The Jump Number of $P \times \mathbf{n}$

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Abstract. In this paper, we calculate the jump number of the product of an ordered set and a chain.

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1 Introduction

An order R on a set is called an *extension* of another order S on the same set if $S \subseteq R$. Let P be a finite ordered set and let $|P|$ be the *number of vertices* in P. We say that b *covers* a, denoted by $b \succ a$, provided that for any $c \in P$, $a < c \leq b$ implies that $c = b$. A *linear extension* of P is a linearly ordered set L such that $a \leq b$ in L whenever $a \leq b$ in P. We write $[x_1, x_2, \cdots, x_n]$ as a linear ordered set such that $x_1 \prec x_2 \prec \cdots \prec x_n$. Szpilarjn [4] showed that any order has a linear extension. Let $\mathcal{L}(P)$ be the set of all linear extensions of P. Then $\mathcal{L}(P)$ is a nonempty set.

Let P and Q be two disjoint ordered sets. The *disjoint sum* $P + Q$ of P and Q is the ordered set on $P \cup Q$ such that $x < y$ if and only if either $x, y \in P$ and $x < y$ in P or $x, y \in Q$ and $x < y$ in Q. The *linear sum* $P \oplus Q$ of P and Q is obtained from $P + Q$ by adding the relation $x < y$ for all $x \in P$ and $y \in Q$.

A (P, L)-*chain* is a maximal sequence of elements $z_1, z_2, \cdots, z_n$ such that $z_1 \prec z_2 \prec \cdots \prec z_n$ in both L and P. Let $c(L)$ be the number of (P, L)-chains. A consecutive pair (x_i, x_{i+1}) of elements in L is a *jump* of P in L if x_i is incomparable to x_{i+1} in P. The jumps induce a decomposition

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$L: C_0 \oplus C_1 \oplus \cdots \oplus C_m$ of L into (P,L)-chains $C_0, C_1, \cdots, C_m$ where $m = c(L)$ and $(\max C_i, \min C_{i+1})$ is a jump of P in L for $i = 0, 1, \cdots, m-1$. Let $s(P,L)$ be the number of jumps of P in L. The *jump number* of P is the minimum number of $s(P,L)$ over all linear extensions L of P, denoted by $s(P)$. If $s(P,L) = s(P)$ then L is called an *optimal linear extension* of P. We denote the set of all optimal linear extensions of P by $\mathcal{O}(P)$. For a positive integer n, we denote by $\mathbf{n} = [1, 2, \cdots, n]$ the n-element chain.

The jump number of a product of ordered sets has been studied by a few authors. Bae, Kim and Lee [2] determined the jump number of the product of generalized crowns as following;

$$s(S_n^k \times S_m^l) = 2(m+l)(n+k) + 2(m-2)(n-2) - 1.$$

Jung [3] studied the jump number of the product of a tree and a chain. Let T be a tree and $C_1 \oplus C_2 \oplus \cdots \oplus C_n$ a greedy optimal linear extension of T. Then $s(T \times \mathbf{n}) = \sum_{i=1}^{n} \min\{|C_i|, n\} - 1$. More generally, we consider the product of an ordered set and a chain.

In this paper, we find an optimal linear extension of $P \times \mathbf{n}$ from the linear extension of P. Furthermore, we introduce a way to find the jump number of the product of an ordered set and a chain. Our main result is the following:

Theorem *Let P be an ordered set and let $L_P = C_0 \oplus C_1 \oplus \cdots \oplus C_r$ be a linear extension of P such that $\sum \max\{0, |D_i| - n\} \leq \sum \max\{0, |C_i| - n\}$ for all $D_0 \oplus D_1 \oplus \cdots \oplus D_t \in \mathcal{L}(P)$. Then*

$$s(P \times \mathbf{n}) = \sum_{i=0}^{r} \min\{|C_i|, n\} - 1.$$

From the above Theorem, we can see that the problem of finding the jump number of $P \times \mathbf{n}$ is the that of finding a linear extension L of P which has maximum of the sum of $|C| - n$ for all (P,L)-chains with $|C| > n$. Hence we have $s(P \times \mathbf{n}) = |P| - 1 - t$, where $t = \sum_{\substack{C:(P,L)-\text{chain} \\ |C|>n}} |C| - n$.

