497

14 Acknowledgements

The idea of using other than ten columns in the main tables in Appendix A arose from a presentation of Eratosthenes' Sieve by Stuart Lynch, a local high school teacher, on "Homework Helper", a program which was produced by my then local cable TV network, Shaw Cable. He used six columns in his presentation, and the patterns were quite striking. Upon seeing non-ten column tables, I realized that much of the arithmetic of producing, or checking, my tables involved the computation of $sm + n$. Using s columns in the table allows the arithmetic to be done more simply, in my case by xeroxing the table onto a transparency, and so the table and the transparency together formed a simple analogue device for computing the value of $sm + n$.

I also wish to thank Frank Bennett and Rudolf Mathon for bringing reference [10] to my attention.

I would like to thank Julian Abel for assisting with the update section; I should also state that, without his interest and urging, it is probable that I would never have gotten round to submitting this paper for publication.

Appendices

There are six appendices. The first covers the main constructions for the classes $A(k)$, $B(k)$, and $C(k)$. Subsequent appendices cover $D(k)$, $RC(k)$, the incomplete transversal design construction, and the completion of Theorem 11.12 for $k = 9$. The final appendix lists the unconstructed cases of Appendix A, and shows the current unconstructed cases.

Key to appendix tables

Code	Authority	Code	Authority	Code	Authority	Code	Authority
a	Table 4.1	h	Table 4.1	If	Table 9.1	P	Section 6
b	Table 4.1	H	Table 7.1	IF	Table 9.1	q	Table 4.4
Ba	Table 4.6	HA	Table 4.2	IG	Table 9.1	r	Lemma 4.14
BA	Table 4.6	HB	Table 4.2	Ir	Table 9.1	R	Lemma 4.7
BB	Table 4.6	Hc	Table 4.2	IR	Table 9.1	RT	Lemma 7.4
BC	Table 4.6	HC	Table 4.2	j	Table 4.3	RW	Theorem 7.1
BD	Lemma 4.10	HD	Table 4.2	J	Table 7.2	S	Section 5
c	Table 4.1	HF	Table 4.2	k	Table 4.4	T	Lemma 4.6
d	Table 4.1	i	Table 4.3	l	Table 4.4	X	Another table
e	Table 4.1	I	Table 7.2	L	Lemma 4.9	Zi	Table 4.5
E	Table 7.1	IA	Table 9.1	m	Table 4.4	Zj	Table 4.5
f	Table 4.1	IB	Table 9.1	n	Table 4.4	ZR	Lemma 4.8
F	Table 7.1	Ic	Table 9.1	o	Table 4.4	ZZ	Section 10
g	Table 4.1	ID	Table 9.1	p	Table 4.4	*n*	Unconstructed

A The Main Constructions

This appendix covers the constructions for the small and medium values for the classes $A(k)$, $B(k)$, and $C(k)$.

Table A.1

Table of Constructions for $A(7)$

		0	1	2	3	4	5	6	7
0	0	S	***1***	***2***	***3***	P	***5***	***6***	ZZ
8	1	P	P	P	P	***12**	***13**	***14**	P
16	2	P	***17**	P	***19**	P	P	***22**	P
24	3	P	P	P	***27**	HF 4	***29**	Ir 5.4	P
32	4	c 4.1	***33**	P	Ir 6.4	HB 5	***37**	P	***39**
40	5	P	P	***42**	***43**	P	Ir 8.4	P	***47**
48	6	P	HF 7	HB 7	P	***52**	Ic 1.4	P	P
56	7	HF 8	HB 8	P	***59**	P	***61**	***62**	P
64	8	P	P	***66**	***67**	P	Ir 12.8	HF 10	T 8.7
72	9	T 8.8	P	P	***75**	T 9.4	HF 11	HB 11	P
80	10	P	T 9.9	***82**	***83**	P	P	P	d 1.11
88	11	P	P	BD 11.9	P	T 11.4	P	IB 2.4	T 11.7
96	12	T 11.8	T 11.9	T 11.10	P	P	P	Ir 18.11	P
104	13	**104**	HF 15	HB 15	**107**	Ir 20.7	Ir 20.8	P	P
112	14	HF 16	HB 16	P	P	IG 17.4	**117**	P	P
120	15	P	ZZ	ZZ	P	P	Ir 24.4	HF 18	R 120.7
128	16	T 16.0	P	R 120.10	P	T 16.4	ZZ	ZZ	T 16.7
136	17	T 16.8	T 16.9	T 16.10	P	P	P	Ir 26.11	P

144	18	T 16.16	ZZ	P	HF 21	P	Ir 26.18	P	P
152	19	ZZ	P	P	HB 22	P	Ir 28.16	P	d 1.20
160	20	c 20.1	P	HB 23	d 2.11	P	e 2.11	IA 4.4	d 1.21
168	21	P	HB 24	IA 4.8	IA 4.9	If 4.4	ID 4.4	P	P
176	22	P	ID 4.8	P	ID 4.10	P	P	HF 26	P
184	23	T 23.0	ID 4.16	If 4.18	ID 4.18	T 23.4	ID 4.20	HB 27	T 23.7
192	24	T 23.8	T 23.9	T 23.10	P	P	HB 28	P	T 23.15
200	25	T 23.16	P	T 23.18	P	T 23.20	T 23.21	Ir 38.15	T 23.23
208	26	P	T 25.9	P	P	Ir 38.21	IR 5.31	Ir 38.23	T 25.15
216	27	T 25.16	HF 31	P	P	P	P	Ir 38.31	T 25.23
224	28	T 25.24	T 25.25	Ic 5.9	Ic 5.10	c 4.8	P	P	ZZ
232	29	R 232.0	P	P	P	R 232.4	Ic 5.20	HF 34	P
240	30	R 232.8	R 232.9	R 232.10	R 232.11	IB 5.28	HF 35	P	R 232.15
248	31	T 31.0	P	P	Ir 46.20	T 31.4	P	Ir 46.23	P
256	32	P	T 31.9	P	T 31.11	T 32.4	Ir 46.30	R 232.30	P
264	33	T 31.16	P	T 31.18	T 32.11	T 31.20	P	c 2.18	P
272	34	T 31.24	P	T 31.26	P	P	T 32.21	P	P
280	35	T 32.24	T 32.25	T 32.26	P	T 32.28	P	T 32.30	T 32.31
288	36	P	Ir 52.28	Ir 56.9	Ir 56.10	R 288.4	Ir 52.32	Ir 54.23	P
296	37	P	R 288.9	P	R 288.11	P	d 11.4	ID 7.7	R 288.15
304	38	P	ID 7.10	R 288.18	Ir 56.26	R 288.20	P	Ic 7.9	P
312	39	R 288.24	P	R 288.26	HF 45	R 288.28	Ic 7.16	R 288.30	P
320	40	P	Ic 7.20	R 288.34	P	R 288.36	g 2.25	P	d 1.41
328	41	T 41.0	Ic 7.28	HB 47	P	T 41.4	Ic 7.32	P	P
336	42	T 41.8	T 41.9	P	T 41.11	P	P	Ic 7.41	P
344	43	P	c 2.23	P	ZZ	T 41.20	T 41.21	HF 50	T 41.23
352	44	T 41.24	P	P	R 344.11	T 41.28	HF 51	T 41.30	T 41.31
360	45	T 41.32	BD 41.8	T 41.34	T 41.35	P	R 344.21	P	R 344.23
368	46	T 41.40	T 41.41	P	P	R 344.28	P	R 344.30	R 344.31
376	47	R 344.32	Ic 8.34	P	R 344.35	R 344.36	Ic 8.38	R 344.38	d 1.48
384	48	P	R 344.41	HB 55	Ic 8.44	R 344.44	P	R 344.46	d 5.11
392	49	T 49.0	HB 56	Ic 8.51	Ic 9.10	P	Ic 8.54	Ic 8.55	P
400	50	T 49.8	P	T 49.10	T 49.11	R 400.4	P	P	T 49.15
408	51	P	R 400.9	T 49.18	R 400.11	T 49.20	T 49.21	P	P
416	52	T 49.24	T 49.25	T 49.26	P	T 49.28	P	T 49.30	T 49.31
424	53	T 49.32	P	T 49.34	T 49.35	P	Ic 9.44	P	R 400.31
432	54	T 49.40	T 49.41	P	T 53.11	P	T 49.45	P	P
440	55	P	P	T 53.18	Ic 9.58	R 400.44	P	R 400.46	T 53.23
448	56	T 56.0	P	R 400.50	R 400.51	T 56.4	R 400.53	P	T 56.7
456	57	T 56.8	T 56.9	P	T 56.11	T 57.4	Ic 10.34	T 53.38	P
464	58	P	P	T 56.18	T 57.11	T 56.20	P	T 53.46	T 56.23
472	59	T 56.24	P	T 56.26	T 53.51	T 56.28	T 57.21	T 56.30	T 56.31
480	60	P	T 57.25	T 56.34	P	T 56.36	Ic 11.16	T 56.38	T 57.31
488	61	T 56.40	T 56.41	T 57.34	T 57.35	T 56.44	P	P	BD 56.9
496	62	T 56.48	T 56.49	T 56.50	P	P	T 56.53	T 56.54	T 56.55
504	63	P	P	T 57.50	T 57.51	ID 11.45	T 57.53	T 57.54	T 57.55
512	64	T 57.56	T 57.57	P	Ic 11.46	P	ZZ	P	P
520	65	P	T 64.9	T 64.10	T 64.11	T 65.4	P	P	T 64.15
528	66	T 64.16	T 65.9	T 64.18	P	T 64.20	T 64.21	Ic 11.65	T 64.23
536	67	T 64.24	T 64.25	T 64.26	P	T 64.28	T 65.21	T 64.30	P
544	68	T 64.32	T 65.25	T 64.34	T 64.35	P	P	T 64.38	P
552	69	T 64.40	P	P	P	T 64.44	T 64.45	T 64.46	d 1.70
560	70	T 64.48	P	T 64.50	T 64.51	P	T 64.53	P	T 64.55
568	71	P	P	T 64.58	T 65.51	T 64.60	T 65.53	P	P

