LIST EDGE COLORINGS OF OUTERPLANAR GRAPHS

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ABSTRACT. The list edge coloring conjecture states that for every graph, the chromatic index equals the choice index. We prove the conjecture for outerplanar graphs with maximum degree at least five.

1. INTRODUCTION

A k-edge coloring of a (simple) graph G is an assignment of colors to the edges of G such that adjacent edges are colored differently. The minimum number k for which a graph G admits a k-edge coloring is the chromatic index $\chi'(G)$ of G.

If a list $L(e)$ is assigned to every edge e of a graph G, an L-edge coloring of G with $L = \{L(e) : e \in E(G)\}$ is defined as an edge coloring of G such that each edge e obtains a color from its list $L(e)$. If G admits a list edge coloring where every list has cardinality at least k then G is said to be k-list edge colorable. The minimum number k for which a graph G is k-list edge colorable is the choice index $ch'(G)$ of G.

The well-known list edge coloring conjecture states that the chromatic index $\chi'(G)$ and the choice index $ch'(G)$ coincide for all graphs G (see [1]). Till now, the list edge coloring conjecture is proved only for some specific classes of graphs, for example for bipartite graphs [6], complete graphs with odd order [7] and for graphs with maximum degree $\Delta(G) = 3$ and $\chi'(G) = 4$ (a consequence of the theorem of brooks for list colorings, see [4] and [8]).

The list edge coloring conjecture is also proved for planar graphs G with maximum degree $\Delta(G) \geq 12$ [2] and for regular planar graphs [3]. In this note we prove the conjecture for outerplanar graphs with maximum degree at least five.

An outerplanar graph is a plane graph such that all vertices are on the boundary of the same face, without loss of generality on the boundary of the outerface. In [5] it is proved that an outerplanar graph G is $\Delta(G)$-edge

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colorable if and only if G is not an odd cycle C_{2n+1}. Therefore, to prove the list edge coloring conjecture for outerplanar graphs it remains to show that $ch'(G) = \Delta(G)$ whenever $G \not\cong C_{2n+1}$ is outerplanar.

2. Result

In this section we prove that G is Δ-list edge colorable if G is outerplanar and has maximum degree $\Delta \geq 5$. For the proof we use the following two lemmas. Lemma 2 is based on a weaker version of a corresponding result in [9].

Lemma 1. *Every 2-connected outerplanar graph G on $p \geq 4$ vertices contains at least two non-adjacent vertices both of degree 2.*

Proof. Since G is outerplanar, it can be drawn in such a way that the vertices of the outerface form a hamiltonian cycle $C = (v_1, v_2, \ldots, v_p, v_1)$. If $p = 4$ or $G = C_p$, the result is obvious. Therefore, let G be an outerplanar graph with $p > 4$ and $\Delta(G) \geq 3$.
Let $v_i \in V(G)$ with $d(v_i) \geq 3$. Then there exists a vertex v_j which is adjacent to v_i ($v_j \sim v_i$) such that the edge $v_iv_j \notin E(C)$. Let $G_1 = \langle\{v_i, v_{i+1}, \ldots, v_{j-1}, v_j\}\rangle$ and $G_2 = \langle\{v_j, v_{j+1}, \ldots, v_{i-1}, v_i\}\rangle$ be induced subgraphs of G. Obviously, G_1 and G_2 are 2-connected and outerplanar. If $|V(G_k)| = 3$ for $k = 1$ or 2 then $V(G_k) = \{x, v_i, v_j\}$ and $d(x) = 2$. Otherwise, we can use the induction hypothesis to prove that both G_1 and G_2 contain at least one vertex y_1 and y_2, respectively, such that $y_k \notin \{v_i, v_j\}$ and $d(y_k) = 2$, $k = 1, 2$. Since $y_1 \not\sim y_2$ in G, the lemma is proved.

□

Lemma 2. *Let G be an outerplanar graph with minimum degree $\delta(G) \geq 2$. Then G has at least one of the following two properties:*

(a) *There exist two vertices of degree 2 having a common neighbor of degree 4.*

(b) *There exists a vertex of degree 2 having a neighbor of degree ≤ 3.*

Proof. Let G be drawn in such a way that the outerface contains every vertex in its boundary. Then G can be partitioned in outerplanar blocks, i.e. 2-connected maximum subgraphs of G. There exists at least one block B that contains only one cut vertex v of G (if G is 2-connected, let v be an arbitrary vertex of $V(G)$). Let D be the graph with vertex set $V(D) = V(B) \cup N(v)$, where $N(v) = \{u : u \sim v,\ u \in V(G)\}$ is the neighborhood of vertex v, and with edge set $E(D) = E(B) \cup \{uv : u \in N(v)\}$. Therefore, $d_G(w) = d_D(w)$ for all $w \in V(B)$. We show by induction that at least one of properties (a) and (b) is already fulfilled for vertices of block B.