2 Preliminaries

In this section, we discuss a property of the linear extension of $P \times Q$ and construct a linear extension of $P \times Q$ for finite ordered sets P and Q. It is easy to determine an upper bound of the jump number of $P \times Q$.

Now, we see that $(a, b) \prec (c, d)$ in $P \times Q$ implies either $a = c$ or $b = d$. In general, if $L = C_0 \oplus C_1 \oplus \cdots \oplus C_t$ is an arbitrary linear extension of $P \times Q$ consisting of $(P \times Q, L)$-chains, then, for $i = 0, 1, \cdots t$,

$$\text{either } C_i = D \times \{y\} \text{ or } C_i = \{x\} \times E$$

for some chains D and E in P and Q, respectively, and for $x \in P$, $y \in Q$. In this case, the $(P \times Q, L)$-chain C_i is said to be *left* $(P \times Q, L)$*-chain* if C_i is of the form $D \times \{y\}$ with $|D| \geq 2$ and *right* $(P \times Q, L)$*-chain* if C_i is of the form $\{x\} \times E$ with $|E| \geq 1$.

Jung showed in [3] that, for all positive integers m and n,

$$s(\mathbf{m} \times \mathbf{n}) = \min\{m, n\} - 1.$$

For some positive integers m and n with $m \geq n$, it is well known that an optimal linear extension L of $\mathbf{m} \times \mathbf{n}$ is of the following form;

$$L = \bigoplus_{k=1}^{n} [(1, k), (2, k), \cdots, (m, k)]. \tag{2.1}$$

Furthermore, we see that $[(1, k), (2, k), \cdots, (m, k)]$ is an $(\mathbf{m} \times \mathbf{n}, L)$-chain for $k = 1, 2, \cdots, n$.

Now, consider an arbitrary linear extensions L_P and L_Q of P and Q, respectively, with $L_P = C_0 \oplus C_1 \oplus \cdots \oplus C_r$ and $L_Q = D_0 \oplus D_1 \oplus \cdots \oplus D_t$. Using the same above method, we will construct a linear extension L of $P \times Q$. Let $L' = \bigoplus_{i=0}^{r} \bigoplus_{j=0}^{t} C_i \times D_j$ and let $L_{i,j}$ be an optimal linear extension of $C_i \times D_j$, which is of the same form in (2.1). Then L' is an extension of $P \times Q$ and $L = \bigoplus_{i=0}^{r} \bigoplus_{j=0}^{t} L_{i,j}$ is a linear extension of $P \times Q$, which is denoted by $L_P * L_Q$. Moreover, we obtain an upper bound of the jump number of $P \times Q$ as follows;

$$s(P \times Q) \leq s(P \times Q, L_P * L_Q) = \sum_{i=0}^{r} \sum_{j=0}^{t} \min\{|C_i|, |D_j|\} - 1. \tag{2.2}$$

A certain question is raised as follows: Does it imply that there are $L_P \in \mathcal{L}(P)$ and $L_Q \in \mathcal{L}(Q)$ such that

$$s(P \times Q) = s(P \times Q, L_P * L_Q)\,?$$

Let $L_1 = C_0 \oplus C_1 \oplus \cdots \oplus C_r$ and $L_2 = D_0 \oplus D_1 \oplus \cdots \oplus D_r$ be linear extensions of P. Then we say that L_1 and L_2 are of the *same type* if $\{|C_i| : 1 \leq i \leq r\} = \{|D_j| : 1 \leq j \leq r\}$ as the same multi-set. Let $\mathcal{O}^*(P)$ be the set of all distinct types of linear extensions of P.

Example 1. (1) Consider the given ordered sets P and Q in Figure 1. Let $L_p = [0,1] \oplus [2] \cong \mathbf{2} \oplus \mathbf{1}$ and $L_Q = [a,b,c] \oplus [d] \cong \mathbf{3} \oplus \mathbf{1}$. Then $L' = ([0,1] \times [a,b,c]) \oplus ([0,1] \times [d]) \oplus ([2] \times [a,b,c]) \oplus ([2] \times [d])$ is an extension of $P \times Q$.

