On the Chromatic Uniqueness of Edge-Gluing of Complete Bipartite Graphs and Cycles

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Abstract

In this paper, we give an alternative proof for the fact that the graph obtained by overlapping the cycle C_m ($m \geq 3$) and the complete bipartite graph $K_{2,s}$ ($s \geq 1$) at an edge is uniquely determined by its chromatic polynomial. This result provides a partial solution to a question raised in [7].

Let $P(G;\lambda)$ denote the chromatic polynomial of a graph G. Then G is said to be *chromatically unique* if it is uniquely determined by its chromatic polynomial. A graph is *vertex-transitive* (respectively *edge-transitive*) if its group of automorphisms acts transitively on the vertex-set (respectively edge-set).

A graph is G *quasi-separable* if it contains a complete subgraph K_n such that $G - K_n$ is disconnected (see [1]). A *quasi-block* Q of G is a maximal subgraph of G that is not quasi-separable.

Suppose G consists of two quasi-blocks Q_1 and Q_2 with $Q_1 \cap Q_2 = K_n$. Then G is said to have property $\mathcal{P}$ if for every $i = 1, 2$, there exists $x_i \in Q_i$ such that x_i is adjacent to all the vertices of $Q_1 \cap Q_2 = K_n$ and that $N(x_i) - K_n$ is nonempty, where $N(x)$ denotes the neighborhood of x.

Theorem 1 *([5], [11])*

Let G be a graph consisting of two quasi-blocks Q_1 and Q_2 with $Q_1 \cap Q_2 = K_2$. Suppose G is chromatically unique. Then

(i) G does not have property $\mathcal{P}$;

(ii) Q_1 and Q_2 are chromatically unique;

(iii) Q_1 and Q_2 are edge-transitive. Further, at least one of Q_1 or Q_2 is vertex-transitive.

Question 5 in [7] asks whether or not the converse of Theorem 1 is true. No counter-examples to this question are found. An example of a family of graphs that satisfies the necessary conditions of Theorem 1 is the edge-gluing of $K_{r,s}$ and C_m denoted $K_{r,s} \cup_2 C_m$. In this paper, we show that $K_{2,s} \cup_2 C_m$ is chromatically unique for all $s \geq 1$ and all $m \geq 3$. This result is also obtained by Xu, Liu and Peng [12] independently by different method.

We need to make use of the following result of Chao and Zhao [3]. By a K_4 – *homeomorph* we mean a subdivision of K_4.

Theorem 2 *([3])*

Let G be a connected graph and let $P(G;\lambda) = (\lambda - 1)T(G;\lambda)$. Then

(i) $|T(G;1)| = 1$ if and only if G is a 2-connected graph and contains no K_4-homeomorph as a subgraph, and

(ii) $|T(G;1)| \geq 2$ if and only if G is a 2-connected graph and contains at least one K_4-homeomorph as a subgraph.

The following lemma is a consequence of Theorem 2 of [8]. Let G be a graph and let A be a subgraph of G. Let $n(A, G)$ denote the number of subgraphs A in G.

Lemma 1 *Let G and Y be two graphs such that $P(G;\lambda) = P(Y;\lambda)$. Then G and Y have the same number of vertices, edges and triangles. Moreover, in the event that G has at most one triangle, then $n(C_4^*, G) = n(C_4^*, Y)$ and*

$$-n(C_5^*, G) + n(K_{2,3}, G) = -n(C_5^*, Y) + n(K_{2,3}, Y).$$

Let G be a connected graph on p vertices and q edges. Then the *cyclomatic number* of G is $q - p + 1$.

Lemma 2 *Let G be a connected graph with cyclomatic number c. Then the number of $K_{2,3}$ in G is at most $\binom{c+1}{3}$.*

Proof: By induction on c.

If $c \leq 2$, the result is trivially true. Suppose the result is true for all connected graphs with cyclomatic number c where $c \geq 2$.

Let G be a connected graph with cyclomatic number $c+1$. Then G contains a cycle C. Delete an edge e from C. The resulting graph $G-e$ is connected and has cyclomatic number c. By the induction hypothesis, the number of $K_{2,3}$ in $G-e$ is at most $\binom{c+1}{3}$.

