The Connectivity of Matching Transformation Graphs of Cubic Bipartite Plane Graphs

Dedicated to the occasion of the 60th anniversary of Professor Derek Holton

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Abstract

Let G be a cubic bipartite plane graph that has a perfect matching. If M is any perfect matching of G, then G has a face that is M-alternating. If f is any face of G then there is a perfect matching M such that f is M-alternating. There is a simple algorithm for visiting all perfect matchings of G begining at one. There are infinitely many cubic plane graphs that have perfect matchings but whose matching transformation graphs are completely disconnected. Several problems are proposed.

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1 Matching Transformation Graphs

Transformation graphs have attracted some attention and they reveal possibilities of certain algorithms ([2], [4] and [1]). In this paper we consider the connectivity of matching transformation graphs of cubic biparitite plane graphs and give examples of matching transformation graphs that are completely disconnected. Througout the paper, $|S|$ denotes the cardinality of a set S. The addition of subgraphs will be performed with respect to their edges over the binary field and that of integers will be in the ring of integers.

Let $G = (V, E)$ be a graph. If F is a spanning subgraph of G, i.e., $V(F) = V(G)$ and $E(F) \subseteq E(G)$, then F is said to be a *factor* of G. A factor F regular of degree r is said to be an *r-factor*. In this paper, a 1-factor is called a *perfect matching*. For brevity, denote $|G| = |V(G)|$.

Let M be a perfect matching of G and C be a cycle of G. If the edges of C alternate between M and $E - M$, then it is said to be an *M-alternating* cycle. If G is a 2-connected graph then each edge of G lies in a cycle.

Let Σ be a surface and G be a graph that has a cellular embedding on Σ. Then each component of $\Sigma - G$ is said to be a *face* of G on Σ. If Σ is the sphere then the graphs that have cellular embeddings on Σ are said to be planar graphs. A *plane* graph is a planar graph embedded on the sphere (or equivalently, on the plane). The set of faces of a plane graph will be denoted by Φ.

Let G be a cubic bipartite planar graph that contains a perfect matching. The *matching transformation graph* of G is defined (see [3]) to be the graph $\mathcal{M}(G)$ with vertex set

$$V(\mathcal{M}(G)) = \{M : M \text{ is a perfect matching of } G\}$$

and two vertices M_i and M_j being adjacent if $M_i + M_j$ consists of boundaries of faces of G.

2 Squares in Cubic Bipartite Plane Graphs

A face of size k is said to be a k-face. Denote by p_k the number of k-faces of G.

LEMMA 2.1 *Every cubic bipartite plane graph has at least six squares.*

Proof: By Euler's formula for polyhedra,

$$\sum_{k \geq 4,\, k \equiv 0 \pmod 2} (6 - k) p_k = 12. \tag{1}$$

Hence,

$$2p_4 = 12 + \sum_{k \geq 6,\, k \equiv 0 \pmod 2} (k - 6) p_k. \tag{2}$$

Therefore, $p_4 \geq 6$. ∎

LEMMA 2.2 *Let f be a face of a cubic bipartite plane graph G. Then every square of G is adjacent to f if and only if $G \simeq C_m \times K_2$ and the boundary of f is one of the two m-cycles $0 \times C_m$ and $1 \times C_m$ for $m \geq 6$ and $m \equiv 0 \pmod 2$.*

Proof: The sufficiency (the "if" part) is obvious.

Necessity (the "only if" part). By Lemma 2.1, G has at least six squares. Clearly, f cannot be a square, since a square has only four sides and if it is to be adjacent to all other squares then it has to be adjacent to at least five squares.

Hence $|\partial f| \geq 6$.

If $|\partial f| = 6$, then G has precisely six squares and f is adjacent to all of them. Clearly, $G \simeq C_6 \times K_2$ with f being either one of the two hexagons.

Let $|\partial f| = m > 6$ and suppose that the "only if" part holds for (H, f') with $|H| < |G|$, $|\partial f'| < m$ and f' adjacent to all the squares of H. Let $s = u_1u_2u_3u_4u_1$ be a square adjacent to f, h_1, g, h_2 in the order given and $u_1, u_2 \in s \cap f$, $u_2, u_3 \in s \cap h_1$, $u_3, u_4 \in s \cap g$ and $u_1, u_4 \in s \cap h_2$.

Claim: $h_1 \neq h_2$.