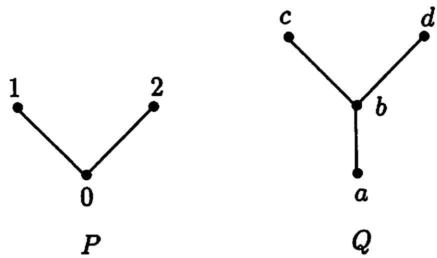


Figure 1.

Thus we have a linear extension $L_P * L_Q$ of $P \times Q$ as follows;

$$\begin{aligned} L_P * L_Q = &[(0,a),\ (0,b),\ (0,c)] \oplus [(1,a),\ (1,b),\ (1,c)] \\ &\oplus [(0,d),\ (1,d)] \\ &\oplus [(2,a),\ (2,b),\ (2,c)] \\ &\oplus [(2,d)]. \end{aligned}$$

In fact, $L_P * L_Q = L_{0,0} \oplus L_{0,1} \oplus L_{1,0} \oplus L_{1,1}$ is an optimal linear extension of $P \times Q$ with $s(P \times Q,\ L_P * L_Q) = s(P \times Q) = 4$. Furthermore, we see that

$$s(P \times Q) = s(P \times Q, L * M)$$

for all $L \in \mathcal{O}^*(P) = \{\mathbf{2} \oplus \mathbf{1}\}$ and $M \in \mathcal{O}^*(Q) = \{\mathbf{3} \oplus \mathbf{1}\}$.

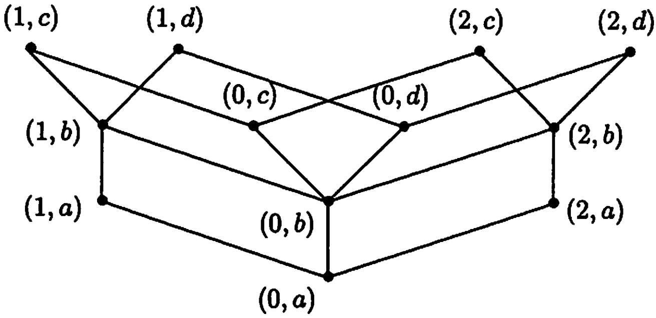


Figure 2. $P \times Q$

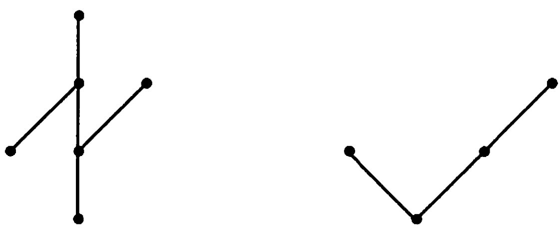
(2) In general, it doesn't need to be true that there are linear extensions L_P and L_Q of P and Q, respectively, with $L_P = C_0 \oplus C_1 \oplus \cdots \oplus C_r$ and $L_Q = D_0 \oplus D_1 \oplus \cdots \oplus D_t$ such that

$$s(P \times Q) = s(P \times Q, L_P * L_Q) = \sum_{i=0}^{r} \sum_{j=0}^{t} \min\{|C_i|, |D_j|\} - 1.$$

Consider the ordered sets P and Q in Figure 3. Then we see that

$$\mathcal{O}^*(P) = \{\mathbf{3} \oplus \mathbf{3}, \mathbf{1} \oplus \mathbf{4} \oplus \mathbf{1}, \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{2}, \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{2}, \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{1}\},$$
$$\mathcal{O}^*(Q) = \{\mathbf{3} \oplus \mathbf{1}, \mathbf{2} \oplus \mathbf{2}, \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1}\}$$

and so $s(P \times Q, L_P * L_Q) = \sum_{i=0}^{r} \sum_{j=0}^{t} \min\{|C_i|, |D_j|\} - 1 \geq rt - 1$ for all $L_P \in \mathcal{O}^*(P)$ and $L_Q \in \mathcal{O}^*(Q)$ with $L_P = C_0 \oplus C_1 \oplus \cdots \oplus C_r$ and $L_Q = D_0 \oplus D_1 \oplus \cdots \oplus D_t$.