576	72	T 64.64	T 65.57	T 65.58	T 71.11	T 65.60	R 512.69	R 512.70	T 65.63
584	73	T 65.64	P	T 71.18	T 72.11	P	T 71.21	P	T 71.23
592	74	T 71.24	P	T 71.26	T 73.11	P	T 72.21	P	T 71.31
600	75	T 71.32	P	T 71.34	T 71.35	T 71.36	T 73.21	P	T 72.31
608	76	P	T 71.41	P	T 72.35	T 71.44	P	T 71.46	T 73.31
616	77	P	T 71.49	T 71.50	T 71.51	P	P	T 71.54	T 71.55
624	78	P	T 71.57	P	T 72.51	T 71.60	T 73.45	T 72.54	T 71.63
632	79	T 71.64	T 71.65	T 72.58	T 73.51	P	T 73.53	T 71.70	P
640	80	P	T 72.65	T 73.58	T 79.11	T 72.68	T 72.69	T 72.70	T 72.71
648	81	T 72.72	T 73.65	T 79.18	T 80.11	T 73.68	P	T 73.70	T 73.71
656	82	T 73.72	T 73.73	T 79.26	T 81.11	T 79.28	P	T 79.30	P
664	83	T 79.32	T 80.25	T 79.34	T 79.35	P	P	T 79.38	P
672	84	T 79.40	T 79.41	P	P	P	T 79.45	P	T 81.31
680	85	T 79.48	T 79.49	T 79.50	P	T 80.44	P	P	T 79.55
688	86	T 79.56	T 79.57	T 79.58	P	T 79.60	T 81.45	T 80.54	P
696	87	T 79.64	T 79.65	T 81.50	T 81.51	P	P	T 79.70	P
704	88	P	P	T 79.74	HF 101	T 79.76	T 79.77	T 79.78	P
712	89	T 80.72	P	P	T 88.11	T 80.76	T 80.77	T 80.78	T 80.79
720	90	P	T 81.73	T 81.74	P	T 81.76	T 81.77	P	T 81.79
728	91	P	T 81.81	T 88.26	P	T 88.28	T 89.21	T 88.30	P
736	92	T 88.32	T 89.25	T 88.34	P	P	P	T 88.38	T 89.31
744	93	T 88.40	T 88.41	P	T 89.35	T 88.44	T 91.21	T 88.46	P
752	94	T 88.48	P	T 88.50	T 88.51	T 89.44	T 89.45	T 88.54	T 88.55
760	95	T 88.56	P	T 88.58	T 89.51	P	T 89.53	P	T 88.63
768	96	P	P	P	R 680.91	T 88.68	P	T 88.70	T 88.71
776	97	T 88.72	T 88.73	T 88.74	P	P	T 88.77	T 88.78	T 88.79
784	98	T 88.80	T 88.81	T 89.74	T 97.11	T 88.84	T 88.85	T 88.86	T 88.87
792	99	T 88.88	T 89.81	T 96.26	ZZ	T 89.84	T 89.85	T 89.86	T 89.87
800	100	T 89.88	T 89.89	T 91.74	T 99.11	T 91.76	T 91.77	T 91.78	T 91.79
808	101	T 91.80	T 91.81	T 97.34	T 100.11	T 91.84	T 91.85	T 91.86	T 91.87
816	102	T 91.88	T 91.89	T 91.90	T 91.91	T 97.44	T 100.21	T 96.54	T 96.55
824	103	T 96.56	T 96.57	T 96.58	T 97.51	T 96.60	T 101.21	T 97.54	T 96.63
832	104	T 96.64	T 96.65	T 97.58	T 103.11	T 96.68	T 99.45	T 96.70	T 96.71
840	105	T 96.72	T 96.73	T 96.74	T 99.51	T 96.76	T 96.77	T 96.78	T 96.79
848	106	T 96.80	T 96.81	T 97.74	T 100.51	T 96.84	T 96.85	T 96.86	T 96.87
856	107	T 96.88	T 96.89	T 96.90	T 96.91	T 96.92	T 96.93	T 97.86	T 96.95
864	108	T 96.96	T 97.89	T 97.90	T 97.91	T 97.92	T 97.93	T 99.78	T 97.95
872	109	T 97.96	T 97.97	T 100.74	T 103.51	T 99.84	T 99.85	T 99.86	T 99.87
880	110	T 99.88	T 99.89	T 99.90	T 99.91	T 99.92	T 99.93	T 100.86	T 99.95
888	111	T 99.96	T 99.97	T 99.98	T 99.99	T 100.92	T 100.93	T 101.86	T 100.95
896	112	T 100.96	T 100.97	T 100.98	T 100.99	T 101.92	T 101.93	T 103.78	T 101.95
904	113	T 101.96	T 101.97	T 101.98	T 101.99	T 103.84	T 103.85	T 103.86	T 103.87
912	114	T 103.88	T 103.89	T 103.90	T 103.91	T 103.92	T 103.93	T 105.78	T 103.95
920	115	T 103.96	T 103.97	T 103.98	T 103.99	T 115.4	T 105.85	T 105.86	T 115.7
928	116	T 105.88	T 105.89	T 105.90	T 105.91	T 105.92	T 105.93	T 112.38	T 105.95
936	117	T 105.96	T 105.97	T 105.98	T 105.99	T 115.20	T 115.21	T 112.46	T 115.23
944	118	T 112.48	T 112.49	T 112.50	T 112.51	T 115.28	T 112.53	T 112.54	T 112.55
952	119	T 112.56	T 112.57	T 112.58	T 113.51	T 112.60	T 109.85	T 115.38	T 112.63
960	120	T 112.64	T 112.65	T 113.58	T 109.91	T 112.68	T 115.45	T 112.70	T 112.71
968	121	T 112.72	T 112.73	T 112.74	T 115.51	T 112.76	T 112.77	T 112.78	T 112.79
976	122	T 112.80	T 112.81	T 113.74	T 121.11	T 112.84	T 112.85	T 112.86	T 112.87
984	123	T 112.88	T 112.89	T 112.90	T 112.91	T 112.92	T 112.93	T 113.86	T 112.95
992	124	T 112.96	T 112.97	T 112.98	T 112.99	T 120.36	T 115.77	T 115.78	T 115.79
1000	125	T 113.96	T 115.81	T 121.34	T 113.99	T 120.44	T 115.85	T 115.86	T 115.87

1008	126	T 115.88	T 115.89	T 120.50	T 115.91	T 115.92	T 115.93	T 120.54	T 115.95
1016	127	T 115.96	T 115.97	T 115.98	T 115.99	T 120.60	T 121.53	T 121.54	T 120.63
1024	128	T 120.64	T 120.65	T 121.58	T 127.11	T 120.68	T 120.69	T 120.70	T 120.71
1032	129	T 120.72	T 120.73	T 120.74	T 128.11	T 120.76	T 120.77	T 120.78	T 120.79
1040	130	T 120.80	T 120.81	T 127.26	T 129.11	T 120.84	T 120.85	T 120.86	T 120.87
1048	131	T 120.88	T 120.89	T 120.90	T 120.91	T 120.92	T 120.93	T 127.38	T 120.95
1056	132	T 120.96	T 120.97	T 120.98	T 120.99	T 121.92	T 121.93	T 127.46	T 121.95
1064	133	T 127.48	T 127.49	T 127.50	T 127.51	T 128.44	T 131.21	T 127.54	T 127.55
1072	134	T 127.56	T 127.57	T 127.58	T 128.51	T 127.60	T 129.45	T 128.54	T 127.63
1080	135	T 127.64	T 127.65	T 128.58	T 129.51	T 127.68	T 129.53	T 127.70	T 127.71
1088	136	T 127.72	T 127.73	T 127.74	T 135.11	T 127.76	T 127.77	T 127.78	T 127.79
		0	1	2	3	4	5	6	7

Table A.2

Table of Constructions for $B(7)$

t	constr.	t	constr.	t	constr.	t	constr.	t	constr.
22	Ir 4.1	24	ZZ	42	Ir 8.1	43	Ir 8.2	53	Ir 10.2
59	Ir 10.8	174	HA 25	179	Ir32.18	203	Ir38.12		

Table A.3

Table of Constructions for $C(7)$

		0	1	2	3	4	5	6	7
0	0	S	P	P	***3***	***4***	P	***6***	P
8	1	P	j 1	P	P	P	P	HF 2	c 2.1
16	2	P	P	P	***19**	P	P	***22**	P
24	3	***24**	P	P	P	P	***29**	P	ZZ
32	4	P	P	***34**	P	HC 5	P	P	***39**
40	5	P	j 5	***42**	***43**	***44**	P	P	P
48	6	P	HF 7	HC 7	P	P	***53**	Ic 1.5	P
56	7	P	P	P	***59**	P	P	P	P
64	8	T 8.0	T 8.1	P	h 5.2	P	T 8.5	P	T 8.7
72	9	P	P	T 9.2	ZZ	P	P	HC 11	T 9.7
80	10	T 9.8	P	***82**	P	HF 12	HC 12	g7.2	P
88	11	P	T 11.1	P	P	HC 13	T 11.5	IF 2.10	P
96	12	P	T 11.9	T 11.10	T 11.11	P	P	P	P
104	13	P	P	T 13.2	P	BD 13.9	T 13.5	P	T 13.7
112	14	P	T 13.9	T 13.10	P	T 13.12	T 13.13	P	HF 17
120	15	R 120.0	P	P	P	ZR 2.5	P	P	R 120.7
128	16	P	T 16.1	T 16.2	P	R 120.12	T 16.5	R 120.14	P
136	17	T 16.8	P	P	T 16.11	P	T 16.13	P	P
144	18	T 16.16	T 17.9	P	P	T 17.12	T 17.13	T 17.14	P
152	19	T 17.16	P	IR 4.7	c 22.1	P	ZZ	g 13.2	IR 4.12
160	20	P	P	HC 23	IR 4.16	IR 4.17	P	P	IR 4.20
168	21	P	j 21	P	h 2.13	P	P	**174**	P
176	22	HC 25	P	P	**179**	c 2.12	P	P	HC 26
184	23	T 23.0	T 23.1	P	P	P	T 23.5	HC 27	T 23.7
192	24	P	T 23.9	T 23.10	P	T 23.12	T 23.13	T 23.14	T 23.15
200	25	P	T 23.17	P	**203**	T 23.20	P	P	T 23.23
208	26	P	T 25.9	T 25.10	T 25.11	T 25.12	P	T 25.14	P
216	27	P	P	T 25.18	IF 5.9	P	P	h 5.7	T 25.23
224	28	T 27.8	T 25.25	T 27.10	T 27.11	P	T 27.13	P	T 27.15

