On the matching polynomials of the complete n-partite graph

Masakazu Nihei
Fujishiro High School
Fujishiro, Ibaraki, 300-1537
Japan

ABSTRACT. Let G be a simple graph with n vertices. $p(G,k)$ denotes the number of ways in which one can select k independent edges in G $(k \geq 1)$. let $p(G,0) = 1$ for all G. Then the matching polynomial $\alpha(G)$ of a graph G is given by

$$\alpha(G) = \alpha(G,x) = \sum_{k=0}^{[n/2]} (-1)^k p(G,k) x^{n-2k}.$$

In this article we give the matching polynomials of the complete n-partite graph with a differential operator.

1 Introduction

We consider finite undirected graphs without loops or multiple edges. We denote the set of vertices and edges of a graph G by $V(G)$ and $E(G)$, respectively.

Let G be a graph with n vertices. Two distinct vertices or edges in a graph G are independent if they are not adjacent in G. $p(G,k)$ denotes the number of ways in which one can select k independent edges in G $(k \geq 1)$. Furthermore, let $p(G,0) = 1$ for all G. Then the matching polynomial $\alpha(G)$ of a graph G is given by

$$\alpha(G) = \alpha(G,x) = \sum_{k=0}^{[n/2]} (-1)^k p(G,k) x^{n-2k}.$$

where the symbol $[x]$, for real number x, represents the greatest integer not exceeding x.

Farrell [3] named $\alpha(G)$ the matching polynomial in 1977. Hosoya [5] used $\alpha(G)$ in chemical thermodynamics.

The matching polynomial referred to in the paper is also known as the acyclic polynomial. It was introduced by Ivan Gutman [6]. The relationship between this matching polynomial and the general matching polynomial was given by Farrell [7].

Now, we present a number of recursion formulas for $\alpha(G)$ which we need in this article.

Let v and e be a vertex and an edge of G, respectively. $G-v$ ($|(G)| \geq 2$) denotes the subgraph with vertex set $V(G)-\{v\}$ and whose edges are all those of G not incident with v, and $G-e$ is the subgraph having vertex $V(G)$ and edge set $E(G)-\{e\}$.

Let e be an edge of G incident to the vertices v and w. Among the $p(G,k)$ selections of independent edges, there are $p(G-e,k)$ selections which do not contain e and p $(G-v-w,k-1)$ selections which contain e. Thus, $p(G,k)=p(G-e,k)+p(G-v-w,k-1)$. This yields

Theorem A. [2,4].

$$\alpha(G)=\alpha(G-e)-\alpha(G-v-w).$$

Corollary B. [2,4]. *Let the vertex v be adjacent to the vertices $w_1,\ldots,w_d$. Let further $H=G-v$. Then*

$$\alpha(G)=x\alpha(H)-\sum_{j=1}^{d}\alpha(H-w_j)$$

Corollary C. [2,4]. *Let $V(G)=\{v_1,\ldots,v_n\}$. Then*

$$d\alpha(G,x)/dx=\sum_{j=1}^{n}\alpha(G-v_j,x),$$

where $d\alpha(G,x)/dx$ is the derivative of $\alpha(G,x)$.

A graph G is n-partite, $n\geq 2$, if it is possible to partition into n subsets $V_1,\ldots,V_n$ (called partite sets) such that every element of $E(G)$ joins a vertex of V_i to a vertex of V_j, $i\neq j$. A complete n-partite graph G is an n-partite graph with partite sets $V_1,\ldots,V_n$ having the added property that if $u\in V_i$ and $v\in V_i$, $i\neq j$, then $uv\in E(G)$. If $|V_i|=p_i$, then this graph is denoted by $K(p_1,\ldots,p_n)$. Then the following results are already known [2].

$$\alpha(K(n,m),x)=(-1)^m m! x^{n-m} L_m^{n-m}(x^2) \ (n\geq m\geq 1),$$

where $L_m^{n-m}(x)$ is the Laguerre polynomials.

In this article, we will give the matching polynomials of the complete n-partite graph with a differential operator.

2 Main Resulls

Firstly we introduce some notations .

The join of graphs G_1 and G_2 is the graph $G = G_1 + G_2$ with the vertex-set $V(G) = V(G_1) \cup V(G_2)$ and the edge-set $E(G) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1) \text{ and } v \in V(G_2)\}$.

The complement G of a graph G is the graph with vertex set $V(G)$ such that two vertices are adjacent in G if and only if these vertices are not adjacent in G. $N_G(v)$ denotes the set of the vertices adjacent to v in G and K_n denotes the complete graph with n vertices.

For the sake of brevity, we denote d/dx by D, and $\sum_{k=0}^{n}(-1)^k\binom{n}{k}x^{n-k}d^k/dx^k$ by $(x-D)^n$. Moreover, let m and n be positive integers. Then we write $(x-D)^m(x-D)^n\alpha(x)$ for $(x-D)^m\{(x-D)^n\alpha(x)\}$. But note that $(x-D)^m\{(x-D)^n\alpha(x)\}$ is not equal to $(x-D)^{m+n}\alpha(x)$. But the following claim holds:

Claim: *Let $\alpha(x)$ be an at least $(n+1)$ times differentiable function. Then*

$$\{x(x-D)^n - (x-D)^n D\}\alpha(x) = (x-D)^{n+1}\alpha(x).$$

Proof:

$$\begin{aligned}
&\{x(x-D)^n - (x-D)^n D\}\alpha(x) \\
&= \{x(\sum_{k=0}^{n}(-1)^k\binom{n}{k}x^{n-k}D^k) - (\sum_{k=0}^{n}(-1)^k\binom{n}{k}x^{n-k}D^k)D\}\alpha(x) \\
&= \{x^{n+1} + \sum_{k=1}^{n}(-1)^k\{\binom{n}{k-1} + \binom{n}{k}\}x^{n-k+1}D^k + (-1)^{n+1}D^{n+1}\}\alpha(x) \\
&= \{x^{n+1} + \sum_{k=1}^{n}(-1)^k\binom{n+1}{k}x^{n-k+1}D^k + (-1)^{n+1}D^{n+1}\}\alpha(x) \\
&= (x-D)^{n+1}\alpha(x).
\end{aligned}$$

□

Here, we present two lemmas.