P Q

Figure 3.

Therefore, we see that $s(P \times Q, L * M) \geq 7$ for all $L \in \mathcal{O}^*(P)$ and $M \in \mathcal{O}^*(Q)$. However, it is known that $s(P \times Q) = 6$.

Now we study the jump number of the product of an ordered set and a chain. If L_P is any linear extension of an ordered set P with $L_P = C_0 \oplus C_1 \oplus \cdots \oplus C_r$, then we obtain from (2.2) that

$$s(P \times \mathbf{n}) \leq \sum_{i=0}^{r} \min\{|C_i|, n\} - 1.$$

In particular, we see that $s(P \times \mathbf{n}) \leq |P| - 1$. Furthermore, let $m(L)$ be the maximum number of $|C_i|$ for $i = 0, 1, \cdots, r$. We say that the *height* of *P with respect to linear extension* is the maximum number of $m(L)$ over all linear extensions L of P, denoted by $h_l(P)$, i.e.,

$$h_l(P) = \max_{L \in \mathcal{L}(P)} \left\{ \max\{|C| : C \text{ is a } (P, L)\text{-chain } \} \right\}.$$

Then it is easy to see that $h_l(P) \leq h(P)$. Jung showed in [3] that if the maximum size of a chain in a ranked ordered set P is at most n, then $s(P \times \mathbf{n}) = |P| - 1$. For a more generalization, we prove the following.

Proposition 1 *$s(P \times \mathbf{n}) = |P| - 1$ if and only if $h_l(P) \leq n$.*

Proof. Suppose that $h_l(P) \leq n$ and let L be any linear extension of $P \times \mathbf{n}$. We see that the length of any $(P \times \mathbf{n}, L)$-chain less than n. Thus $s(P \times \mathbf{n}, L) \geq \frac{|P \times \mathbf{n}|}{n} - 1 = |P| - 1$. Hence $s(P \times \mathbf{n}) = |P| - 1$.

Suppose that $h_l(P) > n$. Then there is a linear extension $L = C_0 \oplus C_1 \oplus \cdots \oplus C_r$ of P such that $|C_{r_0}| > n$ for some $0 \leq r_0 \leq r$. Hence we have $s(P \times \mathbf{n}) \leq \sum_{i=0}^{r} \min\{|C_i|, n\} - 1 < \sum_{i=0}^{r} |C_i| - 1 = |P| - 1$. ■

3 Proof of Theorem

We have the following lemma which is decisive in determining the jump number to its source of the product of ordered sets.

Lemma 2 *Let P be an ordered set and let L be any optimal linear extension of $P \times \mathbf{n}$. Then we have the following properties:*

(1) *The number of left(resp., right) $(P \times \mathbf{n}, L)$-chains in $P \times \{i\}$ is equal to the number of left(resp., right) $(P \times \mathbf{n}, L)$-chains in $P \times \{j\}$ for all i, j with $1 \leq i, j \leq n$.*

(2) *Every $(P \times \mathbf{n}, L)$-chain has the length at least $n - 1$.*

Proof. Let P be an ordered set and L any linear extension of $P \times \mathbf{n}$. Suppose that α_i is the number of left $(P \times \mathbf{n}, L)$-chains in $P \times \{i\}$ and that β_i is the number of right $(P \times \mathbf{n}, L)$-chains in $P \times \{i\}$ for all $i = 1, 2, \cdots, n$. In fact, we see that every right $(P \times \mathbf{n}, L)$-chain in $P \times \{i\}$ is an one element chain of the form $\{x\} \times \{i\}$ for some $x \in P$. Now, for each $i = 1, 2, \cdots, n$, let $L\,|_{P\times\{i\}} = \bigoplus_{k=1}^{\alpha_i+\beta_i} D_{i,k} \times \{i\}$ be the restriction of L to $P \times \{i\}$, where $D_{i,k} \times \{i\}$ is a left $(P \times \mathbf{n}, L)$-chain or one point in a right $(P \times \mathbf{n}, L)$-chain. We will construct a new linear extension L_i of $P \times \mathbf{n}$ with respect to $L\,|_{P\times\{i\}}$.

