On the Girth of Digraphs with High Connectivity

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Abstract

In 1970, Behzad, Chartrand and Wall conjectured that the girth of every r-regular digraph G of order n is at most $\lceil n/r \rceil$. The conjecture follows from a theorem of Menger and Dirac if G has strong connectivity $\kappa = r$. We show that any digraph with minimum indegree and outdegree at least r has girth at most $\lceil n/r \rceil$ if $\kappa = r - 1$. We also find from the literature a family of counterexamples to a conjecture of Seymour.

Let $G = (V, E)$ denote a digraph on n vertices. Loops are permitted but no multiple arcs. If G has at least one (directed) cycle, the minimum length of a cycle in G is called the *girth* of G, denoted $g(G)$. G is said to be *r-regular* if the indegree and outdegree of each vertex both equal r.

Various authors (e.g. [1], [2] and [3]) have considered the problem of determining the minimum number of vertices $f(r, g)$ in an r-regular digraph with girth g. In [1], M. Behzad, G. Chartrand and C. Wall showed that $f(r, g) \leq r(g-1) + 1$ by taking G to be the digraph with vertices v_i, $1 \leq i \leq r(g-1)+1$, where $v_i \to v_j$ if and only if $j = i+1, i+2, \ldots, i+r$, and addition is taken modulo $r(g-1)+1$. In the same paper, they conjectured that $f(r, g) = r(g-1) + 1$. This is equivalent to the following conjecture:

Conjecture 1 ([1]) *Let G be an r-regular digraph of order n. Then* $g \leq \lceil n/r \rceil$.

If we let $\delta^+(G)$ (resp. $\delta^-(G)$) denote the minimum outdegree (resp. indegree) of G, then each of the following conjectures implies Conjecture 1.

Conjecture 2 *Let G be a digraph of order n with* $\min\{\delta^+(G), \delta^-(G)\} \geq r$. *Then $g \leq \lceil n/r \rceil$.*

Conjecture 3 ([5]) *Let G be a digraph of order n with $\delta^+(G) \geq r$. Then $g \leq \lceil n/r \rceil$.*

Some partial results have been obtained. For example, the strongest of the conjectures (Conjecture 3) has been established for $r \leq 5$ in a series of papers [5], [8], [9]. If $n \geq 2r^2 - 3r + 1$, then Conjecture 3 follows from a recent result of Shen [12]. It can also be shown to hold when $n \leq r(3 + \sqrt{7})/2$ [13]. Thus the remaining cases in which Conjecture 3 is open are when $r(3 + \sqrt{7})/2 < n \leq 2r^2 - 3r$, where $r \geq 6$.

Suppose G is strongly connected. A proper subset $T \subset V$ is said to be a *cutset* of G if $G_{V \setminus T}$, the subdigraph induced by $V \setminus T$, is not strongly connected. G is said to be *strongly h-connected* ($h \geq 1$) if $|V| \geq h + 1$ and $G_{V \setminus S}$ is strongly connected for every $S \subset V$ such that $|S| < h$. The *strong connectivity* of G is $\kappa(G) = \max\{h : G \text{ is strongly } h\text{-connected}\}$. If there is no arc $u \to v$ for some pair of distinct vertices u, v, then $\kappa(G)$ is the minimum cardinality of a cutset of G.

Using a theorem of Menger and Dirac, Hamidoune [7, Lemma 4.1] has shown that if G has strong connectivity κ, then $n - 1 \geq \kappa(g - 1)$ or $g \leq \lceil n/\kappa \rceil$. Consequently, if $\kappa = r$ (so, necessarily, $\min\{\delta^+(G), \delta^-(G)\} \geq r$), then Conjectures 1 and 2 hold. Recently Seymour [11] proved that Conjecture 1 is true if $\kappa = r - 1$ and $\lceil n/r \rceil = 3$. By modifying Seymour's argument, we show in Theorem 1 that Conjecture 2 holds whenever $\kappa = r - 1$.

For a subset $S \subseteq V$, let $D_i(S)$ denote the set of vertices whose distance *from* S is equal to i. Thus, $D_0(S) = S$, and, if $i \geq 1$, then $v \in D_i(S)$ if and only if i is the smallest integer such that there is a walk of length i to v from some $u \in S$. Similarly, let $D'_i(S)$ denote the set of vertices whose distance *to* S equals i.

