Asymptotically good choice numbers of multigraphs

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Abstract

For loopless multigraphs G, the total choice number is asymptotically its fractional counterpart as the latter invariant tends to infinity. If G is embedded in the plane, then the edge-face and entire choice numbers exhibit the same "asymptotically good" behaviour. These results are based mainly on an analogous theorem of Kahn [5] for the list-chromatic index. Together with work of Kahn and others, our three results give a complete answer to a natural question: which of the seven invariants associated with list-colouring the nonempty subsets of $\{V, E, F\}$ are asymptotically good?

1 Introduction

We consider the asymptotic behaviour of various list-colouring parameters associated with loopless multigraphs G. The vertex and edge sets of G are denoted by V and E respectively. If G is embedded in the plane, then its face set is denoted by F. Each nonempty subset of $\{V, E, F\}$ corresponds to a list-colouring invariant of G; for example, $\{E\}$ corresponds to the list-chromatic index $\hat{\chi}_e$, $\{V, E\}$ to the total choice number $\hat{\chi}_{ve}$ and $\{V, E, F\}$ to the entire choice number $\hat{\chi}_{vef}$. We define these numbers more carefully below.

Since each of the seven resulting list-colouring invariants has a fractional analogue, a natural question presents itself: which of the seven integral invariants are asymptotic to their fractional counterparts, as the latter invariants tend to infinity? The corresponding question for ordinary (i.e.

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not list) colouring was answered in a short sequence of papers: [4, 6, 7]. The answer to the list version of our question begins with a result of Jeff Kahn [5] placing $\hat{\chi}_e$ on the affirmative side; see Theorem 4.

Kahn's positive result also turns out to be the key to establishing this "asymptotically good" behaviour for three of the other six list-colouring invariants presently under consideration: $\hat{\chi}_{ve}$, $\hat{\chi}_{ef}$ and $\hat{\chi}_{vef}$. The remaining three invariants, $\hat{\chi}_v$, $\hat{\chi}_f$ and $\hat{\chi}_{vf}$, are immediately classified by Thomassen's 5-choosability theorem [10] as uninteresting for our question because each is bounded by a constant (hence so is its fractional version). In this paper we complete the answer to our question by proving that $\hat{\chi}_{ve}$, $\hat{\chi}_{ef}$ and $\hat{\chi}_{vef}$ are asymptotically good.

Terminology etc.

With brevity in mind, we point to the references for any omitted notation or terminology: [1, 12] for general graph theory; [2] for graph colouring and related history; [9] for LP/IP background; [8] for fractional concepts. By *multigraph*, we mean loopless finite multigraph. We use Δ for maximum degree.

The entire chromatic and choice numbers are representative parameters starting with which the reader can easily infer definitions of other (list-) colouring invariants and concepts. An *entire colouring* of a plane multigraph G is a map $\sigma : V \cup E \cup F \to S$ — where S is a set of "colours" — such that $\sigma(X) \neq \sigma(Y)$ whenever X, Y are incident or adjacent elements, i.e. a pair of adjacent vertices, a vertex-edge pair with the edge incident on the vertex, an edge-face pair with the edge on the boundary of the face, etc.; a face touching either another face or an edge only in a vertex is not considered an adjacency. The *entire chromatic number*, χ_{vef}, is the least size of an S admitting such a colouring. For example, if G consists of two copies of K_3 joined at a single common vertex, then $\chi_{vef}(G) = 6$. The *entire choice number*, $\hat{\chi}_{vef}$, is the least integer k such that if $L(A)$ is a set (*list*) of size k for each $A \in V \cup E \cup F$, then there exists an entire colouring σ of G with $\sigma(A) \in L(A)$ for each such A. In this case we call σ an *L-colouring*. Following [12], we use hats to distinguish between a colouring parameter χ and its list version $\hat{\chi}$. We hope the subscript notation to indicate which entities of a graph are being coloured is clear; in any case, it is consistent with [11] and other articles.