For $k = 1, 2, \cdots, \alpha_i + \beta_i$, we define a chain $L_{i,k}$ in $P \times \mathbf{n}$ as follows:

$$(3.1) \qquad L_{i,k} = \begin{cases} D_{i,k} \times \{1\} \oplus D_{i,k} \times \{2\} \oplus \cdots \oplus D_{i,k} \times \{n\} & \text{if } |D_{i,k}| \geq 2 \\ D_{i,k} \times \mathbf{n} & \text{if } |D_{i,k}| = 1. \end{cases}$$

Let $L_i = \bigoplus_{k=1}^{\alpha_i+\beta_i} L_{i,k}$. Hence L_i is a linear extension of $P \times \mathbf{n}$ and we have the following properties:

$$(3.2) \qquad s(P \times \mathbf{n}, L) \geq -1 + \beta_j + \sum_{i=1}^{n} \alpha_i \text{ for all } j = 1, 2, \cdots, n,$$

$$(3.3) \qquad s(P \times \mathbf{n}, L_i) = -1 + \beta_i + n \cdot \alpha_i.$$

Now, there exist integers i_0 and j_0 with $1 \leq i_0$, $j_0 \leq n$ such that

$$n \cdot \alpha_{i_0} + \beta_{i_0} = \min\{n \cdot \alpha_i + \beta_i \mid 1 \leq i \leq n\} \text{ and } \beta_{j_0} = \max\{\beta_1, \beta_2, \cdots, \beta_n\}.$$

Therefore, we have

$$n \cdot \alpha_i + \beta_{j_0} \geq n \cdot \alpha_i + \beta_i \geq n \cdot \alpha_{i_0} + \beta_{i_0}$$

and so

$$\alpha_i + \frac{\beta_{j_0}}{n} \geq \alpha_{i_0} + \frac{\beta_{i_0}}{n}.$$

Hence we see that

$$\beta_{j_0}+\sum_{i=1}^{n}\alpha_i=\sum_{i=1}^{n}(\alpha_i+\frac{\beta_{j_0}}{n})\geq\sum_{i=1}^{n}(\alpha_{i_0}+\frac{\beta_{i_0}}{n})=n\cdot(\alpha_{i_0}+\frac{\beta_{i_0}}{n})=n\cdot\alpha_{i_0}+\beta_{i_0}.$$

Furthermore, we obtain from (3.2) that

$$s(P\times\mathbf{n}\,,\,L)\geq-1+\beta_{j_0}+\sum_{i=1}^{n}\alpha_i\geq-1+n\cdot\alpha_{i_0}+\beta_{i_0}=s(P\times\mathbf{n}\,,L_{i_0}).$$

(1) Suppose that L is an optimal linear extension $P\times\mathbf{n}$. Since $L\in\mathcal{O}(P\times\mathbf{n})$, it follows from (3.2) that, for all $k=1,2,\cdots,n$,

$$\beta_{j_0}+\sum_{i=1}^{n}\alpha_i-1\leq s(P\times\mathbf{n}\,,L)\leq s(P\times\mathbf{n}\,,L_k)=n\cdot\alpha_k+\beta_k-1. \tag{3.4}$$

Thus summing both sides of (3.4) from $k=1$ to n, we obtain

$$\sum_{k=1}^{n}(\beta_{j_0}+\sum_{i=1}^{n}\alpha_i)\leq\sum_{k=1}^{n}(n\cdot\alpha_k+\beta_k)$$

and so $n\cdot\beta_{j_0}\leq\sum_{k=1}^{n}\beta_k$. By the maximality of β_{j_0}, we see that

$$\beta_{j_0}=\beta_k$$

for all $k=1,2,\cdots,n$.