Assume that $h_1 = h_2$. Then $S = \{u_1u_4,\ u_2u_3\}$ is a cyclic 2-cut. (Cyclic means that each component of $G - S$ has a cycle.) Let L and R be the two cyclic components of $G - S$, with $u_1, u_2 \in L$ and $u_3, u_4 \in R$. Let v_i be the neighbour of u_i that is not on s. Now if g is a square then it is a square not adjacent to f, a contradiction to the assumption of the lemma. Hence $|g| \geq 6$. Let

$$H(g) = R - \{u_3, u_4\} \cup \{v_3v_4\}.$$

Then $H(g)$ is a cubic bipartite plane graph. By Lemma 2.1, $H(g)$ has at leas six squares, at least one of which is a square of G not adjacent to f. This is a contradiction. Hence the claim is proved.

Assume that $f \neq g$ and let

$$H = G - \{u_i : 1 \leq i \leq 4\} + v_1v_2 + v_3v_4$$

and

$$\partial f' = \partial f + v_1u_1u_2v_2 + v_1v_2.$$

Then H is a cubic bipartite plane graph in which f' is a face adjacent to all the squares of H, $|H| < |G|$ and $|\partial f'| < |\partial f| = m$. Moreover, $m = |\partial f'| + 2$ and hence $|\partial f'| \equiv m \equiv 0 \pmod 2$ and $|\partial f'| \geq 6$. By the inductive hypothesis, $H \simeq C_{m-2} \times K_2$ with $\partial f'$ being one of the two $(m-2)$-cycles for some $m - 2 \geq 6$ and $m - 2 \equiv 0 \pmod 2$. Clearly, $G \simeq C_m \times K_2$ with ∂f being one of the two m-cycles, where $\partial f = \partial f' + v_1v_2 + v_1u_1u_2v_2$.

If $f = g$, then the proof is exactly the same except that

$$\partial f' = \partial f + v_1u_1u_2v_2 + v_3u_3u_4v_4 + v_1v_2 + v_3v_4,$$

$$H \simeq C_{m-4} \times K_2$$

and $|\partial f'| = |\partial f| - 4$. ■

3 Matchings and Alternating Faces

LEMMA 3.1 *Let G be a cubic bipartite plane graph that has a perfect matching. Then for each perfect matching M of G, G has an M-alternating face.*

Proof: Let G be a cubic bipartite plane graph and let M be any perfect matching of G. The proof is by induction on the order $|G|$ of G. Each cubic graph satisfies $3|V| = 2|E|$. Hence it is routine to verify that the least order of a cubic bipartite planar graph is 8 and there is precisely one graph of this order, namely the graph of the 3-dimensional cube: $K_2 \times K_2 \times K_2 = C_4 \times K_2$, where $\times$ denotes the cartesian product of graphs. For each perfect matching of this graph, there is clearly a face that is alternating.

Assume that for each cubic bipartite plane graph H with $8 < |V(H)| < |V(G)|$ that has a perfect matching, the statement of lemma holds. As shown in Lemma 2.1, G has at least six 4-faces. In particular, there are two 4-faces that are not adjacent. Let f be any one of these 4-faces of G.

If the edges of f are M-alternating, then f is a face that satisfies the assertion of the lemma. Assume therefore that the edges of f are not M-alternating. Since M is a perfect matching, each vertex of f is M-saturated (i.e., is a vertex of M). Let $f = u_1u_2u_3u_4u_1$ and let the neighbour of u_i not on f itself be denoted v_i. By symmetry, consider the following two distinct cases.

1. $u_iv_i \in M$ for $1 \leq i \leq 4$. In this case, let

$$H = G - \{u_i : 1 \leq i \leq 4\} \cup \{v_1v_4, v_2v_3\}.$$

Then H is a cubic bipartite plane graph with $|V(H)| < |V(G)|$. Let

$$M' = M + v_1v_4 + v_2v_3 + \{u_iv_i : 1 \leq i \leq 4\}.$$

Then clearly, M' is a perfect matching of H. By the inductive assumption, H has a face h whose boundary, denoted by ∂h, is M'-alternating.

If $\partial h \cap \{v_1v_4, v_2v_3\} = \emptyset$, then h itself is a face of G and ∂h is an M-alternating cycle of G.

If $\partial h \cap \{v_1v_4, v_2v_3\} = v_1v_4$, then $\partial h + v_1v_4 + u_1v_1 + u_1u_2 + u_4v_4$ is an M-alternating cycle that is the boundary of a face of G.

If $\partial h \cap \{v_1v_4, v_2v_3\} = v_2v_3$, then $\partial h + v_2v_3 + u_2v_2 + u_2u_3 + u_3v_3$ is an M-alternating cycle that is the boundary of a face of G.

If $\partial h \cap \{v_1v_4, v_2v_3\} = \{v_1v_4, v_2v_3\}$, then both ∂f_1 and ∂f_2 are M-alternating, where f_1 is the face of G that is adjacent to f at u_1u_2 and f_2 is the one adjacent to f at u_3u_4.