Two non-asymptotic conjectures

Though our focus is on asymptotic estimates, we take a short digression in an attempt to place some of our results into the context of other research on restricted graph colouring. Considering the case when all the lists $L(A)$ are identical shows that $\hat{\chi}_v \geq \chi_v$, $\hat{\chi}_e \geq \chi_e$, $\hat{\chi}_{ve} \geq \chi_{ve}$, and so on. Some of these bounds can be far from sharp (see [2]), but the one for edge colouring seems to be tight.

Conjecture 1 *Every multigraph satisfies* $\hat{\chi}_e = \chi_e$.

Establishing Conjecture 1, the so-called "List-colouring Conjecture", is Problem 12.20 in [2], to which the reader should turn for details. The problem dates at least to 1975 and has stimulated vigorous activity, especially in recent years, which have witnessed the proofs of several special cases. More recently, Juvan et al. [3] proposed a total colouring cousin to LCC:

Conjecture 2 *Every multigraph satisfies* $\hat{\chi}_{ve} = \chi_{ve}$.

Corollary 5 below shows that Conjecture 2 is at least asymptotically correct.

Fractional colouring

As mentioned above, all of these colouring and list-colouring invariants have fractional counterparts, their linear relaxations. We use asterisks to denote fractional parameters, so, e.g., χ_e^* denotes the fractional chromatic index and $\hat{\chi}_e^*$ the fractional list-chromatic index. A few concepts underlying detailed definitions of χ_e^*, χ_{ve}^* and χ_{ef}^* are needed. Writing $\mathfrak{M}$ for the set of matchings in G, we call an $f : \mathfrak{M} \to [0,1]$ satisfying

$$\sum_{A \in M \in \mathfrak{M}} f(M) = 1 \quad \text{for each } A \in E$$

a *fractional edge colouring* of G. Note that an ordinary (integral) edge colouring arises if we restrict the range of f to $\{0,1\}$. Now

$$\chi_e^*(G) = \min\left\{ \sum_{M \in \mathfrak{M}} f(M) : f \text{ is a fractional edge colouring of } G \right\} \quad (1)$$

makes the LP defining χ_e^* explicit.

Likewise we may define χ_{ve}^* and χ_{ef}^*, but the generalizations of $\mathfrak{M}$ are more complicated. For $I \subseteq V$, let $\delta(I)$ denote the set of edges of G with exactly one end in I. A *total stable set* of G is a subset of $E \cup V$ of the form $M \cup I$, where $M \subseteq E$ is a matching, $I \subseteq V$ is a stable (independent) set, and $M \cap \delta(I) = \varnothing$. We write $\mathfrak{T}$ for the family of total stable sets of G and call an $f : \mathfrak{T} \to [0,1]$ satisfying

$$\sum_{A \in T \in \mathfrak{T}} f(T) = 1 \quad \text{for each } A \in E \cup V$$

a *fractional total colouring* of G. Then

$$\chi^*_{ve}(G) = \min\left\{\sum_{T\in\mathfrak{T}} f(T) : f \text{ is a fractional total colouring of } G\right\}. \quad (2)$$

For $\Phi \subseteq F$, let $\partial_e(\Phi)$ denote the set of edges of G on the boundary of some face in Φ. An *edge-face stable set* of G is a subset of $E \cup F$ of the form $M \cup \Phi$, where $M \subseteq E$ is a matching, $\Phi \subseteq F$ is a collection of faces, no two sharing a common boundary edge, and $M \cap \partial_e(\Phi) = \varnothing$. Writing $\mathfrak{S}$ for the family of edge-face stable sets of G, we call an $f : \mathfrak{S} \to [0,1]$ satisfying

$$\sum_{A\in S\in\mathfrak{S}} f(S) = 1 \quad \text{for each } A \in E \cup F$$

a *fractional edge-face colouring* of G, and

$$\chi^*_{ef}(G) = \min\left\{\sum_{S\in\mathfrak{S}} f(S) : f \text{ is a fractional edge-face colouring of } G\right\}. \quad (3)$$

We often abbreviate the objective functions in (1)–(3) to $f(G)$.

If the reader wonders why we have strayed from our main topic, list-colouring, it is because of the following fact.

Lemma 3 *Every multigraph satisfies* $\chi^*_v = \hat{\chi}^*_v$.