232	29	T 27.16	P	T 27.18	BD 27.8	T 27.20	P	P	T 27.23
240	30	R 232.8	P	P	P	R 232.12	P	R 232.14	P
248	31	P	T 31.1	T 31.2	ZR 4.20	R 232.20	T 31.5	g 21.2	P
256	32	T 31.8	P	P	T 31.11	T 31.12	P	T 31.14	P
264	33	T 31.16	T 31.17	P	T 32.11	P	T 31.21	P	P
272	34	T 32.16	T 31.25	T 31.26	T 31.27	P	P	P	T 31.31
280	35	P	T 32.25	P	P	T 32.28	IG 41.5	T 32.30	P
288	36	T 32.32	R 288.1	P	P	P	P	ZR 5.7	R 288.7
296	37	P	P	P	R 288.11	P	T 37.5	R 288.14	T 37.7
304	38	T 37.8	P	T 37.10	T 37.11	P	T 37.13	P	P
312	39	P	P	T 37.18	HF 45	T 37.20	T 37.21	R 288.30	T 37.23
320	40	R 288.32	T 37.25	P	T 37.27	T 37.28	P	T 37.30	T 37.31
328	41	T 37.32	T 37.33	T 41.2	P	P	P	L 41.14	P
336	42	P	T 41.9	P	T 41.11	T 41.12	T 41.13	P	T 41.15
344	43	T 41.16	T 41.17	T 41.18	P	P	T 41.21	HF 50	T 41.23
352	44	P	T 41.25	T 41.26	P	P	P	T 41.30	T 41.31
360	45	P	T 41.33	R 344.18	P	T 41.36	T 41.37	P	P
368	46	P	T 41.41	P	R 344.27	R 344.28	P	R 344.30	P
376	47	T 47.0	T 47.1	P	R 344.35	P	P	P	T 47.7
384	48	T 47.8	P	T 47.10	T 47.11	T 47.12	T 47.13	P	P
392	49	T 47.16	T 47.17	T 47.18	P	P	P	P	T 47.23
400	50	P	T 47.25	T 47.26	T 47.27	T 47.28	T 49.13	P	T 47.31
408	51	T 47.32	T 47.33	T 49.18	P	P	T 47.37	T 47.38	T 49.23
416	52	T 47.40	P	T 49.26	T 49.27	P	T 47.45	T 47.46	P
424	53	T 49.32	P	P	T 49.35	T 49.36	T 49.37	T 49.38	R 400.31
432	54	P	T 49.41	HF 62	R 400.35	P	T 49.45	T 49.46	T 49.47
440	55	T 49.48	P	HC 63	P	ZR 7.45	P	P	P
448	56	P	T 56.1	T 56.2	P	P	P	ZR 7.55	P
456	57	T 56.8	T 56.9	P	T 56.11	T 56.12	P	T 56.14	T 56.15
464	58	T 56.16	P	P	P	P	T 56.21	T 57.14	T 56.23
472	59	P	T 56.25	T 56.26	P	P	T 57.21	T 56.30	T 56.31
480	60	T 56.32	P	T 57.26	T 56.35	T 56.36	T 56.37	P	T 57.31
488	61	T 56.40	T 56.41	T 61.2	T 57.35	P	T 56.45	T 56.46	P
496	62	T 56.48	T 56.49	T 56.50	T 56.51	P	T 57.45	T 57.46	P
504	63	T 56.56	T 57.49	P	T 57.51	P	T 61.21	P	P
512	64	T 57.56	T 57.57	T 64.2	T 61.27	T 61.28	T 64.5	T 61.30	T 61.31
520	65	T 61.32	T 61.33	T 65.2	T 61.35	T 61.36	T 61.37	T 61.38	T 65.7
528	66	T 61.40	T 61.41	T 65.10	T 65.11	T 65.12	T 61.45	T 61.46	T 61.47
536	67	T 61.48	T 61.49	T 61.50	T 61.51	T 61.52	T 65.21	T 61.54	T 61.55
544	68	T 64.32	T 64.33	T 65.26	T 64.35	T 64.36	T 64.37	T 64.38	T 65.31
552	69	T 64.40	T 64.41	T 67.18	T 65.35	T 65.36	T 64.45	T 64.46	T 64.47
560	70	T 64.48	T 64.49	T 64.50	T 64.51	T 64.52	T 65.45	T 65.46	T 64.55
568	71	T 64.56	T 64.57	T 64.58	T 65.51	T 64.60	T 64.61	T 64.62	T 64.63
576	72	T 64.64	T 65.57	T 65.58	T 71.11	T 65.60	T 65.61	T 65.62	T 65.63
584	73	T 65.64	T 65.65	T 67.50	T 67.51	T 67.52	T 71.21	T 72.14	T 67.55
		0	1	2	3	4	5	6	7

Table A.4

Table of Constructions for $A(8)$

		0	1	2	3	4	5	6	7
0	0	S	S	***2***	***3***	***4***	***5***	***6***	***7***
8	1	P	p 1.0	p 1.1	***11**	***12**	***13**	***14**	r 2.0
16	2	r 2.1	p 2.0	P	***19**	***20**	***21**	***22**	***23**

24	3	***24**	***25**	***26**	***27**	***28**	r 4.0	r 4.1	***31**
32	4	***32**	m 4.1	***34**	***35**	r 5.0	r 5.1	***38**	***39**
40	5	***40**	p 5.0	p 5.1	r 6.0	r 6.1	***45**	***46**	***47**
48	6	***48**	p 6.0	p 6.1	r 7.1	***52**	***53**	Zj 6	j 6
56	7	P	p 7.0	p 7.1	***59**	***60**	***61**	***62**	Zj 7
64	8	n 8.0	p 8.0	P	***67**	***68**	***69**	***70**	r 10.0
72	9	n 8.8	p 8.8	p 9.1	***75**	***76**	***77**	***78**	k 9.8
80	10	n 9.8	p 9.8	p 9.9	ZZ	ZZ	r 12.0	r 12.1	k 10.8
88	11	n 10.8	p 10.8	p 10.9	p 10.10	ZZ	r 12.8	r 12.9	r 12.10
96	12	***96**	p 12.0	p 12.1	r 14.0	r 14.1	**101**	**102**	**103**
104	13	m 12.8	p 12.8	p 12.9	p 12.10	r 14.9	r 14.10	**110**	P
112	14	Ir 17.9	P	p 14.1	r 15.9	r 15.10	i 13	Ir 17.15	**119**
120	15	n 15.0	p 14.8	p 14.9	p 14.10	ZZ	ZZ	**126**	r 18.0
128	16	n 15.8	p 15.8	p 15.9	p 15.10	ZZ	ZZ	r 19.0	n 15.15
136	17	p 15.15	p 16.8	p 16.9	p 16.10	ZZ	ZZ	k 16.15	n 16.15
144	18	p 16.15	p 16.16	p 17.9	P	r 21.0	r 19.15	k 17.15	n 17.15
152	19	p 17.15	p 17.16	p 17.17	p 18.10	r 21.8	r 21.9	k 18.15	n 18.15
160	20	p 18.15	p 18.16	p 18.17	p 18.18	r 21.16	r 21.17	r 21.18	m 19.15
168	21	p 19.15	p 19.16	p 19.17	p 19.18	ZZ	P	**174**	**175**
176	22	m 21.8	p 21.8	p 21.9	p 21.10	BD 19.10	**181**	**182**	m 21.15
184	23	p 21.15	p 21.16	p 21.17	p 21.18	**188**	Zj 21	j 21	**191**
192	24	**192**	p 24.0	p 24.1	**195**	**196**	**197**	**198**	**199**
200	25	m 24.8	p 24.8	p 24.9	p 24.10	r 29.0	r 29.1	Zi 23	m 24.15
208	26	p 24.15	p 24.16	p 24.17	p 24.18	r 29.8	r 29.9	r 29.10	Ir 33.16
216	27	Zj 24	j 24	d 31.1	r 29.15	r 29.16	r 29.17	r 29.18	**223**
224	28	Zi 25	r 32.0	r 30.15	r 30.16	r 30.17	r 30.18	**230**	**231**
232	29	n 29.0	p 29.0	p 29.1	r 32.10	ZZ	ZZ	ZZ	k 29.8
240	30	n 29.8	p 29.8	P	p 29.10	ZZ	ZZ	k 29.15	n 29.15
248	31	p 29.15	p 29.16	p 29.17	p 29.18	i 28	r 36.0	k 30.15	n 30.15
256	32	p 30.15	p 30.16	p 30.17	p 30.18	P	n 29.29	p 29.29	m 31.15
264	33	p 31.15	p 31.16	p 31.17	p 31.18	k 30.29	n 30.29	p 30.29	p 30.30
272	34	p 32.15	p 32.16	p 32.17	p 32.18	r 35.30	m 31.29	P	p 31.30
280	35	p 33.15	p 33.16	p 33.17	p 33.18	r 39.10	m 32.29	p 32.29	p 32.30
288	36	p 34.15	p 34.16	p 34.17	p 34.18	k 33.29	n 33.29	p 33.29	p 33.30
296	37	p 35.15	p 35.16	p 35.17	p 35.18	BD 31.10	m 34.29	p 34.29	p 34.30
304	38	p 36.15	p 36.16	p 36.17	p 36.18	ZZ	m 35.29	p 35.29	p 35.30
312	39	p 37.15	p 37.16	p 37.17	p 37.18	k 36.29	n 36.29	p 36.29	p 36.30

Table A.5

Table of Constructions for $B(8)$

t	constr.	t	constr.
53	q 6.4	195	q 24.2

Table A.6

Table of Constructions for $C(8)$

		0	1	2	3	4	5	6	7
0	0	S	S	S	***3***	S	P	P	S
8	1	S	P	P	***11**	P	***13**	P	P
16	2	S	P	E 1.2	F 2.1	***20**	P	***22**	***23**
24	3	P	***25**	***26**	***27**	***28**	P	P	r 4.2
32	4	P	H 4.1	E 2.2	P	P	F 4.1	r 5.2	P
40	5	r 5.4	H 5.1	P	r 6.0	P	P	F 5.1	r 6.4