Let $\{K_{2,s_1}, \ldots, K_{2,s_t}\}$ denote the set of all subgraphs (which are complete bipartite graphs) in G containing the edge e. Here $s_i \geq 3$ for $i = 1, \ldots, t$. Notice that $(s_1 - 1) + \cdots + (s_t - 1) \leq c + 1$. Then the number of $K_{2,3}$ in G containing the edge e is $\sum_{i=1}^{t} \binom{s_i - 1}{2}$. It is a routine exercise to show that this number is no more than $\binom{c+1}{2}$. Consequently, the number of $K_{2,3}$ in G is at most $\binom{c+1}{3} + \binom{c+1}{2} = \binom{c+2}{3}$ and this furnishes the proof. □

Lemma 3 *Suppose $m_i \geq 3$ is an integer for $i = 1, \ldots t$. Then*

$$\sum_{i=1}^{t} \binom{m_i}{3} \leq \binom{(\sum_{i=1}^{t} m_i) - 2(t-1)}{3} - 2(t-1).$$

Proof: The lemma is trivially true if $t = 1$. It is routine to verify that

$$\binom{m_1}{3} + \binom{m_2}{3} \leq \binom{m_1 + m_2 - 2}{3} - 2.$$

By repeatedly applying the above inequality, the lemma follows. □

Let H be a graph containing a subgraph of the form $K_{2,l}$ for some $l \geq 2$. Let x be a vertex in $H - K_{2,l}$. Then x is called a *t-vertex* to $K_{2,l}$ if x is adjacent to only two vertices of $K_{2,l}$ so that the resulting subgraph $K_{2,l} \cup \{x\}$ is isomorphic to $K_{2,l+1}$.

Theorem 3 *For any $s \geq 1$ and $m \geq 3$, the graph $G = K_{2,s} \cup_2 C_m$ is uniquely determined by its chromatic polynomial.*

Proof: Let Y be a graph such that $P(Y;\lambda) = P(G;\lambda)$. Then Y is a 2-connected graph on $s+m$ vertices and $2s+m-1$ edges. By Theorem 2, Y contains no K_4-homeomorph as a subgraph because G contains no such subgraph.

When $s = 1$, $K_{2,s} \cup_2 C_m$ is the vertex-gluing of K_2 and C_m. It is chromatically unique (see [4]).

When $s = 2$, $K_{2,s} \cup_2 C_m$ is the θ-graph and is chromatically unique (see [2] and [10]). The case $s = 3$ has been treated in [6] where it is shown that $K_{2,3} \cup_2 C_m$ is chromatically unique. So we may assume that $s \geq 4$.

Since G has at most one triangle and $n(K_{2,3}, G) = \binom{s}{3}$, by Lemma 1, $n(K_{2,3}, Y) \geq \binom{s}{3}$ if $m \neq 5$ and $n(K_{2,3}, Y) \geq \binom{s}{3} - 1$ if $m = 5$. In either case, we see that Y contains a subgraph $K_{2,3}$. Let K denote this subgraph.

Let J be the graph $Y - K$ and assume that there are e edges joining K to J. Now note that J has $s + m - 5$ vertices and $2s + m - 7 - e$ edges and so

$$|E(J)| - |V(J)| = s - e - 2.$$

Let $J_1, \ldots, J_k$ be the connected components of J, $k \geq 1$. Suppose there are e_i edges joining K and J_i, $i = 1, \ldots, k$.

We make the following observations:

$(O1)$: Each J_i contains at most one t-vertex to K. This is because if there are two t-vertices x_1 and x_2 from J_i to K, then there is a path in J_i connecting x_1 and x_2. This path together with K contains a K_4-homeomorph as a subgraph which is impossible.

$(O2)$: If $e_i = 2$, then J_i contains a t-vertex only if J_i is an isolated vertex because Y is 2-connected.

Let c_i denote the cyclomatic number of J_i, $i = 1, \ldots, k$. Then $\sum_{i=1}^{k} c_i = s - e - 2 + k$. Consequently, $e \leq s - 2 + k$. Since $e \geq 2k$, it follows that $1 \leq k \leq s - 2$. Let β denote the number of J_i's that are isolated vertices. Then clearly, $\beta \leq k - 1$.