Proof. See, e.g., [8, Theorem 3.8.1]. ∎

By considering line graphs, total graphs and the like, it is easy to check the validity of Lemma 3 for edge colouring, total colouring, and so on.

2 Results: one old, three new

The model and main tool for our three results was proved by Kahn in 1995 [5], though at this writing the proof is yet to appear in published form:

Theorem 4 *For multigraphs,*

$$\hat{\chi}_e \sim \hat{\chi}^*_e \quad \textit{as} \quad \hat{\chi}^*_e \to \infty. \quad (4)$$

Since $\hat{\chi}^*_e \leq \hat{\chi}_e$ always holds — the left side is the optimal value of the linear relaxation of the integer program defining the right — one should read (4) as: for each $\gamma > 0$ there exists $B = B(\gamma)$ such that every multigraph G with $\hat{\chi}^*_e(G) > B$ satisfies $\hat{\chi}_e(G) < (1+\gamma)\hat{\chi}^*_e(G)$.

Our first result is an analogue of Theorem 4 for the total choice number.

Theorem 5 *For multigraphs,*

$$\hat{\chi}_{ve} \sim \hat{\chi}^*_{ve} \quad \textit{as} \quad \hat{\chi}^*_{ve} \to \infty. \tag{5}$$

The convergence in (5) is in the same sense as that in (4), but we spell out the quantifiers for reference in the proof: for each $\varepsilon > 0$ there exists $D = D(\varepsilon)$ such that every multigraph G with $\hat{\chi}^*_{ve}(G) > D$ satisfies

$$\hat{\chi}_{ve}(G) < (1+\varepsilon)\hat{\chi}^*_{ve}(G). \tag{6}$$

Using Theorem 5, we can prove an asymptotic version of Conjecture 2.

Corollary 5 *For multigraphs,*

$$\hat{\chi}_{ve} \sim \chi_{ve} \quad \textit{as} \quad \chi_{ve} \to \infty.$$

Notice that the analogous corollary of Theorem 4, which we have not stated, shows that Conjecture 1 is also asymptotically correct.

Although we are always careful to specify the condition upon which our asymptotic estimates depend, such care is really unnecessary. The chain of inequalities $\Delta + 1 \leq \chi_{ve} \leq \hat{\chi}_{ve} \leq 2\Delta + 1$ shows that $\hat{\chi}_{ve}$ and χ_{ve} never differ in ratio by more than a factor of (about) 2, so, e.g., an equivalent formulation of Corollary 5 could replace $\chi_{ve} \to \infty$ by $\hat{\chi}_{ve} \to \infty$. Similar remarks apply to each of our main results and their corollaries.

We are getting a little ahead of ourselves, but here is the *Proof of* Corollary 5. The main result of [6] is that $\chi_{ve} \sim \chi^*_{ve}$ as $\chi^*_{ve} \to \infty$. This fact, together with Lemma 3 and Theorem 5, proves the assertion. ∎

The rest of our results concern plane multigraphs. First we see that the edge-face choice number is asymptotically good.

Theorem 6 *For plane multigraphs,*

$$\hat{\chi}_{ef} \sim \hat{\chi}^*_{ef} \quad \textit{as} \quad \hat{\chi}^*_{ef} \to \infty.$$

Using Lemma 3 and a theorem from [7], we obtain

Corollary 6 *For plane multigraphs,*

$$\hat{\chi}_{ef} \sim \chi_{ef} \quad \textit{as} \quad \chi_{ef} \to \infty.$$ ∎

We also establish that the entire choice number is asymptotically good.

Theorem 7 *For plane multigraphs,*

$$\hat{\chi}_{vef} \sim \hat{\chi}^*_{vef} \quad \textit{as} \quad \hat{\chi}^*_{vef} \to \infty.$$

Of course, we have an analogue of Corollaries 5, 6 for entire colouring:

Corollary 7 *For plane multigraphs,*

$$\hat{\chi}_{vef} \sim \chi_{vef} \quad \textit{as} \quad \chi_{vef} \to \infty.$$ ∎

3 Proofs

Since we indicated how to prove Corollaries 5–7, it remains to prove their sibling theorems.