Since L is an optimal, $s(P\times\mathbf{n}\,,L)=s(P\times\mathbf{n}\,,L_{i_0})=n\cdot\alpha_{i_0}+\beta_{i_0}-1$. By inserting this into (3.4), we have

$$\beta_{j_0}+\sum_{i=1}^{n}\alpha_i-1\leq n\cdot\alpha_{i_0}+\beta_{i_0}-1\leq n\cdot\alpha_i+\beta_i-1.$$

Since $\beta_{i_0}=\beta_i=\beta_{j_0}(i=1,2,\cdots,n)$, it follows that $\sum_{i=1}^{n}\alpha_i\leq n\cdot\alpha_{i_0}$ and $\alpha_{i_0}\leq\alpha_i$ for all $i=1,2,\cdots,n$. By the minimality of α_{i_0}, we see that

$$\alpha_i=\alpha_j$$

for all $i,j=1,2,\cdots,n$.

(2) Without loss of generality, we may assume from (1) that $\alpha = \alpha_i$ and $\beta = \beta_i$ for $i = 1, 2, \cdots, n$. Then $s(P \times \mathbf{n}, L) = \beta + n \cdot \alpha - 1$ for $L \in \mathcal{O}(P \times \mathbf{n})$. Since every $(P \times \mathbf{n}, L)$-chain is either right $(P \times \mathbf{n}, L)$-chain or left $(P \times \mathbf{n}, L)$-chain, it follows that β is the number of right $(P \times \mathbf{n}, L)$-chains and $n \cdot \alpha$ is the number of left $(P \times \mathbf{n}, L)$-chains.

Suppose that $C = [(a, j), (a, j+1), \cdots, (a, j+k-1)]$ is a right $(P \times \mathbf{n}, L)$-chain. If $j > 1$, then there are distinct β right $(P \times \mathbf{n}, L)$-chains penetrating $P \times \{j-1\}$, which are different from C. Hence the number of right $(P \times \mathbf{n}, L)$-chains is greater than β, which is a contradiction. Similarly, we induce a contradiction in the case $j + k - 1 < n$. Hence the length of every right $(P \times \mathbf{n}, L)$-chain is exactly $n - 1$.

Furthermore, suppose that there is a left $(P \times \mathbf{n}, L)$-chain $D \times \{i\}$ with $D = [d_1, d_2, \cdots, d_r]$ and $r < n$. Then we see from (1) that $L_i = \bigoplus_{k=1}^{\alpha+\beta} L_{i,k}$ is an optimal linear extension of $P \times \mathbf{n}$, where $L_{i,k}$ is defined in (3.1). Now, there is k with $1 \leq k \leq \alpha + \beta$ such that $D = D_{i,k}$ and so $L_{i,k} = D_{i,k} \times \{1\} \oplus D_{i,k} \times \{2\} \oplus \cdots \oplus D_{i,k} \times \{n\}$. Let $L_i^* = L_{i,1} \oplus \cdots \oplus L_{i,k-1} \oplus L'_{i,k} \oplus L_{i,k+1} \oplus \cdots \oplus L_{i,\alpha+\beta}$, where $L'_{i,k} = \{d_1\} \times \mathbf{n} \oplus \{d_2\} \times \mathbf{n} \oplus \cdots \oplus \{d_r\} \times \mathbf{n}$. Then L_i^* is a linear extension of $P \times \mathbf{n}$ and $L'_{i,k}$ consists of r $(P \times \mathbf{n}, L_i^*)$-chains and so

$$s(P \times \mathbf{n}, L_i^*) = -1 + \beta + r + (n-1) \cdot \alpha < -1 + \beta + n \cdot \alpha = (P \times \mathbf{n}, L_i),$$

which is a contradiction as L_i is optimal. Hence we have the result. ∎

Now we are ready to proof the main result:

Proof of Theorem. Let L be any optimal linear extension of $P \times \mathbf{n}$. We see from Lemma 2 that there are exactly distinct $n \cdot \alpha$ left $(P \times \mathbf{n}, L)$-chains and β right $(P \times \mathbf{n}, L)$-chains such that $s(P \times \mathbf{n}, L) = n \cdot \alpha + \beta - 1$.