2. $u_1u_4, u_2v_2, u_3v_3 \in M$. Let H be the graph given in the above case and let

$$M' = M + v_2v_3 + u_1u_4 + u_2v_2 + u_3v_3.$$

Then the proof is exactly the same as that of case 1.

In summary, G has a face f which is M-alternating. ■

LEMMA 3.2 *Let G be a cubic bipartite plane graph that has a perfect matching. If f is any face of G then there is a perfect matching M such that f is M-alternating.*

Proof: In the proof of the previous lemma, take f to be any face of G. Then it has been shown that G has a perfect matching M, for which f is M-alternating except the case that f is a 4-face. However, if f is a 4-face, then take a 4-face g that is not adjacent to f and consider a reduction on g as in the proof of the above lemma. Now by the inductive hypothesis, there is a perfect matching M for which f is M-alternating. ■

Note that both these results improve upon the results of [3].

4 The Connectivity of Matching Transformation Graphs

The following theorem is the main result of [3]. The proof is included here for completeness.

THEOREM 4.1 *If G is a cubic bipartite plane graph with a perfect matching, then the matching transformation graph $\mathcal{M}(G)$ of G is connected.*

Proof: If G has only one perfect matching then the matching transformation graph of G has just one vertex and by definition it is connected. Hence let M_1 and M_2 be any two perfect matchings of G. It will be shown that there is a path connecting M_1 and M_2 in $\mathcal{M}(G)$.

Let

$$M_1 + M_2 = \{C_1, C_2, \cdots, C_s\} \cup \{D_1, D_2, \cdots, D_t\},$$

where each cycle C_i $(i = 1, 2, \cdots, s)$ bounds a face and each cycle D_i $(i = 1, 2, \cdots, t)$ does not.

Let

$$T = \bigcup_{i=1}^{t} \text{int}(D_i) = T(M_1, M_2),$$

where $\text{int}(D)$ denotes the topological interior of cycle D.

We use induction on $|T|$.

If $|T| = 0$, then every cycle in $M_1 + M_2$ bounds a face of G and by the definition of matching transformation graphs there is an edge joining M_1 and M_2 in $\mathcal{M}(G)$.

Assume inductively that for $|T| < n$ the graph $\mathcal{M}(G)$ has a path connecting M_1 and M_2. We then prove that the assertion holds also for $|T| = n$.

Suppose without loss of generality that D_1 is a cycle whose interior does not contain any other D_i's. Since D_1 is not the boundary of any face of G, and since G has a perfect matching, $\text{int}(D_1)$ contains at least two vertices. Since D_1 is M_1-alternating as well as M_2-alternating, no edge that has an end in $\text{int}(D_1)$ and an end on D_1 belongs to M_1 or to M_2. Therefore, the induced subgraph $G|_{\text{int}(D_1)}$ has both a perfect matching $M_1 \cap \text{int}(D_1)$ and a perfect matching $M_2 \cap \text{int}(D_1)$. Hence, each vertex in $\text{int}(D_1)$ is saturated by both M_1 and M_2. Hence, $\text{int}(D_1)$ has vertices v_1 and v_2 adjacent to vertices of D_1, that are connected by an M_1-alternating path Q whose terminal edges are in M_1.

Let u_i be the vertex of D_1 that is adjacent to v_i ($i = 1, 2$). Assign a temporary orientation to D_1 and let $P_1 = D_1^+(u_1, u_2)$ and $P_2 = D_1^-(u_1, u_2)$. Since D_1 is both M_1-alternating and M_2-alternating, hence both terminal edges of P_1 and those of P_2 belong to M_1 or to M_2, respectively; for otherwise, a contradiction to the assumption that G is bipartite. It may then be assumed that both terminal edges of P_1 are edges of M_1. Then both terminal edges of P_2 are edges of M_2. Let $D = P_1 \cup Q \cup \{u_1v_1, u_2v_2\}$. Then D is an M_1-alternating cycle. (Note that D is not necessarily M_2-alternating.)

Let $M_3 = M_1 + D$. Then M_3 is also a perfect matching of G. By the definition of M_3, we have $M_1 + M_3 = D$. Also, $\text{int}(D) \subset \text{int}(D_1)$, i.e., $|T(M_1, M_3)| < |T(M_1, M_2)| = n$. By the inductive assumption, $\mathcal{M}(G)$ has a path connecting M_1 and M_3.