48	6	r 6.5	H 6.1	P	P	r 7.2	***53**	P	F 6.1
56	7	P	P	n 7.2	r 8.2	n 7.4	n 7.5	H 2.4	E 1.7
64	8	S	P	P	ZR 1.4	E 2.4	n 8.5	F 4.2	P
72	9	P	H 9.1	P	P	n 9.4	P	P	n 9.7
80	10	P	RT 9.0	RT 9.1	RT 9.2	P	RT 9.4	RT 9.5	RT 9.6
88	11	RT 9.7	RT 9.8	RT 9.9	P	H 2.6	r 12.8	r 12.9	P
96	12	P	H 12.1	E 6.2	r 14.0	n 12.4	P	n 12.6	n 12.7
104	13	n 12.8	P	n 12.10	P	n 12.12	r 14.10	P	r 14.12
112	14	n 14.0	n 14.1	n 14.2	r 15.9	P	P	n 14.6	P
120	15	P	n 14.9	P	r 16.10	n 14.12	n 15.5	P	n 15.7
128	16	S	P	n 15.10	P	n 15.12	n 16.5	n 15.14	n 16.7
136	17	n 16.8	P	n 16.10	r 19.5	n 16.12	P	n 16.14	n 16.15
144	18	P	n 17.9	n 17.10	ZZ	n 17.12	P	P	n 17.15
152	19	P	n 19.1	n 19.2	r 21.7	P	n 19.5	n 19.6	n 19.7
160	20	n 19.8	P	P	r 21.15	n 19.12	P	n 19.14	n 19.15
168	21	n 19.16	n 19.17	P	P	n 21.4	n 21.5	n 21.6	n 21.7
176	22	P	n 21.9	n 21.10	P	n 21.12	r 24.12	n 21.14	n 21.15
184	23	n 21.16	P	P	n 21.19	r 24.19	n 21.21	r 24.21	**191**
192	24	n 24.0	n 24.1	P	**195**	n 24.4	n 24.5	n 24.6	n 24.7
200	25	P	P	n 24.10	g 4.7	n 24.12	r 29.1	n 24.14	P
208	26	n 24.16	n 24.17	P	n 24.19	P	n 24.21	r 29.10	P
216	27	n 24.24	r 30.6	r 29.14	P	r 29.16	r 30.10	P	r 30.12
224	28	Zi 25	P	r 30.15	r 30.16	r 29.24	r 30.18	r 32.5	P
232	29	n 29.0	n 29.1	P	r 32.10	P	n 29.5	n 29.6	P
240	30	n 29.8	n 29.9	n 29.10	r 32.18	n 29.12	n 30.5	n 29.14	n 29.15
248	31	n 29.16	P	n 29.18	n 29.19	P	n 29.21	n 30.14	n 30.15
256	32	S	n 30.17	n 30.18	n 30.19	P	P	n 32.6	n 32.7
		0	1	2	3	4	5	6	7

Table A.7

Table of Constructions for $A(9)$

		0	1	2	3	4	5	6	7	8
0	0	S	P	***2**	***3**	***4**	***5**	***6**	***7**	***8**
9	1	n 1.0	n 1.1	**11**	**12**	P	**14**	**15**	P	**17**
18	2	P	P	**20**	**21**	**22**	P	**24**	P	P
27	3	**27**	P	P	P	**31**	**32**	P	**34**	P
36	4	P	**37**	**38**	P	**40**	P	**42**	**43**	P
45	5	**45**	P	**47**	P	P	**50**	P	**52**	**53**
54	6	P	HB 6	**56**	r 7.0	P	**59**	**60**	**61**	**62**
63	7	**63**	p 7.0	p 7.1	r 8.1	**67**	**68**	P	P	P
72	8	**72**	p 8.0	P	**75**	**76**	**77**	**78**	P	Zj 8
81	9	n 9.0	n 9.1	p 9.1	**84**	P	**86**	**87**	P	d 11.1
90	10	P	P	**92**	**93**	**94**	P	**96**	**97**	P
99	11	P	p 11.0	p 11.1	*102**	P	P	P	*106**	*107**
108	12	m 11.9	p 11.9	P	P	Ir14.13	*113**	P	*115**	P
117	13	n 13.0	n 13.1	*119**	P	P	Ir 16.9	Ir16.10	P	P
126	14	n 13.9	n 13.10	*128**	Zi 13	n 13.13	P	*132**	*133**	P
135	15	P	HB 15	d 17.1	*138**	P	Ir18.13	*141**	*142**	Ir18.16
144	16	P	n 16.1	P	*147**	P	P	*150**	*151**	k 16.9
153	17	n 16.9	n 16.10	P	k 16.13	n 16.13	*158**	P	n 16.16	P
162	18	m 17.9	m 17.10	p 17.10	P	P	p 17.13	P	m 17.16	P
171	19	n 19.0	n 19.1	P	*174**	P	*176**	*177**	Ir 24.9	P
180	20	n 19.9	P	Ir24.13	P	P	Ir24.16	*186**	n 19.16	P

189	21	n 19.18	P	*191**	*192**	d 24.1	*194**	*195**	*196**	*197**
198	22	m 21.9	Zi 20	P	*201**	m 21.13	*203**	*204**	m 21.16	*206**
207	23	n 23.0	n 23.1	*209**	P	P	*212**	Ir 29.9	Ir29.10	*215**
216	24	n 23.9	P	p 24.1	*219**	n 23.13	P	*222**	P	P
225	25	n 23.18	P	p 24.10	P	Ba23.10	P	P	Ir29.28	d 29.1
234	26	n 25.9	P	P	Ir30.26	P	P	p 24.23	n 25.16	k 26.9
243	27	P	P	*245**	P	*247**	n 25.23	P	n 25.25	k 26.18
252	28	k 26.19	P	P	Ir34.16	P	d 32.1	k 26.25	k 26.26	k 28.9
261	29	P	n 29.1	p 29.1	P	P	Ir36.13	k 28.16	Ir34.29	k 28.18
270	30	P	n 29.10	p 29.10	Ir37.13	n 29.13	P	k 28.25	n 29.16	p 29.16
279	31	P	P	P	k 30.13	Ir36.30	n 29.23	P	P	n 29.26
288	32	p 29.26	P	n 29.29	p 29.29	k 30.23	Ir39.19	P	k 30.26	P
297	33	n 33.0	n 33.1	P	P	P	p 32.13	P	m 32.16	P
306	34	n 33.9	n 33.10	p 32.19	Zi 31	n 33.13	m 32.23	p 32.23	n 33.16	m 32.26
315	35	n 33.18	n 33.19	m 32.29	m 32.30	p 32.30	P	Ir41.33	n 33.25	n 33.26
324	36	n 35.9	n 33.28	P	n 33.30	n 35.13	P	P	P	k 36.9
333	37	P	P	P	k 36.13	d 42.1	P	k 36.16	P	n 35.26
342	38	k 36.19	P	n 35.29	P	k 36.23	IR 5.23	P	k 36.26	n 35.35
351	39	n 39.0	n 39.1	p 36.28	p 36.29	P	P	Ir49.13	p 36.33	P
360	40	n 39.9	n 39.10	p 39.10	k 39.13	P	p 39.13	k 39.16	n 39.16	P
369	41	P	P	P	*372**	k 39.23	n 39.23	p 39.23	P	n 39.26
378	42	n 41.9	n 39.28	P	n 39.30	n 41.13	k 39.33	n 39.33	n 41.16	P
387	43	n 39.36	n 41.19	p 42.10	P	p 39.39	n 41.23	P	n 41.25	p 42.16
396	44	P	n 41.28	P	P	Zj 40	k 41.33	n 41.33	P	n 41.35
405	45	n 41.36	p 44.9	p 42.28	n 41.39	p 42.30	n 41.41	Ir55.25	p 42.33	p 44.16
414	46	n 46.0	P	p 44.19	IR 6.30	P	P	P	P	p 44.25
423	47	n 46.9	P	p 44.28	p 44.29	n 46.13	P	k 46.16	n 46.16	P
432	48	n 46.18	P	P	P	P	n 46.23	p 44.41	n 46.25	n 46.26
441	49	n 48.9	n 46.28	n 46.29	n 46.30	P	k 46.33	n 46.33	P	n 46.35
450	50	P	n 48.19	p 50.1	n 46.39	n 49.13	P	k 48.25	n 48.25	n 46.44
459	51	n 49.18	n 46.46	n 48.29	n 48.30	Ir62.28	n 49.23	n 48.33	n 49.25	n 48.35
468	52	n 48.36	n 49.28	n 49.29	n 48.39	n 51.13	n 48.41	n 49.33	n 51.16	n 48.44
477	53	n 49.36	n 48.46	p 50.28	n 48.48	p 50.30	n 49.41	IF 6.51	n 51.25	n 49.44
486	54	p 50.35	n 49.46	n 51.29	n 49.48	n 49.49	Ir66.28	n 51.33	Ir66.30	n 51.35
495	55	n 51.36	p 54.9	p 50.46	n 51.39	p 50.48	n 51.41	Ir70.10	Ir66.39	n 51.44
504	56	Ir66.41	n 51.46	p 54.19	n 51.48	n 51.49	p 55.13	n 51.51	Ir66.48	p 55.16
513	57	n 57.0	n 57.1	p 54.28	p 54.29	p 54.30	p 56.13	p 55.23	p 54.33	p 55.25
522	58	n 57.9	n 57.10	p 55.28	p 55.29	n 57.13	Ir70.36	p 54.41	n 57.16	p 56.25
531	59	n 57.18	n 57.19	p 54.46	p 56.29	p 54.48	n 57.23	p 55.41	n 57.25	n 57.26
540	60	p 55.44	n 57.28	n 57.29	n 57.30	p 55.48	p 55.49	n 57.33	p 55.51	n 57.35
549	61	n 57.36	p 55.54	p 55.55	n 57.39	p 56.48	n 57.41	n 58.33	p 56.51	n 57.44
558	62	p 58.35	n 57.46	p 56.55	n 57.48	n 57.49	k 57.51	n 57.51	k 58.44	k 57.54
567	63	n 57.54	n 57.55	p 58.46	n 57.57	p 58.48	p 58.49	n 58.51	p 58.51	k 58.54
576	64	n 58.54	n 58.55	p 58.55	k 58.58	p 58.57	p 58.58	r 71.13	Ir78.36	k 64.9
585	65	n 64.9	n 64.10	p 64.10	k 64.13	n 64.13	p 64.13	k 64.16	n 64.16	p 64.16
594	66	n 64.18	n 64.19	p 64.19	r 71.28	k 64.23	n 64.23	p 64.23	n 64.25	n 64.26
603	67	p 64.26	n 64.28	n 64.29	n 64.30	p 64.30	k 64.33	n 64.33	p 64.33	n 64.35
612	68	n 64.36	p 64.36	k 64.39	n 64.39	p 64.39	n 64.41	k 66.25	k 64.44	n 64.44
621	69	n 69.0	n 64.46	p 64.46	n 64.48	n 64.49	k 64.51	n 64.51	k 66.35	k 64.54
630	70	n 64.54	n 64.55	p 64.55	n 64.57	n 64.58	p 64.58	p 66.41	n 69.16	k 70.9
639	71	n 69.18	n 64.64	p 64.64	k 66.49	p 66.48	n 69.23	k 70.16	n 69.25	n 69.26
648	72	n 71.9	n 69.28	n 69.29	n 69.30	n 71.13	p 71.13	n 69.33	n 71.16	n 69.35
657	73	n 69.36	n 71.19	p 70.28	n 69.39	p 70.30	n 69.41	p 71.23	n 71.25	n 69.44
666	74	n 73.9	n 69.46	n 71.29	n 69.48	n 69.49	L 73.23	n 69.51	n 73.16	n 71.35