Now, consider the restriction linear extension $L|_{P \times \{i\}} = \bigoplus_{k=1}^{\alpha+\beta} C_k \times \{i\}$ of L to $P \times \{i\}$, where $C_k \times \{i\}$ is a left $(P \times \mathbf{n}, L)$-chain or one point in a right $(P \times \mathbf{n}, L)$-chain. Then each $C_k (1 \leq k \leq \alpha + \beta)$ is a chain in P with either $|C_k| \geq n$ or $|C_k| = 1$. Hence $L_P = \bigoplus_{k=1}^{\alpha+\beta} C_k$ is a linear extension of P and $s(L_P * \mathbf{n}, P \times \mathbf{n}) = n \cdot \alpha + \beta - 1 = s(P \times \mathbf{n})$, that is,

$$\begin{aligned} s(P \times \mathbf{n}) &= n \cdot \alpha + \beta - 1 \\ &= \overbrace{n + n + \cdots + n}^{\alpha} + \overbrace{1 + 1 + \cdots + 1}^{\beta} - 1 \\ &= \sum_{k=1}^{\alpha+\beta} \min\{|C_k|, n\} - 1. \end{aligned}$$ ∎

Corollary 3 *Let P be an ordered set. Then we have*

$$s(P \times \mathbf{n}) = -1 + \min_{L \in \mathcal{L}(P)} \left\{ |P| - \sum_{i=1}^{r} \max\{0, |C_i| - n\} \ : \ L = C_1 \oplus \cdots \oplus C_r \right\}.$$

4 Concluding Remarks

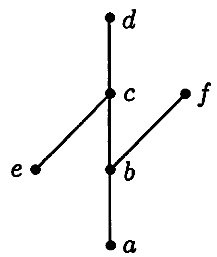
1. Let P be any ordered set with $n = h_l(P)$. Then we see that

$$s(P) = s(P \times \mathbf{1}) \leq s(P \times \mathbf{2}) \leq \cdots \leq s(P \times \mathbf{n}) = \cdots = |P| - 1$$

2. Observe that it need not be true that

$$\sum \min\{|D_i|, n\} \leq \sum \min\{|C_i|, n\}$$

for each linear extension $C_0 \oplus C_1 \oplus \cdots \oplus C_r$ of an ordered set P and each optimal linear extension $D_0 \oplus D_1 \oplus \cdots \oplus D_t$ of an ordered set P. For instance, let P be an ordered set as in Figure 4.



P

Figure 4.

Consider the optimal linear extension L_1 and non-optimal linear extension L_2 of P, where $L_1 = D_0 \oplus D_1 = [a, b, f] \oplus [e, c, d] \cong \mathbf{3} \oplus \mathbf{3}$ and $L_2 = C_0 \oplus C_1 \oplus C_2 = [e] \oplus [a, b, c, d] \oplus [f] \cong \mathbf{1} \oplus \mathbf{4} \oplus \mathbf{1}$. In fact, we have $5 = \sum_{i=0}^{2} \min(|C_i|, n) = s(P \times \mathbf{n}, L_2 * \mathbf{n}) < s(P \times \mathbf{n}, L_1 * \mathbf{n}) = \sum_{i=0}^{1} \min(|D_i|, n) = 6$ for some integer $n = 3$. Furthermore, we see that

$$s(P \times \mathbf{2}) = s(P \times \mathbf{2}, L_1 * \mathbf{2}) = s(P \times \mathbf{2}, L_2 * \mathbf{2}) = -1 + |P| - 2 = 3.$$

3. We see from main Theorem that we can construct an optimal linear extension of $P \times \mathbf{n}$ from the linear extension of P. And this linear extension of P depends on the value of n. Because of this fact, the statement of the example (2) in section 1 is not true. In fact, in the example, we take a linear extension of Q of the type $\mathbf{3} \oplus \mathbf{1}$ and let L be a linear extension of $P \times Q$ such that $L = ((\mathbf{1} \oplus \mathbf{4} \oplus \mathbf{1}) * \mathbf{3}) \oplus ((\mathbf{3} \oplus \mathbf{3}) * \mathbf{1})$. Then L is an optimal linear extension of $P \times Q$. Hence we give a following question: Is there a linear extension L_Q of Q with $L_Q = D_0 \oplus D_1 \oplus \cdots \oplus D_t$ such that $L = \oplus_{i=0}^{t} L_i$ is an optimal linear extension of $P \times Q$ for $L_i \in \mathcal{O}(P \times D_i)$?

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