Now $M_2 + M_3 \subseteq \{D', C_2, \cdots, C_s\} \cup \{D', D_2, \cdots, D_t\}$, where D' is a cycle contained in the interior of D_1 and $\text{int}(D') \subset \text{int}(D_1)$. Hence, $|T(M_2, M_3)| < n$. By the inductive assumption, $\mathcal{M}(G)$ has a path connecting M_2 and M_3. Therefore, $\mathcal{M}(G)$ has a path connecting M_1 and M_2. ■

It was also noted in [3] that all 1-factors of a cubic bipartite plane graph can be visited starting from a given 1-factor since the matching transformation graph is connected. This provides an algorithm. This algorithm works by means of an operation called *rotation*. Let M_1 be any perfect matching of G. Then by the lemma above, G has a face f that is M_1-alternating. Let $M_2 = M_1 + \partial f$. Then M_2 is also a perfect matching of G. By means of rotations, all perfect matchings of G can be visited, though some will be visited more than once. Therefore, a natural question is: What conditions on G guarantee that $\mathcal{M}(G)$ is hamiltonian? If the hamiltonicity is difficult to establish, then how about 2-connectedness?

It would be of interest also to know whether the above theorem achieves a kind of sharpness. In the next section, cubic plane graphs will be constructed, whose matching transformation graphs are completely disconnected.

5 Cubic Plane Graphs

Let M be a perfect matching of a cubic plane graph G. A natural way for a face f not to be M-alternating is that it to have an odd size. If all the faces of G are odd, then the matching transformation graph of G consists of isolated vertices, i.e., it is completely disconnected.

Consider a 3-connected cubic plane graph G with all faces odd.

LEMMA 5.1 *If G is a cubic plane graph without even face, then*

$$|G| \equiv 0 \pmod 4.$$

Proof: Since G must be of even order as it is cubic, there is an integer n such that $|V(G)| = 2n, |E(G)| = 3n$ and $|\Phi(G)| = n+2$ by Euler's formula. Since any plane graph has an even number of odd faces,

$$|\Phi(G)| = n + 2 \equiv 0 \pmod 2.$$

That is,

$$n \equiv 0 \pmod 2,$$

and therefore

$$|G| = 2n \equiv 0 \pmod 4.$$

■

Denote, as above, by p_k the number of k-faces of G. Then

$$3p_3 + 2p_4 + p_5 = 12 + \sum_{k \geq 6} (k-6)p_k.$$

Since all faces are of odd sizes,

$$p_k = 0, \quad \text{for } k \equiv 0 \pmod 2.$$

Hence,

$$3p_3 + p_5 = 12 + \sum_{k \geq 7,\, k \equiv 1 \pmod 2} (k-6)p_k.$$

Clearly,

$$3p_3 + p_5 \geq 12.$$

Consider, for example, $3p_3 + p_5 = 12$. In this case, if $p_3 = 0$ then G is isomorphic to the dodecahedron. $p_3 = 1, 2, 3$ result in contradictions, and $p_3 = 4$ gives $p_5 = 0$ and $G \simeq K_4$.

Let $p = 2k + 1$ be any sufficiently large odd integer, $k \geq 3$. Then we may construct a cubic plane graph with

$$p_5 = 4k + 2, \quad (3)$$

$$p_{2k+1} = 2 \quad (4)$$

and

$$p_l = 0, \quad \text{for } l \neq 5,\ 2k + 1.$$

Construct a cubic plane graph as follows.

Let

$$U = \{u_i : 1 \leq i \leq 2k + 1\},\ V = \{v_i : 1 \leq i \leq 2k + 1\},$$

$$X = \{x_i : 1 \leq i \leq 2k + 1\},\ Y = \{y_i : 1 \leq i \leq 2k + 1\}.$$

On the plane draw a polygon

$$C_U = u_1 u_2 u_3 \cdots u_{2k+1} u_1.$$

Then another concentric polygon

$$C_{X \cup Y} = x_1 y_1 x_2 y_2 \cdots x_{2k+1} y_{2k+1} x_1$$

in the interior of C_U. Then a third concentric polygon

$$C_V = v_1 v_2 v_3 \cdots v_{2k+1} v_1$$

in the exterior of $C_{X \cup Y}$. Finally, introduce edges $u_i x_i$ and $v_i y_i$ for $i = 1, 2, \cdots, 2k + 1$. Denote the resulting graph by G_{2k+1}. Then G_{2k+1} is a cubic plane graph with $4k + 2$ pentagonal faces and two $(2k + 1)$-faces. This realizes the above sequence of numbers of faces, justifying the sharpness of Theorem 4.1.

To see that G_{2k+1} has a perfect matching, consider the following hamiltonian cycle:

$$u_1 x_1 y_1 v_1 v_2 y_2 x_2 u_2 u_3 x_3 y_3 v_3 v_4 \cdots$$

$$\cdots u_{2k-1} x_{2k-1} y_{2k-1} v_{2k-1} v_{2k} v_{2k+1} y_{2k+1} x_{2k+1} y_{2k} x_{2k} u_{2k} u_{2k+1} u_1.$$

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