675	75	n 69.54	n 69.55	p 70.46	n 69.57	n 69.58	n 71.41	p 71.41	n 73.25	n 71.44
684	76	m 75.9	n 69.64	n 69.65	n 69.66	n 71.49	p 70.58	n 69.69	m 75.16	n 73.35
693	77	n 71.54	n 71.55	p 70.64	n 71.57	n 71.58	n 73.41	k 70.70	m 75.25	n 73.44
702	78	m 77.9	n 71.64	n 71.65	n 71.66	n 73.49	p 72.58	n 71.69	n 71.70	n 71.71
711	79	n 73.54	n 73.55	p 72.64	n 73.57	n 73.58	m 75.41	L 73.69	m 77.25	m 75.44
720	80	n 79.9	n 73.64	n 73.65	n 73.66	n 79.13	p 74.58	n 73.69	n 73.70	n 73.71
		0	1	2	3	4	5	6	7	8

Table A.8

Table of Constructions for $B(9)$

		0	1	2	3	4	5	6	7	8
0	0	X	X	***2**	***3**	***4**	***5**	X	X	X
9	1	X	X	X	**12**	**13**	**14**	X	Ir 2.1	X
18	2	**18**	**19**	**20**	X	**22**	**23**	X	**25**	**26**
27	3	**27**	**28**	X	X	**31**	X	**33**	**34**	X
36	4	X	**37**	**38**	X	**40**	**41**	X	**43**	X
45	5	X	**46**	**47**	**48**	X	X	X	**52**	**53**
54	6	X	X	X	X	X	**59**	**60**	**61**	**62**
63	7	X	X	X	X	**67**	**68**	ZZ	X	X
72	8	X	X	X	X	**76**	X	X	X	X
81	9	X	X	q 9.1	X	**85**	X	X	X	X
90	10	X	X	**92**	**93**	**94**	X	X	l 10.8	l 10.9
99	11	X	X	X	*102**	*103**	a 13.1	X	X	X
108	12	X	X	X	q 11.11	Ir 15.6	Ir 15.7	*114**	*115**	X
117	13	X	o 13.1	X	X	X	X	o 13.6	o 13.7	o 13.8
126	14	X	o 13.10	o 13.11	X	X	X	*132**	Ir 18.6	Ir 18.7
135	15	X	X	X	Ir18.11	*139**	Ir 19.6	X	Ir18.15	l 16.0
144	16	X	X	*146**	*147**	*148**	X	X	o 16.7	X
153	17	X	X	o 16.11	X	*157**	l 16.15	X	X	X
162	18	X	X	X	X	*166**	*167**	X	X	X
171	19	X	o 19.1	*173**	*174**	Ir 24.6	X	o 19.6	o 19.7	X
180	20	X	o 19.10	o 19.11	*183**	a 23.1	X	X	o 19.16	o 19.17
189	21	X	X	*191**	*192**	X	X	X	X	X
198	22	X	X	X	X	*202**	*203**	X	*205**	X
207	23	X	o 23.1	*209**	X	BC21.10	X	o 23.6	o 23.7	X
216	24	X	X	o 23.11	X	Ir 30.9	Ir30.10	X	q 24.6	o 23.17
225	25	X	o 25.1	q 24.10	o 23.21	*229**	*230**	X	o 25.7	X
234	26	X	o 25.10	X	*237**	q 24.21	X	X	X	o 25.17
243	27	l 26.10	l 26.11	Ir 34.6	o 25.21	Ir 34.8	l 26.15	X	l 26.17	l 28.0
252	28	X	*253**	l 26.21	Ir34.16	Ir34.17	X	l 28.7	l 28.8	X
261	29	X	X	q 29.1	X	*265**	X	X	X	X
270	30	X	X	X	q 29.11	Ir36.21	l 28.24	X	q 29.15	X
279	31	X	X	X	X	q 29.21	X	X	q 29.24	X
288	32	X	X	X	X	q 30.21	X	X	X	X
297	33	X	X	X	X	q 30.30	X	X	o 33.7	X
306	34	o 33.9	o 33.10	o 33.11	X	X	*311**	X	X	o 33.17
315	35	X	X	X	X	q 32.30	X	X	X	X
324	36	X	X	X	X	a 41.1	X	X	X	X
333	37	X	X	X	X	X	l 36.15	X	X	X
342	38	q 36.17	X	X	X	q 36.21	X	X	X	X
351	39	X	X	X	X	q 36.30	l 39.6	X	X	X
360	40	X	X	X	q 39.11	Ir 51.6	l 39.15	X	q 39.15	X
369	41	X	X	X	X	q 39.21	l 39.24	X	o 41.7	o 41.8

378	42	X	X	X	X	q 39.30	X	o 41.15	q 42.6	X
387	43	X	q 42.9	X	X	q 39.39	X	X	X	X
396	44	X	X	X	X	X	X	X	q 42.24	X
405	45	X	X	X	X	X	X	X	q 44.15	l 46.0
414	46	X	o 46.1	X	X	q 42.39	X	X	X	X
423	47	X	o 46.10	o 46.11	X	q 44.30	l 46.15	X	l 46.17	X
432	48	X	X	l 46.21	X	X	l 46.24	X	X	X
441	49	X	X	X	o 46.30	l 46.32	o 46.32	X	X	X
450	50	X	X	X	X	*454**	X	X	X	X
459	51	X	X	X	X	X	X	X	X	X
468	52	X	X	X	X	q 50.21	X	X	q 50.24	X
477	53	X	Ir66.15	X	X	X	X	X	Ir66.21	X
486	54	X	X	X	X	X	X	X	X	X
495	55	X	X	q 54.10	X	Ir66.36	X	X	q 54.15	X
504	56	X	X	X	X	X	X	X	X	X
513	57	X	X	q 56.10	X	q 54.30	X	X	X	X
522	58	X	X	X	X	q 54.39	X	X	X	X
531	59	X	X	X	X	q 55.39	X	X	q 54.51	X
540	60	X	X	X	X	X	X	X	X	X
549	61	X	X	X	X	q 58.30	X	X	X	X
558	62	X	X	X	X	X	X	X	q 58.42	X
567	63	X	X	X	X	X	X	X	q 58.51	X
576	64	X	X	X	X	X	X	X	X	X
585	65	X	X	X	X	Ir78.42	X	X	X	X
594	66	X	X	X	X	X	X	X	X	X
603	67	X	X	X	X	q 64.30	X	X	l 64.35	X
612	68	X	X	X	X	X	X	X	X	X
621	69	X	X	X	X	X	X	X	X	X
630	70	X	X	X	X	X	X	X	q 70.6	X
639	71	X	X	X	X	X	X	X	X	X
648	72	X	X	X	X	q 70.21	o 69.32	X	X	X
657	720	X	X	X	X	X	X	X	X	X

Table A.9

Table of Constructions for $C(9)$

		0	1	2	3	4	5	6	7	8
0	0	S	P	***2**	***3**	***4**	***5**	P	S	S
9	1	P	n 1.1	P	**12**	**13**	**14**	P	**16**	P
18	2	**18**	**19**	**20**	P	**22**	**23**	P	**25**	**26**
27	3	**27**	**28**	P	P	**31**	P	**33**	**34**	P
36	4	P	**37**	**38**	P	**40**	**41**	P	**43**	P
45	5	P	**46**	**47**	**48**	g 6.1	P	P	**52**	**53**
54	6	P	HC 6	P	P	r 7.1	**59**	**60**	**61**	**62**
63	7	r 7.6	r 7.7	P	P	**67**	**68**	**69**	BB7.10	P
72	8	P	r 8.8	P	P	**76**	P	P	r 9.6	P
81	9	n 9.0	n 9.1	**83**	P	**85**	ZR 1.6	n 9.6	n 9.7	n 9.8
90	10	n 9.9	P	**92**	**93**	**94**	P	P	**97**	**98**
99	11	n 11.0	n 11.1	P	*102**	*103**	*104**	P	n 11.7	P
108	12	n 11.9	n 11.10	P	*111**	*112**	*113**	*114**	*115**	P
117	13	P	*118**	P	P	g 15.1	P	*123**	*124**	*125**
126	14	P	*127**	*128**	P	i 13	P	*132**	*133**	*134**
135	15	HF 15	HC 15	P	*138**	*139**	*140**	P	*142**	*143**
144	16	P	c16.1	*146**	*147**	*148**	P	P	*151**	P