Following Hamidoune [7], we call a subset F of a strongly connected digraph G a *positive fragment* if $D_1(F)$ is a minimum cutset, that is, if $|D_1(F)| = \kappa(G)$ and $F \cup D_1(F)$ is a proper subset of V. A *negative fragment* of G is a positive fragment of the digraph obtained by reversing the direction of all arcs in G. A positive (resp. negative) fragment of minimum cardinality among all the positive and negative fragments is said to be a *positive* (resp. *negative*) *atom.*

Lemma 1 *Suppose G is a digraph with girth g and strong connectivity $\kappa \geq 1$. Suppose that G has a positive atom A and let $r = \min\{\delta^+(G), \kappa+1\}$. If $u \in A$, then $|D_i(u)| \geq r$ for all $1 \leq i \leq g-1$.*

Proof. We may assume that $g \geq 2$. If $|A| = 1$, then $\delta^+(G) = \kappa \leq \delta^-(G)$ as A is a positive atom. Thus $\kappa = r$. Let u be any vertex in G. Whenever $D_{i+1}(u) \neq \emptyset$, its predecessor $D_i(u)$ must be a cutset and so $|D_i(u)| \geq \kappa = r$. Because G has girth g, $D_{g-1}(u) \neq \emptyset$. Thus we need only consider the case where $D_g(u) = \emptyset$. But, in that case, all in-arcs at u must originate in $D_{g-1}(u)$ and so $|D_{g-1}(u)| \geq |D_1'(u)| \geq \delta^-(G) \geq r$. Consequently, the statement of the lemma actually holds for all vertices u in G if $|A| = 1$.

Now suppose $|A| \geq 2$. Then $\kappa < \min\{\delta^+(G), \delta^-(G)\}$ by the definition of an atom. Thus $r = \kappa+1$. Since A is an atom, the subdigraph of G induced by A is strongly connected with girth at least g. Let C be the minimum cutset $D_1(A)$ and let $B = V \setminus (A \cup C)$. Let $u \in A$. For $0 \leq i \leq g-1$, let $A_i = A \cap D_i(u)$, $B_i = B \cap D_i(u)$ and $C_i = C \cap D_i(u)$. We abuse our notation slightly and define $A_g = A \setminus \cup_{i=0}^{g-1} A_i$, $B_g = B \setminus \cup_{i=0}^{g-1} B_i$ and $C_g = C \setminus \cup_{i=0}^{g-1} C_i$. Let $t = \max\{i : B_i \neq \emptyset\}$.

Claim 1: $|A_i \cup \cup_{j=0}^{i} C_j| \geq \kappa + 1$ for all $1 \leq i \leq g-1$. Since there is no arc from A to B, we have $D_1(\cup_{j=0}^{i-1} A_j) \subseteq A_i \cup \cup_{j=0}^{i} C_j$. Also $\cup_{j=0}^{i-1} A_j \cup (A_i \cup \cup_{j=0}^{i} C_j) \subseteq V \setminus B \neq V$. Thus $D_1(\cup_{j=0}^{i-1} A_j)$ is a cutset of G. Since $\cup_{j=0}^{i-1} A_j \subseteq A \setminus A_{g-1} \neq A$, the definition of an atom implies $|D_1(\cup_{j=0}^{i-1} A_j)| \geq \kappa + 1$ and Claim 1 follows.

Claim 2: $|B_i \cup \cup_{j=i}^{g} C_j| \geq \kappa$ for all $1 \leq i \leq t-1$. Since no arc into B originates in A, we have $D_1'(\cup_{j=i+1}^{g} B_j) \subseteq B_i \cup \cup_{j=i}^{g} C_j$. Also $(\cup_{j=i+1}^{g} B_j) \cup$

$(B_i \cup \cup_{j=i}^{g} C_j) \subseteq V \setminus A \neq V$. Thus $D_1'(\cup_{j=i+1}^{g} B_j)$ is a cutset of G, and so $|B_i \cup \cup_{j=i}^{g} C_j| \geq |D_1'(\cup_{j=i+1}^{g} B_j)| \geq \kappa$.

Claim 3: $|D_i(u)| \geq \kappa + 1$ for all $1 \leq i \leq g-1$. If $1 \leq i \leq t-1$, by Claims 1 and 2,

$$|D_i(u)| = |A_i| + |B_i| + |C_i| = |A_i \cup \cup_{j=0}^{i} C_j| + |B_i \cup \cup_{j=i}^{g} C_j| - |C| \geq \kappa + 1.$$

Thus it may be supposed that $t \leq i \leq g-1$. The proof of Claim 3 now reduces to the following two cases:

Case 1: $i = g-1$. If $D_g(u) = \emptyset$, then, as before, $D_{g-1}(u) \supseteq D_1'(u)$ and so $|D_{g-1}(u)| \geq \delta^-(G) \geq \kappa + 1$. On the other hand, if $D_g(u) \neq \emptyset$, then $D_{g-1}(u)$ is a cutset, which separates $\cup_{j \geq g} D_j(u)$ from the remaining vertices of G. Since $t \leq g-1$, we have $B_g = \emptyset$. Thus $0 < |\cup_{j \geq g} D_j(u)| = |A_g| + |C_g| = |A_g| + \kappa - |\cup_{j=0}^{g-1} C_j| \leq |A_g| + |A_{g-1}| - 1 \leq |A| - 1$, where the second last inequality follows from Claim 1. By the definition of an atom, $|D_{g-1}(u)| \geq \kappa + 1$.