Proof of Theorem 5

Together with Theorem 4, we need the following elementary connections between the total choice numbers and the list-chromatic indices (in (7), k is a positive constant and the multigraph must be non-empty):

$$\begin{aligned} \hat{\chi}^*_{ve} &\leq k\hat{\chi}^*_e; & (7)\\ \hat{\chi}_{ve} &\leq \hat{\chi}_e + 2; & (8)\\ \hat{\chi}^*_e &\leq \hat{\chi}^*_{ve}. & (9) \end{aligned}$$

Proof of (7). We have no need to optimize k, so we do not attempt to do so. Greedy colouring yields $\hat{\chi}_{ve} \leq 2\Delta + 1$, which, along with the obvious $\hat{\chi}^*_{ve} \leq \hat{\chi}_{ve}$ and $\Delta \leq \hat{\chi}^*_e$, gives (7). ∎

Proof of (8). Easy, and well-known; see e.g. [3]. ∎

Proof of (9). By Lemma 3, $\chi^* = \hat{\chi}^*$ holds for the relevant colouring parameters, so it suffices to prove

$$\chi^*_e \leq \chi^*_{ve} \quad (10)$$

instead of the list-colouring version. From an optimal fractional total colouring $f : \mathfrak{T} \to [0,1]$, we may obtain a fractional edge colouring $h : \mathfrak{M} \to [0,1]$ by shifting the weight $f(T)$ from each total stable set $T = M \cup I$ to the matching M in the natural way. This yields an h with $h(G) = f(G) = \chi^*_{ve}(G)$, and (10) follows since $\chi^*_e(G) \leq h(G)$. ∎

To prove Theorem 5, we need to establish (6) for arbitrary $\varepsilon > 0$ and sufficiently large $\hat{\chi}^*_{ve}$. Given $\varepsilon > 0$, let $\gamma = \varepsilon/2$, and choose B so large (according to Theorem 4) that

$$\hat{\chi}^*_e > B \quad \text{implies} \quad \hat{\chi}_e < (1+\gamma)\hat{\chi}^*_e. \quad (11)$$

With k as in (7), if $\hat{\chi}^*_{ve} > D := \max\{kB, 4k/\varepsilon\}$, then $\hat{\chi}^*_e$ exceeds both B and $4/\varepsilon = 2/\gamma$. Thus, provided $\hat{\chi}^*_{ve} > D$, we have

$$\hat{\chi}_{ve} \leq \hat{\chi}_e + 2 < (1+\gamma)\hat{\chi}^*_e + \gamma\hat{\chi}^*_e = (1+\varepsilon)\hat{\chi}^*_e \leq (1+\varepsilon)\hat{\chi}^*_{ve}$$

(justifying the inequalities, respectively, by: (8); the preceding sentence and (11); and (9)), as desired. ∎

Proof of Theorem 6

Again we lean on Theorem 4, now augmented by three easy relationships between the edge-face choice numbers and the list-chromatic indices:

$$\hat{\chi}^*_{ef} \leq \hat{\chi}^*_e + 5; \tag{12}$$
$$\hat{\chi}_{ef} \leq \hat{\chi}_e + 5; \tag{13}$$
$$\hat{\chi}^*_e \leq \hat{\chi}^*_{ef}. \tag{14}$$

Proof of (12). Again invoking Lemma 3 ($\chi^* = \hat{\chi}^*$), we prove the superficially simpler

$$\chi^*_{ef} \leq \chi^*_e + 5. \tag{15}$$

We may obtain a fractional colouring h of $E \cup F$ by fractionally χ^*_e-colouring E and (integrally) colouring F with a set C of additional colours. The five colour theorem guarantees that $|C|$ need not exceed 5; of course, the trivial six colour theorem (or the less-trivial four colour theorem) would serve equally well for our purposes below. Since $h(G) \leq \chi^*_e + 5$, (15) follows, and therefore so does (12). ∎