153	17	n 17.0	n 17.1	*155**	P	*157**	*158**	n 17.6	n 17.7	P
162	18	P	n 17.10	n 17.11	P	*166**	*167**	n 17.15	g 21.1	P
171	19	P	*172**	*173**	*174**	*175**	P	*177**	*178**	P
180	20	i 18	*181**	*182**	*183**	*184**	P	P	*187**	*188**
189	21	HF 21	HC 21	*191**	*192**	g 24.1	P	n 21.6	n 21.7	n 21.8
198	22	n 21.9	Zi 20	P	P	*202**	*203**	n 21.15	*205**	n 21.17
207	23	P	*208**	*209**	P	*211**	P	*213**	*214**	P
216	24	HF 24	HC 24	*218**	P	*220**	*221**	P	*223**	*224**
225	25	P	*226**	*227**	*228**	*229**	*230**	P	*232**	g 29.1
234	26	P	*235**	P	*237**	*238**	P	BB24.10	g 30.1	*242**
243	27	*243**	*244**	*245**	*246**	*247**	*248**	P	*250**	*251**
252	28	P	*253**	*254**	*255**	*256**	g 32.1	*258**	*259**	P
261	29	P	n 29.1	*263**	P	*265**	P	P	n 29.7	P
270	30	P	n 29.10	n 29.11	*273**	*274**	*275**	P	*277**	n 29.17
279	31	Zi 28	i 28	j 28	n 29.21	*283**	P	P	*286**	P
288	32	n 32.0	n 32.1	n 29.29	IR 4.30	*292**	IR 4.32	n 32.6	n 32.7	n 32.8
297	33	P	n 32.10	P	P	*301**	P	n 32.15	*304**	P
306	34	*306**	*307**	*308**	P	i 31	*311**	n 32.24	g39.1	*314**
315	35	P	n 35.1	n 32.29	n 32.30	*319**	n 32.32	n 35.6	n 35.7	n 35.8
324	36	P	n 35.10	P	P	*328**	P	n 35.15	IR 5.7	P
333	37	IR 5.9	IR 5.10	IR 5.11	P	g 42.1	*338**	P	i 34	P
342	38	*342**	g6.7	P	n 35.30	*346**	P	IR 5.24	h 6.7	P
351	39	P	n 39.1	g 44.1	P	*355**	*356**	P	n 39.7	n 39.8
360	40	n 39.9	n 39.10	P	*363**	*364**	*365**	n 39.15	*367**	n 39.17
369	41	P	i 37	P	n 39.21	*373**	*374**	P	*376**	*377**
378	42	HF 42	HC 42	P	P	*382**	n 39.32	*384**	*385**	P
387	43	n 39.36	*388**	Zi 39	P	*391**	P	g 49.1	IR 6.7	IR 6.8
396	44	P	HC 44	h 6.8	g 7.7	Zj 40	P	P	*403**	IR 6.17
405	45	HF 45	HC 45	P	IR 6.21	Zi 41	i 41	IR 6.24	*412**	*413**
414	46	P	*415**	P	IR 6.30	*418**	IR 6.32	P	j 42	IR 6.35
423	47	IR 6.36	*424**	*425**	IR 6.39	*427**	*428**	P	*430**	P
432	48	P	g 54.1	*434**	P	IR 6.49	*437**	IF 6.6	IF 6.7	BB44.10
441	49	P	n 49.1	L 49.11	*444**	*445**	*446**	n 49.6	n 49.7	P
450	50	n 49.9	n 49.10	P	L 49.21	*454**	g 8.7	n 47.15	g 57.1	n 49.17
459	51	P	n 51.1	j 46	P	h 7.8	L 49.32	P	n 51.7	n 51.8
468	52	n 51.9	n 51.10	P	P	*472**	n 49.32	P	*475**	n 49.35
477	53	n 49.36	*478**	P	n 49.39	L 49.49	L 49.50	n 49.42	*484**	P
486	54	P	HC 54	n 51.29	n 51.30	n 49.49	P	IG 55.6	IG 55.7	n 51.35
495	55	n 51.36	HC 55	*497**	n 51.39	*499**	P	n 51.42	*502**	n 51.44
504	56	n 51.45	HC 56	P	P	n 51.49	P	n 51.51	g 9.7	c 7.8
513	57	n 57.0	n 57.1	*515**	P	*517**	ZR 7.6	P	n 57.7	n 57.8
522	58	P	n 57.10	n 57.11	P	*526**	P	n 57.15	n 58.7	P
531	59	n 58.9	n 58.10	n 58.11	P	*535**	P	P	*538**	n 58.17
540	60	i 54	j 54	P	n 57.30	ZR 7.32	n 57.32	n 58.24	ZR 7.35	n 57.35
549	61	n 57.36	BB55.10	P	n 57.39	*553**	n 58.32	P	ZR 7.44	P
558	62	n 57.45	Zi 56	P	P	n 57.49	n 57.50	P	*565**	n 58.44
567	63	n 57.54	n 57.55	n 57.56	P	n 58.49	n 58.50	n 58.51	*574**	r 71.6
576	64	n 58.54	n 58.55	n 58.56	n 58.57	n 58.58	P	P	n 64.7	P
585	65	n 64.9	n 64.10	n 64.11	L 64.21	*589**	P	P	r 73.7	n 64.17
594	66	r 73.9	r 73.10	r 73.11	n 64.21	r 71.29	P	P	r 71.32	P
603	67	L 64.36	r 71.35	n 64.29	n 64.30	*607**	n 64.32	r 73.24	*610**	P
612	68	n 64.36	r 71.44	P	n 64.39	ZR 8.32	P	n 64.42	r 71.50	n 64.44
621	69	P	L 64.55	r 71.54	P	n 64.49	P	n 64.51	ZR 8.44	r 73.44
630	70	P	n 64.55	n 64.56	n 64.57	n 64.58	P	r 73.51	*637**	ZR 8.54

639	71	P	n 64.64	r 73.56	r 73.57	r 73.58	P	n 71.6	n 71.7	n 71.8
648	72	n 71.9	n 71.10	n 71.11	P	*652**	*653**	P	r 73.70	n 71.17
657	73	n 73.0	n 73.1	P	P	j 66	P	n 71.24	n 73.7	n 73.8
666	74	P	n 73.10	n 71.29	n 71.30	r 81.21	n 71.32	n 73.15	r 81.24	P
675	75	n 71.36	n 75.1	P	n 71.39	r 81.30	P	P	n 75.7	n 71.44
684	76	n 71.45	n 75.10	n 73.29	n 73.30	n 71.49	n 71.50	P	r 81.42	n 73.35
693	77	n 71.54	n 71.55	n 71.56	P	n 71.58	r 81.49	n 73.42	n 77.7	n 73.44
702	78	n 71.63	n 71.64	n 71.65	P	n 73.49	P	n 73.51	n 71.70	n 71.71
711	79	P	n 73.55	n 73.56	n 73.57	n 73.58	IR10.77	P	n 79.7	n 75.44
720	80	n 73.63	n 73.64	n 73.65	n 73.66	n 75.49	P	n 75.51	n 73.70	n 73.71
		0	1	2	3	4	5	6	7	8

B Constructions for D(k)

In this appendix, we give the details of some constructions for the class $D(k)$.

In Table B.1 we establish $Q \subset D(7)$ for the Q of Lemma 12.13. We also construct $D(7)$ designs for other values in Ling et al.'s exception list.

Table B.1

Constructions for $t \in D(7)$.

t	constr.	t	constr.	t	constr.	t	constr.	t	constr.
0	S	1	S	7	[4]	8	c 1.1	9	j 1
15	c 2.1	17	j 2	28	b 4.1	33	j 4	36	e 5.1
41	j 5	96	i 12	97	j 12	99	c 14.1	104	i 13
105	b 15.1	106	c 15.1	140	b 20.1	147	b 21.1	148	e 21.1
155	c 22.1	161	b 23.1	162	c 23.1	168	b 24.1	175	b 25.1
176	e 25.1	182	b 26.1	183	c 26.1	184	i 23	193	j 24
267	c 38.1								

All values of t safisfy $t \in D(7)$ with the possible exception of the following values for t: 2 6, 10 14, 16, 18 27, 29 32, 34, 35, 37 40, 42 48, 51 55, 59 62, 93 95, 98, 100 103, 107 111, 116 118, 138, 139, 146, 152, 154, 156 160, 163 167, 170-174, 177 181, 185 189, 191, 192, 194, 195, 199 201, 207 209, 215, 216, 219, 221, 228 230, 269, 270, 272, 275 278, 283, 285, 286, 326, 334, 339, 342.

In Table B.2 we establish $Q \subset D(8)$ for the Q of Lemma 12.15. We also construct $D(8)$ designs for other values in Ling and Colbourn's exception list.

Table B.2

Constructions for $t \in D(8)$.

t	constr.	t	constr.	t	constr.	t	constr.	t	constr.
0	S	1	S	8	Zi 1	9	i 1	10	j 1
15	g 2.1	16	r 2.1	17	c 2.1	29	g 4.1	30	r 4.1
33	c 4.1	36	g 5.1	37	r 5.1	41	c 5.1	43	g 6.1
44	r 6.1	49	c 6.1	50	g 7.1	51	r 7.1	53	Zi 6
54	Zj 6	55	j 6	58	r 8.1	63	Zj 7	71	g 10.1

79	r 10.8	104	b 13.1	106	g 15.1	107	r 14.8	108	r 14.9
109	r 14.10	113	c 14.1	115	r 15.9	116	Zi 13	117	i 13
126	i 14	177	r 24.8	178	r 24.9	179	r 24.10	180	Zj 20
184	b 23.1	185	r 24.16	186	r 24.17	189	Zj 21	190	j 21
193	c 24.1	204	g 29.1	205	r 29.1	206	Zi 23	207	i 23
211	g 30.1	212	r 29.8	213	r 29.9	214	r 29.10	216	Zj 24
217	j 24	218	g 31.1	219	r 30.8	220	r 30.9	221	r 30.10
224	Zi 25	225	i 25	226	r 32.1	227	r 30.16	228	r 30.17

All values of t safisfy $t \in D(8)$ with the possible exception of the following values for t: 2 7, 11 14, 18 28, 31, 32, 34, 35, 38 40, 42, 45 48, 52, 56, 59 62, 66 70, 75 78, 101 103, 105, 111, 112, 118, 119, 173 176, 181 183, 187, 188, 191, 192, 194 203, 208 210, 215, 222, 223, 229 231.

Table B.3

Constructions of $t \in D(9)$ for $t < 100$.

t	constr.	t	constr.	t	constr.	t	constr.	t	constr.
0	S	1	S	9	n 1.0	10	i 1	17	§12
19	§12	49	g 6.1	54	b 6.1	55	c 6.1	57	r 7.0
58	r 7.1	64	c 7.1	65	r 8.0	66	r 8.1	72	b 8.1
73	r 9.0	74	r 9.1	80	Zj 8	81	r 10.0	82	r 10.1
89	g 11.1	90	Zj 9	91	r 10.10				

C Resolvable Constructions

In this appendix, we give the detailed constructions for the class $RC(k)$ for smaller (< 100) values.

Table C.1

Constructions for $t \in RC(k)$.

t	constr.	t	constr.	t	constr.	t	constr.	t	constr.
$k = 6$									
0	S	4	S	5	S	6	RW 1A	8	P
10	P	12	P	14	[2]	16	P	18	P
20	P	24	RW 4A	26	P	28	P	30	P
32	P	36	RW 6A	38	P	40	P	42	RW 7A
46	P	48	P	52	P	54	RW 9A	56	P
57	See §12	58	P	60	P	62	P	66	P
68	P	70	P	72	P	76	P	84	RW 14A
86	[2]	88	P	90	P	92	[2]	96	P
98	[2]								
$k = 7$									
0	S	1	S	8	I 1	9	J 1	17	J 2
28	RW 4A	33	J 4	41	J 5	49	J 6	56	RW 8A
57	J 7	63	RW 9A	64	I 8	65	J 8	70	RW 10A
72	I 9	73	J 9	77	RW 11A	80	I 10	81	J 10
88	I 11	89	I 12	91	RW 13A	96	I 12	97	J 12

Table C.1 (continued)

Constructions for $t \in RC(k)$.

t	constr.	t	constr.	t	constr.	t	constr.	t	constr.
$k=9$									
0	S	1	S	7	S	8	S	9	RW 1A
10	H 1.1	37	See §12	54	RW 6A	64	H 7.1	71	F 1.7
72	RW 8A	73	H 8.1	81	F 1.8	82	H 9.1	90	E 1.9
91	H 10.1								

This paper originally was written without knowing of constructions for the whole of Table C.2. Constructions were later found for the parenthesized entries, and the entire set of constructions is presented in [28]. Constructions of the smaller values of $RC(8)$ (excluding parenthesized cases) have been incorporated into the constructions of $C(8)$ in Appendix Table A.6.