Lemma 1. *Let G be a graph. Then*

$$\alpha(K_1 + G, x) = (x-D)\alpha(G, x).$$

Proof: Let $V(G) = \{w_1, \ldots, w_p\}$ and $K_1 = \{v\}$. Put $K_1 + G = H$. Then, from Corollary B we obtain

$$\alpha(H,x) = x\alpha(H-v,x) - \sum_{j=1}^{p}\alpha(H-v-w_j,x) = x\alpha(G) - \sum_{j=1}^{p}\alpha(G-w_j,x).$$

Thus, we have the desired result from $N_H(v) = \{w_1, \ldots, w_p\}$ and Corollary C. □

In general, the following result holds.

Lemma 2. *Let G be a graph. Then*

$$\alpha(\overline{K}_n + G, x) = (x - D)^n \alpha(G, x).$$

Proof: The proof is by induction on the number of vertices of $\overline{K}_n$. If $p = 1$, Lemma 2 holds by Lemma 1.

Now, assume that Lemma 2 is true for $n \leq m$. Then we may prove that Lemma 2 holds for $n = m + 1$.

Let $V(G) = \{w_1, \ldots, w_p\}$. From Corollary B we have

$$\alpha(\overline{K}_{m+1} + G, x) = x\alpha(\overline{K}_m + G, x) - \sum_{j=1}^{p} \alpha(\overline{K}_m + G - w_j, x).$$

Consequently, from the induction hypothesis and $\overline{K}_m + (G - w_j) = (\overline{K}_m + G) - w_j$, we obtain

$$\begin{aligned}\alpha(\overline{K}_{m+1} + G, x) &= x\alpha(\overline{K}_m + G, x) - \sum_{j=1}^{p} \alpha(\overline{K}_m + (G - w_j), x)\\ &= x(x - D)^m \alpha(G, x) - \sum_{j=1}^{p} (x - D)^m \alpha(G - w_j, x)\\ &= x(x - D)^m \alpha(G, x) - (x - D)^m \sum_{j=1}^{p} \alpha(G - w_j))\\ &= x(x - D)^m \alpha(G, x) - (x - D)^m D\alpha(G, x).\end{aligned}$$

From Claim, we have

$$\alpha(\overline{K}_{m+1} + G, x) = (x - D)^{m+1} \alpha(G, x).$$

Hence, the proof by induction is complete. □

$\psi(G)$ denotes the number of ways in which one can select perfect machings in G. Then, as $\psi(G) = |\alpha(G, 0)|$, we have the following result from Lemma 2.

Corollary. *Let G be a graph. Then*

$$\psi(\overline{K}_n + G) = |\alpha^{(n)}(G, 0)|,$$

where $\alpha^{(n)}(x) = d^k \alpha(G, x)/dx^k$.

Theorem. $\alpha(K(p_1,\dots,p_n),x) = (x-D)^{p_n}\cdots(x-D)^{p_2}x^{p_1}$

Proof: If $n=1$, Theorem is trivial. Hence, without loss of generality we can assume $n \geq 2$.

The proof is induction on n. If $n=2$, then we have $\alpha(K(p_1,p_2)) = \alpha(\overline{K}_{p_1} + \overline{K}_{p_2}) = (x-D)^{p_2}\alpha(\overline{K}_{p_1}) = (x-D)^{p_2}x^{p_1}$ from Lemma 2.

Now, assume that Theorem is true for $n-1$. Since $K(p_1,\dots,p_n) = \overline{K}_{p_n} + K(p_1,\dots,p_{n-1})$, we obtain $\alpha(K(p_1,\dots,p_n) = (x-D)^{p_n}\alpha(K(p_1,\dots,p_{n-1})$ from Lemma 2. Thus, by the induction hypothesis

$$\alpha(K(p_1,\dots,p_n),x) = (x-D)^{p_n}\cdots(x-D)^{p_2}x^{p_1}$$

This completes the proof by induction. □

Lastly, let us give concrete example.

Example.

(1) $\alpha(K(1,2,3) = (x-D)^3(x-D)^2x = (x-D)^3\{(x-D)^2x\} = (x-D)^3(x^3-2x) = x^6 - 11x^4 + 24x^2 - 6.$

From Corollary, we also have

$$\psi(K(1,2,3,4) = \psi(\overline{K}_4 + K(1,2,3)) = |\alpha^{(4)}(K(1,2,3),0)| = 264.$$

(2) $\alpha(K(3,3,3) = (x-D)^3(x-D)^3x^3 = (x-D)^3\{(x-D)^3x^3\} = (x-D)^3\{x^6 - 9x^4 + 18x^2 - 6\} = x^9 - 27x^7 + 216x^5 - 558x^3 + 324x.$

$$\psi(K(3,3,3,3) = \psi(\overline{K}_3 + K(3,3,3)) = |\alpha^{(3)}(K(3,3,3),0)| = 3348.$$

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