Proof of (13). Though the proof is similar to that of (8), it differs enough to warrant providing the details. Given lists $L(A)$, for $A \in E \cup F$, of size $\hat{\chi}_e + 5$, start by L-colouring the faces. The dual form of Thomassen's 5-choosability theorem [10] guarantees that this is possible. If $A \in E$, then remove from $L(A)$ the colour(s) assigned to the face(s) separated by A. This yields modified edge lists of size at least $\hat{\chi}_e + 3$, so the face colouring may be extended to an L-colouring of $E \cup F$. ∎

Proof of (14). Similar to the proof of (9). ∎

To complete the proof, we need to show that $\hat{\chi}_{ef} < (1+\varepsilon)\hat{\chi}^*_{ef}$ holds for any given $\varepsilon > 0$, provided $\hat{\chi}^*_{ef}$ is sufficiently large. Given $\varepsilon > 0$, again let $\gamma = \varepsilon/2$, and choose B large enough to give (11). If $\hat{\chi}^*_{ef} > D := \max\{B+5, 10\varepsilon^{-1}+5\}$, then, since $\hat{\chi}^*_e \geq \hat{\chi}^*_{ef} - 5$ (by (12)), we see that $\hat{\chi}^*_e$ exceeds both B and $10/\varepsilon = 5/\gamma$. Thus, as long as $\hat{\chi}^*_{ef} > D$, we have

$$\hat{\chi}_{ef} \leq \hat{\chi}_e + 5 < (1+\gamma)\hat{\chi}^*_e + \gamma\hat{\chi}^*_e = (1+\varepsilon)\hat{\chi}^*_e \leq (1+\varepsilon)\hat{\chi}^*_{ef}$$

(justifying the inequalities, respectively, by: (13); the preceding clause and (11); and (14)). ∎

Proof of Theorem 7

Since this proof mirrors that of Theorem 6, we simply provide a sketch. The following inequalities are analogous to (12)–(14) and may be proved similarly (in (16) and (17), C is any *constant* upper bound on $\hat{\chi}_{vf}$):

$$\hat{\chi}^*_{vef} \leq \hat{\chi}^*_e + C; \quad (16)$$
$$\hat{\chi}_{vef} \leq \hat{\chi}_e + C; \quad (17)$$
$$\hat{\chi}^*_e \leq \hat{\chi}^*_{vef}. \quad (18)$$

To prove Theorem 7, one may now use the proof of Theorem 6 with the following replacements:

$$(\hat{\chi}_{ef}, \hat{\chi}^*_{ef}, \text{constant } 5, (12)\text{–}(14)) \mapsto (\hat{\chi}_{vef}, \hat{\chi}^*_{vef}, \text{constant } C, (16)\text{–}(18)). \ \blacksquare$$

4 Closing remarks

Our results shed light on the asymptotic behaviour of three list-colouring parameters for multigraphs. We have chosen to present these results from a perspective highlighting asymptotically good behaviour — integral/fractional pairs of invariants agreeing asymptotically. A different angle would be to establish that each of $\hat{\chi}_{ve}$, $\hat{\chi}_{ef}$ and $\hat{\chi}_{vef}$ is asymptotic to $\hat{\chi}_e$ or to $\hat{\chi}^*_e = \chi^*_e$ (say, as $\chi^*_e \to \infty$). These facts are easily deduced from our results, and vice versa.

Knowing that $\hat{\chi}_e$, $\hat{\chi}_{ve}$, $\hat{\chi}_{ef}$ and $\hat{\chi}_{vef}$ are asymptotically good makes one wonder just which choice numbers enjoy this property. Though we have given a complete answer to this question in the case of plane multigraphs, still unsettled is the general (non-planar) case. Thus, it is natural to pose the following

Problem 8 *Determine conditions on a multigraph G to ensure that*

$$\hat{\chi}_v(G) \sim \hat{\chi}^*_v(G) \quad \textit{as} \quad \hat{\chi}^*_v \to \infty. \quad (19)$$

Several partial solutions to Problem 8 have appeared in this paper. Theorem 4 shows that "G is the line graph of a multigraph" suffices for (19), while Theorem 5 gives the sufficient condition "G is the total graph of a multigraph". Theorems 6, 7 can also be stated in this language. A complete characterization of which graphs enjoy the property (19) remains elusive.

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