Table C.2

Values of t for which $t \in RC(8)$ is unknown.

3	11	13	20	22	23	25	26	27	28
31	38	(40)	43	47	(48)	52	53	58	59
(60)	61	67	69	76	79	93	(94)	(99)	102
103	(106)	111	112	115	(118)	123	124	125	133
134	139	140	(142)	143	(147)	(166)	174	182	(184)
191	192	195	197	(198)	199	203	(205)	208	209
211	213	(214)	218	220	223	224	227	229	230
232	237	243	(244)	(245)	(246)	247	248	250	253
254	(259)	(364)	(382)	391	(412)	(427)	437	(454)	(472)
(488)	(489)	(826)	(832)	(1039)					

D Incomplete Transversal Design Constructions

In this appendix, we give the detailed constructions promised in Section 9. We use the constructions of Table 9.1 to provide a $B[\{k, f^*\}, 1; v]$. The value of m is used in this construction. The value of n is given by $n = \lfloor (f - a + ka)/(k(k-1)) \rfloor$. The needed incomplete transversal design can be obtained from the form in the last column, and application of one of the lemmas in Section 9, in particular Lemmas 9.2, 9.4, and 9.8, which we use without further comment. The final 6 entries are used in Section 12.

Table D.1

	k	m	f	v	a	$(v-f)$ $\div (k-1)$	n	Result	$v-f+a$
IA	7	4	169	1141	0	162	4A	166A	$972 + a = 7 \cdot 137 + 13 + a$
IA	7				28	162	8A	170A	
IA	7				35	162	9A	171A	
IB	7	2	91	631	13	90	4A	94A	$540 + a = 7 \cdot 73 + 29 + a$

Table D.1 (continued)

	k	m	f	v	a	$(v-f)$ $\div (k-1)$	n	Result	$v-f+a$
IB	7	5	217	1513	160	216	28A	244A	$1296+a=7\cdot185+1+a$
Ic	7	1	49	343	20	49	4A	53A	$294+a=7\cdot43+a-7$
Ic	7				28	49	5C	54C	
Ic	7	5	217	1519	27	217	9A	226A	$1302+a=7\cdot186+a$
Ic	7				34	217	10A	227A	
Ic	7				104	217	20A	237A	
Ic	7	7	301	2107	13	301	9A	310A	$1806+a=7\cdot258+a$
Ic	7				62	301	16A	317A	
Ic	7				90	301	20A	321A	
Ic	7				146	301	28A	329A	
Ic	7				174	301	32A	333A	
Ic	7				237	301	41A	342A	
Ic	7	8	343	2401	181	343	34A	377A	$2058+a=7\cdot294+a$
Ic	7				209	343	38A	381A	
Ic	7				251	343	44A	387A	
Ic	7				300	343	51A	394A	$2058+300=7\cdot300+258$
Ic	7				321	343	54A	397A	$2058+321=7\cdot321+132$
Ic	7				328	343	55A	398A	$2058+328=7\cdot328+90$
Ic	7	9	385	2695	6	385	10A	395A	$2310+a=7\cdot330+a$
Ic	7				244	385	44A	429A	
Ic	7				342	385	58A	443A	$2310+342=7\cdot342+258$
Ic	7	10	427	2989	167	427	34A	461A	$2562+a=7\cdot366+a$
Ic	7	11	469	3283	34	469	16A	485A	$2814+a=7\cdot402+a$
Ic	7				244	469	46A	515A	
Ic	7				377	469	65A	534A	
ID	7	4	169	1183	0	169	4A	173A	$1014+a=7\cdot143+13+a$
ID	7				28	169	8A	177A	
ID	7				42	169	10A	179A	
ID	7				84	169	16A	185A	
ID	7				98	169	18A	187A	
ID	7				112	169	20A	189A	
ID	7	7	295	2065	0	295	7A	302A	$1770+a=7\cdot251+13+a$
ID	7				21	295	10A	305A	
ID	7	11	463	3241	238	463	45A	508A	$2778+a=7\cdot395+13+a$
If	7	4	169	1177	0	168	4A	172A	$1008+a=7\cdot144+a$
If	7				98	168	18A	186A	
IF	7	2	91	595	56	84	10C	94C	$504+a=7\cdot72+a$
IF	7	5	217	1477	28	210	9C	219C	$1260+a=7\cdot180+a$
IG	7	17	49	721	20	112	4A	116A	$672+a=7\cdot96+a$
IG	7	41	49	1729	28	280	5C	285C	$1680+a=7\cdot240+a$
Ir	7	4	25	151	4	21	1B	22B	$126+a=7\cdot17+7+a$
Ir	7	5	31	187	23	26	4A	30A	179; see Corollary 9.7
Ir	7	6	37	223	22	31	4A	35A	$186+a=7\cdot25+11+a$
Ir	7	8	49	295	0	41	1B	42B	$246+a=7\cdot31+29+a$
Ir	7				7	41	2B	43B	
Ir	7				20	41	4A	45A	
Ir	7	10	61	367	5	51	2B	53B	$306+a=7\cdot41+19+a$
Ir	7				47	51	8B	59B	$306+47=7\cdot47+23+1$
Ir	7	12	73	439	44	61	8A	69A	$366+a=7\cdot49+23+a$
Ir	7	18	109	655	59	91	11A	102A	$546+a=7\cdot78+a$

Table D.1 (continued)

	k	m	f	v	a	$(v-f) \div (k-1)$	n	Result	$v-f+a$
Ir	7	20	121	727	29	101	7A	108A	$606+a=7\cdot 83+25+a$
Ir	7				36	101	8A	109A	
Ir	7	24	145	871	4	121	4A	125A	$726+a=7\cdot 101+19+a$
Ir	7	26	157	943	51	131	11A	142A	$786+a=7\cdot 109+23+a$
Ir	7				100	131	18A	149A	
Ir	7	28	169	1015	84	141	16A	157A	$846+a=7\cdot 117+27+a$
Ir	7	32	193	1159	95	161	18B	179B	$966+a=7\cdot 138+a$
Ir	7	38	229	1375	47	191	12B	203B	$1146+a=7\cdot 161+19+a$
Ir	7				67	191	15A	206A	
Ir	7				109	191	21A	212A	
Ir	7				123	191	23A	214A	
Ir	7				179	191	31A	222A	$1146+179=7\cdot 179+72$
Ir	7	46	277	1663	94	231	20A	251A	$1386+a=7\cdot 198+a$
Ir	7				115	231	23A	254A	
Ir	7				164	231	30A	261A	
Ir	7	52	313	1879	144	261	28A	289A	$1566+a=7\cdot 221+19+a$
Ir	7				172	261	32A	293A	
Ir	7	54	325	1951	107	271	23A	294A	$1626+a=7\cdot 229+23+a$
Ir	7	56	337	2023	7	281	9A	290A	$1686+a=7\cdot 239+13+a$
Ir	7				14	281	10A	291A	
Ir	7				126	281	26A	307A	
IR	7	4	175	1057	21	147	7C	154C	$882+a=7\cdot 126+a$
IR	7				56	147	12C	159C	
IR	7				84	147	16C	163C	
IR	7				91	147	17C	164C	
IR	7				112	147	20C	167C	
IR	7	5	217	1309	181	182	31A	213A	$1092+181=7\cdot 181+13-7$
Ir	8	17	120	841	55	103	9A	112A	$721+a=7\cdot 103+a$
Ir	8				103	103	15A	118A	
Ir	8	33	232	1625	95	199	16A	215A	$1393+a=7\cdot 199+a$
IF	9	6	441	3897	0	432	6C	438C	$3456+a=8\cdot 432+a$
IF	9				9	432	7C	439C	
IF	9				404	432	51A	483A	
IG	9	55	81	3969	45	486	6C	492C	$3888+a=16\cdot 243+a$
IG	9				54	486	7C	493C	
Ir	9	2	17	137	8	15	1B	16B	$120+8=8\cdot 16$; Lemma 9.6
Ir	9	14	113	905	103	99	13A	112A	$792+103=8\cdot 103+71$
Ir	9	15	121	969	40	106	6B	112B	$848+a=16\cdot 53+a$
Ir	9				49	106	7B	113B	
Ir	9	16	129	1033	65	113	9A	122A	$904+a=8\cdot 113+a$
Ir	9				74	113	10A	123A	
Ir	9	18	145	1161	37	127	6B	133B	$1016+a=8\cdot 127+a$
Ir	9				46	127	7B	134B	
Ir	9				82	127	11B	138B	
Ir	9	18	145	1161	99	127	13A	140A	
Ir	9				118	127	15B	142B	
Ir	9				126	127	16A	143A	
Ir	9	19	153	1225	36	134	6B	140B	$1072+a=16\cdot 67+a$
Ir	9	24	193	1545	31	169	6B	175B	$1352+a=8\cdot 169+a$
Ir	9				57	169	9A	178A	