There are two cases that we need to consider.

Case (1): All the J_i's are trees.

Assume that $e_i = 2$ for $i = 1, \ldots, k$. Then $k = s - 2$. From $(O2)$, each isolated vertex of J could be a t-vertex to K and so

$$n(K_{2,3}, Y) \leq \binom{3+\beta}{3} \leq \binom{s}{3}.$$

Clearly, the second inequality holds if $\beta = k - 1$. When $\beta \leq k - 2$, $\binom{3+\beta}{3} < \binom{s}{3} - 1$.

Suppose $\beta = k - 1$. Then one of the J_i, say J_k, is the path on $m - 2$ vertices and $J_1, \ldots, J_{k-1}$ are t-vertices to K. Now, the two edges joining J_k

and K are not incident to a common vertex in K or in J_k. Moreover, these two edges must join the two end vertices of J_k to two adjacent vertices in K. This is because otherwise either Y contains a K_4-homeomorph as a subgraph or $P(Y;\lambda) \neq P(G;\lambda)$. But then $Y \cong G$.

Assume that $e_i \geq 3$ for some i. Then $k \leq s-3$. Since each isolated vertex in J contributes at most one t-vertex to K, we have

$$n(K_{2,3},Y) \leq \binom{3+\beta+1}{3} \leq \binom{s}{3}.$$

Clearly, the second inequality holds if $\beta = k-1$ and $k = s-3$. When $\beta \leq k-2$, $\binom{4+\beta}{3} < \binom{s}{3} - 1$.

Suppose $\beta = s-4 = k-1$. Then one of the J_i, say J_k, is a tree on $m-2$ vertices and $J_1,\dots,J_{k-1}$ are t-vertices to K. Since one of the end vertices of J_k is a t-vertex to K, J_k is a path. Now, the other end vertex of J_k must be adjacent to a vertex in K which is not of degree 2 because otherwise Y contains a K_4-homeomorph as a subgraph. But then $Y \cong G$.

Case (2): Not all the J_i's are trees.

Assume that $J_1,\dots,J_t$ are not trees and $J_{t+1},\dots,J_k$ are trees so that $c_1,\dots,c_t \geq 1$ and $c_{t+1},\dots,c_k = 0$ for some $t \geq 1$.

Consider the subgraph induced by the vertices of $J_i \cup K$. For each i, let H_i denote the graph obtained from $J_i \cup K$ by deleting all the edges in K. Let α_i denote the number of isolated vertices in H_i. Then $\alpha_i \leq 3$.

Let H_i' denote the graph obtained from H_i by deleting all the α_i isolated vertices. Then H_i' is a connected graph with cyclomatic number $c_i + e_i - 5 + \alpha_i \leq c_i + e_i - 2$. By Lemma 2, $n(K_{2,3},H_i') \leq \binom{c_i+e_i-1}{3}$.

By $(O1)$, since each J_i contributes at most one t-vertex, we have

$$\begin{aligned} n(K_{2,3},Y) &\leq \binom{k+3}{3} + \sum_{i=1}^{t}\binom{c_i+e_i-1}{3} \\ &\leq \binom{k+3}{3} + \binom{\sum_{i=1}^{t}(c_i+e_i-1) - 2(t-1)}{3} - 2(t-1) \end{aligned}$$

by Lemma 3.

Now observe that

$$\sum_{i=1}^{t}(c_i+e_i-1) - 2(t-1) = \sum_{i=1}^{k} c_i + \sum_{i=1}^{t} e_i - 3t + 2$$

$$
\begin{aligned}
&= (s-e-2+k)+e-\sum_{i=t+1}^{k} e_i - 3t + 2 \\
&\le (s-2+k) - 2(k-t) - 3t + 2 \\
&= s-t-k \\
&\le s-k-1.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
n(K_{2,3}, Y) &\le \binom{k+3}{3} + \binom{s-k-1}{3} - 2(t-1) \\
&\le \binom{s}{3} - 2t \qquad \text{by Lemma 3} \\
&\le \binom{s}{3} - 2 \qquad \text{because } t \ge 1.
\end{aligned}
$$

This completes the proof of the theorem. □

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