Obviously, if $|V(B)| \leq 4$ then property (b) holds. Let $|V(B)| \geq 5$. Due to Lemma 1, there exist two non-adjacent vertices of degree 2 in block B of which at least one, say x, fulfils $d_B(x) = d_D(x)$. Let $N(x) = \{y, z\}$ and, without loss of generality, let $d_B(z) = d_D(z)$ and $d_B(y) = d_D(y) - k$, $k \geq 0$. If $d_D(y)$ or $d_D(z)$ is at most 3 then property (b) holds. Therefore, let both $d_D(y)$ and $d_D(z)$ be at least 4. Define graph D' by

$$D' = \begin{cases} D - x & \text{if } yz \in E(D) \\ D - x + yz & \text{otherwise.} \end{cases}$$

In both cases D' is outerplanar, $|V(D')| = |V(D)| - 1$, and D' contains block B' which is analogously defined by

$$B' = \begin{cases} B - x & \text{if } yz \in E(B) \\ B - x + yz & \text{otherwise.} \end{cases}$$

Therefore, we can use the induction hypothesis for D'.

a) Consider first the case that D' fulfils property (a), i.e. there exist vertices $u_1, u_2 \in V(D')$ with $d_{D'}(u_1) = d_{D'}(u_2) = 2$ which have a common neighbor w with $d_{D'}(w) = 4$. Since B' is 2-connected and outerplanar, there is a hamiltonian cycle C' in B'. Because y and z are neighbors on C' and $d_{D'}(y)$ as well as $d_{D'}(z)$ are at least 3, we have $w \notin \{y, z\}$. Therefore, the degrees of u_1, u_2 and w in D, respectively, equal the appropriate degrees in D'. This implies that property (a) also holds in graph D.

b) If otherwise property (b) is fulfilled in D', then there exists a vertex u of degree $d_{D'}(u) = 2$ which is adjacent to a vertex w of degree ≤ 3. Suppose first that $D' = D - x$. If $w = y$ or $w = z$ in D', then $d_D(w) = d_{D'}(w) + 1 \leq 4$ and therefore $d_D(w) = 4$, which implies that x and v are vertices of degree 2 in D having a common neighbor w of degree 4. Thus property (a) is fulfilled in D. If $w \notin \{y, z\}$ then $d_D(v) = d_{D'}(v)$ and $d_D(w) = d_{D'}(w)$. Hence property (b) also holds in D.
If $D' = D - x + yz$, then $w \notin \{y, z\}$ since y and z both have degree ≥ 4 in D'. This implies that property (b) is fulfilled in D, which concludes the proof.

□

We use Lemma 1 and Lemma 2 to confirm the list edge coloring conjecture for outerplanar graphs with maximum degree at least 5.

Theorem. *If G is an outerplanar graph with maximum degree $\Delta(G) \geq 5$, then $ch'(G) = \Delta(G)$.*

Proof. We assume that the edges of G are not list colorable with $\Delta(G)$ colors. Let G' be a subgraph of G with minimum number of edges such

that $ch'(G) > \Delta(G)$, and let $L = \{L(e) : e \in E(G')\}$ be a family of lists such that $|L(e)| = \Delta(G)$ for all $e \in E(G')$ and G' is not L-edge colorable. Obviously, G' is outerplanar.
If there was a vertex of degree 1, then G' would contain an edge e adjacent to at most $\Delta(G) - 1$ edges in G'. According to the minimality of G', the graph $G' - e$ would have an L-list coloring assigning at most $\Delta(G) - 1$ different colors to the adjacent edges of e. Therefore, there would be at least one additional color of $L(e)$ which could be assigned to edge e, implying an L-edge coloring of G' in contradiction to the assumption. Hence $\delta(G') \geq 2$.
According to Lemma 2, G' fulfils at least one of the properties (a) and (b). It follows that there exists an edge e in G with at most $4 \leq \Delta(G) - 1$ adjacent edges, which implies as above that G' is L-edge colorable, contradicting the assumption. Hence G is $\Delta(G)$-list edge colorable.

□

3. Remarks

Considering outerplanar graphs G with $\Delta(G) = 4$, Lemma 2 again implies that at least one of properties (a) and (b) is fulfilled.
In the case that (a) holds, i.e. there exists a vertex of degree 2 in G adjacent to a vertex of degree at most 3, it can be proved in exactly the same way that $ch'(G) = \Delta(G)$.
If property (a) does not hold, then the method of the proof is not sufficient to confirm the list edge coloring conjecture for G. The case $\Delta = 3$ also needs a different treatment.
As we have been recently informed, there is an independent unpublished proof of the main result for all outerplanar graphs, using different methods, by Juvan and Mohar.

References

[1] B. Bollobás and A. J. Harris, *List-colourings of graphs.* Graphs and Combinatorics 1 (1985), 115–127.

[2] O. V. Borodin, A. V. Kostochka and D. R. Woodall, *List edge and list total colourings of multigraphs.* Journal of Combinatorial Theory, Series B 71 (1997), 184–204.

[3] M. N. Ellingham and L. Goddyn, *List edge colourings of some 1-factorable multigraphs.* Combinatorica 16 (1996), 343–352.

[4] P. Erdös, A. L. Rubin and H. Taylor, *Choosability in graphs.* Proceeding of the West Coast Conference on Combinatorics, Graph Theory and Computing, Congressus Numerantium 26 (1979), 125–157.

[5] S. Fiorini, *On the chromatic index of outerplanar graphs.* Journal of Combinatorial Theory, Series B 18 (1975), 35–38.

[6] F. Galvin, *The list chromatic index of a bipartite multigraph.* Journal of Combinatorial Theory, Series B 63 (1995), 153–158.

[7] R. Häggkvist and J. C. M. Janssen, *New bounds on the list-chromatic index of the complete graph and other simple graphs.* Research Report 19, Department of Mathematics, University of Umeå (1993).

[8] V. G. Vizing, *Coloring the vertices of a graph in prescribed colors (in Russian).* Diskret. Analiz. 29, Metody Diskret. Anal. v. Teorii Kodov i Shem 101 (1976), 3–10.

[9] H. P. Yap, *Total colourings of graphs.* Lecture Notes in Mathematics 1623, Springer-Verlag, Berlin (1996).

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