Table D.1 (continued)

	k	m	f	v	a	$(v-f) \div (k-1)$	n	Result	$v-f+a$
Ir	9	24	193	1545	93	169	13A	182A	
Ir	9				120	169	16A	185A	
Ir	9	29	233	1865	52	204	9A	213A	$1632+a=16\cdot 103+a-16$
Ir	9				61	204	10A	214A	
Ir	9				223	204	28A	232A	$1632+223=8\cdot 223+71$
Ir	9	30	241	1929	52	211	9B	220B	$1688+a=8\cdot 211+a$
Ir	9				61	211	10B	221B	
Ir	9				204	211	26A	237A	
Ir	9	34	273	2185	21	239	6B	245B	$1912+a=8\cdot 239+a$
Ir	9				39	239	8B	247B	
Ir	9				110	239	16A	255A	
Ir	9				111	239	16B	255B	
Ir	9				120	239	17B	256B	
Ir	9				227	239	29A	268A	
Ir	9	36	289	2313	81	253	13A	266A	$2024+a=8\cdot 253+a$
Ir	9				154	253	21B	274B	
Ir	9				234	253	30A	283A	
Ir	9	37	297	2377	80	260	13A	273A	$2080+a=16\cdot 131+a-16$
Ir	9	39	313	2505	132	274	19A	293A	$2192+a=16\cdot 137+a$
Ir	9	41	329	2633	256	288	33A	321A	$2304+a=8\cdot 288+a$
Ir	9	49	393	3145	68	344	13A	357A	$2752+a=16\cdot 173+a-16$
Ir	9	51	409	3273	4	358	6B	364B	$2864+a=16\cdot 179+a$
Ir	9	55	441	3529	170	386	25A	411A	$3088+a=16\cdot 193+a$
Ir	9	62	497	3977	190	435	28A	463A	$3480+a=8\cdot 435+a$
Ir	9	66	529	4233	70	463	15B	478B	$3704+a=8\cdot 463+a$
Ir	9				124	463	21B	484B	
Ir	9				186	463	28A	491A	
Ir	9				204	463	30A	493A	
Ir	9				259	463	36B	499B	
Ir	9				285	463	39A	502A	
Ir	9				303	463	41A	504A	
Ir	9				366	463	48A	511A	
Ir	9	70	561	4489	20	491	10A	501A	$3928+a=8\cdot 491+a$
Ir	9				254	491	36A	527A	
Ir	9	78	625	5001	246	547	36A	873A	$4376+a=8\cdot 547+a$
Ir	9				301	547	42B	589B	
IR	9	4	297	2385	234	261	30C	291C	$2088+a=8\cdot 261+a$
IR	9				252	261	32C	293C	
IR	9	5	369	2961	18	324	7C	331C	$2592+a=16\cdot 163+a-16$
IR	9				36	324	9C	333C	
IR	9				45	324	10C	334C	
IR	9				54	324	11C	335C	
IR	9				161	324	23A	347A	
IR	9				171	324	24C	348C	$2592+171=16\cdot 171+27$
IR	9	6	441	3537	9	387	7C	394C	$3096+a=8\cdot 387+a$
IR	9				18	387	8C	395C	
IR	9				99	387	17C	404C	
IR	9				135	387	21C	408C	
IR	9				162	387	24C	411C	
IR	9				215	387	30A	417A	

Table D.1 (continued)

	k	m	f	v	a	$(v-f) \div (k-1)$	n	Result	$v-f+a$
IR	9	6	441	3537	216	387	30C	417C	
IR	9				234	387	32C	419C	
IR	9				261	387	35C	422C	
IR	9				270	387	36C	423C	
IR	9				297	387	39C	426C	
IR	9				387	387	49C	436C	
IR	9	10	729	5841	603	639	77C	716C	$5112 + a = 8 \cdot 639 + a$
IG	7	7	49	301	7	42	2C	44C	$252 + 7 = 7 \cdot 37$
Ir	9	12	97	777	60	85	8A	93A	$680 + 60 = 56 \cdot 11 + 64 + 60$; See § 12
Ir	9	14	113	905	58	99	8A	107A	$792 + a = 8 \cdot 99 + a$
Ir	9	17	137	1097	55	120	8A	128A	$960 + a = 8 \cdot 121 + a - 8$
Ir	9	29	233	1865	43	204	8A	212A	$1632 + a = 16 \cdot 103 + a - 16$
Ir	9	30	241	1929	42	211	8A	219A	$1688 + a = 8 \cdot 211 + a$

E Completion of Theorem 11.12

In this appendix, we provide the missing constructions for $A(9)$ and $B(9)$ that arise from the exceptions in Lemmas 11.9, 11.10, and 11.11, and from the gaps between the ends of Tables A.7 or A.8, and the first valid construction in Tables 11.2 and 11.3. Finally, note that the sole exceptional case in Lemma 11.9 can be covered using construction m or o, as 203 is not an exceptional case in Lemma 11.8.

Table E.1

Exceptions from Lemma 11.10.

res mod 18	0 A	7 A	15 A	13 B	16 B	17 B
$(n+18-1)/9$	2.0	4.7	9.6	5.4	3.7	5.8
76	m 77.9	n 79.16	X	n 75.58	n 79.7	l 82.0
84	n 85.9	m 87.16	X	n 83.58	n 87.7	l 90.0
92	p 90.35	k 88.80	k 100.16	l 88.86	n 87.79	l 98.0
94	p 88.71	X	n 100.33	l 90.86	n 97.7	l 100.0
174	X	X	n171.114	o 173.58	n 177.7	l 180.0
192	X	X	n189.114	o183.130	n 187.79	l 190.72
194	X	X	m185.168	n 193.58	n 189.79	n 199.8
204	X	X	m201.114	n189.184	n 207.7	l 210.0

Table E.2

Exceptions from Lemma 11.11.

res mod 18	2 A	4 A	5 A	11 A	13 A	16 A
$(n+18+1)/9$	2.2	6.4	14.5	3.2	5.4	5.7
92	m 87.65	m 97.13	X	m 87.74	p 90.66	n 95.25
146	n 139.83	m 151.13	n149.104	n137.110	n 145.58	m 149.25

Table E.2 (continued)

Exceptions from Lemma 11.11.

res mod 18	1 B	3 B	4 B	6 B	7 B
$(n+18+1)/9$	6.1	8.3	4.4	10.6	2.7
92	q 90.72	n 97.30	q 90.57	n 97.51	q 90.42
146	q 152.0	n 151.30	q 144.57	n 151.51	q 144.42

Table E.3

Constructions to fill gaps.

res=5 A					
82.5 90.5	n 80.23	p 78.58	n 82.41	p 82.58	p 84.58
92.5 100.5	p 86.58	p 88.58	p 90.58	m 96.23	m 96.41
102.5 110.5	p 96.58	p 98.58	p 100.58	p 99.85	p 104.58
112.5 120.5	p 106.58	p 108.58	p 110.58	m107.104	n109.104
122.5 130.5	n111.104	p 118.58	p 120.58	n117.104	m119.104
132.5 140.5	n121.104	n123.104	n125.104	n127.104	n129.104
142.5	n131.104				
res=8 A					
82.8 90.8	n 79.35	n 81.35	n 83.35	n 85.35	m 87.35
92.8 96.8	n 89.35	m 87.71	n 89.71		
res=15 A					
81.6 89.6	k 80.16	k 82.16	n 82.33	k 80.70	k 88.16
91.6 95.6	k 90.16	n 88.51	k 88.70		
res=6 B					
82.6 90.6	n 81.15	n 83.15	o 85.15	n 87.15	n 89.15
92.6 100.6	n 87.51	n 89.51	n 87.87	n 97.15	n 99.15
102.6 104.6	n 97.51	n 99.51			
res=11 B					
81.2	n 75.56				
res=14 B					
81.5 89.5	l 80.15	q 78.49	n 77.77	n 79.77	n 81.77
91.5 99.5	n 83.77	o 85.77	n 87.77	n 89.77	l 98.15
101.5 103.5	q 96.49	q 98.49			

F The Unconstructed Cases

In this appendix, we summarize the unconstructed cases in Appendix A; we parenthesize values for which a construction is now known.

Table F.1

Values of t for which $t \in A(7)$ is unconstructed.

1	2	3	5	6	12	(13)	14	17	19
22	27	(29)	33	37	39	42	(43)	47	(52)
59	(61)	62	(66)	(67)	(75)	(82)	(83)	(104)	(107)
(117)									

Table F.2

Values of t for which $t \in B(7)$ is unconstructed.

3	(4)	(6)	19	(29)	34	39	(44)	(82)

Table F.3

Values of t for which $t \in C(7)$ is unconstructed.

3	(4)	(6)	19	(22)	24	(29)	34	39	(42)
(43)	(44)	(53)	(59)	(82)	(174)	(179)	(203)		

Table F.4

Values of t for which $t \in A(8)$ is unconstructed.

2	3	4	5	6	7	(11)	(12)	(13)	14
19	20	21	22	(23)	24	25	26	27	28
31	32	34	35	(38)	39	40	(45)	46	(47)
(48)	52	(53)	59	(60)	61	62	67	(68)	(69)
(70)	(75)	(76)	(77)	(78)	(96)	(101)	(102)	(103)	(110)
(119)	(126)	(174)	(175)	(181)	(182)	(188)	(191)	(192)	(195)
(196)	(197)	(198)	(199)	(223)	(230)	(231)			

Table F.5

Values of t for which $t \in B(8)$ is unconstructed.

3	11	13	20	22	23	25	26	27	28
(191)									

Table F.6

Values of t for which $t \in C(8)$ is unconstructed.

3	11	13	20	22	23	25	26	27	28
(53)	(191)	(195)							

Table F.7

Values of t for which $t \in A(9)$ is unconstructed.

2	3	4	5	(6)	7	(8)	11	12	(14)
15	(17)	20	21	22	24	27	31	32	34
37	38	40	42	43	45	47	50	52	53
56	(59)	60	61	62	(63)	67	68	(72)	75
76	(77)	(78)	84	(86)	(87)	92	(93)	94	96
(97)	102	(106)	(107)	(113)	(115)	(119)	(128)	132	(133)
(138)	(141)	(142)	(147)	(150)	(151)	(158)	174	(176)	(177)
(186)	191	(192)	194	(195)	196	(197)	201	(203)	204
(206)	209	(212)	(215)	(219)	(222)	(245)	(247)	(372)	

Table F.8

Values of t for which $t \in B(9)$ is unconstructed.

2	3	4	5	12	13	14	18	(19)	(20)
22	23	25	26	27	28	31	33	34	(37)
38	40	41	43	46	47	(48)	52	(53)	59
(60)	61	62	67	68	76	85	(92)	93	94
102	103	(114)	(115)	(132)	139	(146)	(147)	148	(157)
(166)	(167)	(173)	174	183	(191)	192	202	203	(205)
209	229	(230)	(237)	(253)	(265)	(311)	(454)		

For $C(9)$, there were 200 exceptions in Table A.9. Construction u eliminated most larger values, with 41 exceptions. Of these 41, construction r eliminated 17 values, IR eliminated eight, and with the aid of [25, 26], three more were eliminated: (j 120, i 133, Zj 238); the 13 remaining large exceptions were 754, 886, 895, 931, 985, 1006, 1057, 1084, 1093, 1165, 1246, 1624, 1726. Constructions are now known for almost all values not in Table F.8; the current exceptions are given in Table F.9.

Table F.9

Values of t with $t \in B(9)$, but $t \in C(9)$ unconstructed.

16	20	48	60	92	104	147	166	187	191
205									

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