Case 2: $t \leq i \leq g-2$. Since $D_{g-1} \neq \emptyset$, we know that $D_i(u)$ is a cutset, which separates $\cup_{j \geq i+1} D_j(u)$ from the remaining vertices of G. Also $0 < |\cup_{j \geq i+1} D_j(u)| = |\cup_{j=i+1}^{g} A_j(u)| + |\cup_{j=i+1}^{g} C_j(u)| = |\cup_{j=i+1}^{g} A_j(u)| + \kappa - |\cup_{j=0}^{i} C_j| \leq |\cup_{j=i}^{g} A_j(u)| - 1 \leq |A| - 1$, where the second last inequality follows from Claim 1. The definition of an atom implies $|D_i(u)| \geq \kappa + 1$.

When $|A| \geq 2$, Lemma 1 follows from Claim 3, since $r = \kappa + 1$. □

Theorem 1 *Suppose G is a digraph with strong connectivity $\kappa \geq r - 1$ where $r = \min\{\delta^+(G), \delta^-(G)\}$. Then $g \leq \lceil n/r \rceil$.*

Proof. We may suppose that $\kappa \geq 1$. By reversing the directions of all arcs if necessary, we may suppose that G has a positive atom. Then, because the sets $D_i(u)$, $0 \leq i \leq g-1$, are disjoint and $D_0(u) = u$, Lemma 1 implies that $n = |V| \geq 1 + r(g-1)$. Thus $g \leq \lceil n/r \rceil$. □

Remark. In proving the bound $n - 1 \geq \kappa(g-1)$ in [7], Hamidoune uses the Menger-Dirac theorem to show that, for each vertex u in a digraph G with strong connectivity κ, there are κ (directed) cycles C_i through u such that $C_i \cap C_j = \{u\}$ for all $1 \le i < j \le \kappa$. This leads to the following conjecture of Seymour that is quoted in [4].

Conjecture 4 *Any digraph has a vertex u and* $\deg^+(u)$ *cycles C_i such that* $C_i \cap C_j = \{u\}$, $1 \le i < j \le \deg^+(u)$.

Thomassen [14] used the following construction to disprove a conjecture of Hamidoune and another conjecture of Seymour. In Lemma 2, we see that his construction also disproves Conjecture 4.

Lemma 2 ([14]) *For each natural number r, there exists a digraph H_r of minimum outdegree r such that, for any vertex u, H_r does not contain three cycles having precisely u in common pair by pair.*

Proof. H_1 can be any cycle. Suppose H_r $(r \ge 1)$ exists. Form a digraph H_{r+1} having H_r as an induced subdigraph as follows. For each vertex v in H_r, create three new vertices v_1, v_2, v_3 and arcs $v \to v_1$, $v_1 \to v_2$, $v_2 \to v_3$, $v_3 \to v_1$ as well as arcs $v_i \to w$ for each $w \in D_1(v)$ and each $i = 1, 2, 3$. Then the minimum outdegree of H_{r+1} is $r + 1$. If H_{r+1} contained a vertex u and three cycles having precisely u in common pair by pair, then u would appear in H_r and so H_r would have contained three cycles having precisely u in common pair by pair. Thus Lemma 2 follows by using induction. □

In the first paragraph of the proof of Lemma 1, we showed that $|D_i(u)| \ge \kappa$ for all u and all $1 \le i \le g - 1$. This implies that $n - 1 \ge \kappa(g - 1)$ and so gives a direct proof of the inequality $g \le \lceil n/\kappa \rceil$ without using the Menger-Dirac theorem. It also suggests the following conjecture:

Conjecture 5 *Any digraph G has a vertex u such that, for all $1 \le i \le g-1$,*

$$\sum_{j=1}^{i} |D_j(u)| \ge i \cdot \delta^+(G).$$

We note that Conjecture 5 holds for tournaments [6] ($g = 3$ for any non-transitive tournament [10]). Conjecture 5, if true, would imply Conjecture 3.

It is also natural to ask about the girths of digraphs with low strong connectivities. We are able to prove that Conjecture 2 holds for all digraphs with $\kappa = 1$. However, as the following lemma indicates, proving the stronger Conjecture 3 for the case $\kappa = 1$ would be as difficult as proving it for all digraphs.

Lemma 3 *Suppose Conjecture 3 holds for digraphs with strong connectivity* $\kappa = 1$. *Then* $g \leq \lceil (n+1)/r \rceil$ *for every digraph* G *of order* n *with* $\delta^+(G) \geq r$.

Proof. Without loss of generality, it may be supposed that G is strongly connected; otherwise, we could replace G by a strong component with minimum outdegree at least r. Suppose $u \to v$ is an arc in G. Form a digraph G' having G as an induced subdigraph as follows. Create a new vertex v' and arcs $u \to v'$ as well as $v' \to w$ for each $w \in D_1(v)$. Then G' is strongly connected, $g(G') = g(G)$ and $\delta^+(G') = \delta^+(G)$. Since $\deg^-(v') = 1$ in G', we have $\kappa(G') = 1$. Therefore the inequality $g(G) = g(G') \leq \lceil (n+1)/r \rceil$ follows from the assumption that Conjecture 3 holds for G'. □

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