Some Identities Related to Ramanujan's Tau Function

Neville Robbins
Mathematics Department
San Francisco State University
San Francisco, CA 94132 USA
robbins@math.sfsu.edu

ABSTRACT: Let $\tau(n)$ denote Ramanujan's tau function . We obtain an identity that involves $\tau(n)$ and $\sigma(n)$, as well as some apparently new congruence properties of $\tau(n)$ with respect to the moduli 23 and 5.

KEY WORDS AND PHRASES: Ramanujan's tau function, partitions

AMS CLASSIFICATION: 11P83

Introduction: Ramanujan's tau function, denoted $\tau(n)$, has fascinated number theorists since its discovery in 1916. In this note, using elementary means, we obtain an apparently new identity involving $\tau(n)$ and the sum-of-divisors function, $\sigma(n)$. We also present some congruence properties of $\tau(n)$ with respect to the moduli 23 and 5.

Preliminaries:

Definition 1: Let $\tau(n)$ denote Ramanujan's tau function.

Definition 2: Let $p(n)$ denote the number of partitions of n.

Definition 3: Let $\omega(n) = n(3n-1)/2$ (the n^{th} pentagonal number).

Definition 4: If $r \geq 2$, let $b_r(n)$ denote the number of *r-regular* partitions of n, that is, the number of partitions of n such that no part occurs r or more times (or equivalently, the number of partitions of n such that no part is a multiple of r).

Let x denote a complex variable such that $|x| < 1$. Then the following identities hold:

$$\sum_{n=1}^{\infty} \tau(n)x^n = x \prod_{n=1}^{\infty} (1-x^n)^{24} \tag{1}$$

$$\tau(mn) = \tau(m)\tau(n) \;\; if \;\; (m,n) = 1 \tag{2}$$

If p is prime and $k \geq 1$, then

$$\tau(p^{k+1}) = \tau(p)\tau(p^k) - p^{11}\tau(p^{k-1}) \tag{3}$$

$$(n-1)\tau(n) = \sum_{k\geq 1}(-1)^{k-1}(2k+1)(n-1-9\frac{k(k+1)}{2})\tau(n-\frac{k(k+1)}{2}) \tag{4}$$

$$(n-1)\tau(n) = \sum_{k\geq 1}(-1)^{k-1}\{(n-1-25\omega(\pm k))\tau(n-\omega(\pm k)) \tag{5}$$

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty}(1-x^n)^{-1} \tag{6}$$

If $r \geq 2$, then

$$\sum_{n=0}^{\infty} b_r(n)x^n = \prod_{n=1}^{\infty} \frac{1-x^{rn}}{1-x^n} \tag{7}$$

$$\prod_{n=1}^{\infty}(1-x^n) = 1 + \sum_{k=1}^{\infty}(-1)^k(x^{\omega(k)} + x^{\omega(-k)}) \tag{8}$$

If $A \subset N$ and $f(n)$ is a given function, let

$$\prod_{n\in A}(1-x^n)^{-\frac{f(n)}{n}} = 1 + \sum_{n=1}^{\infty} p_{A,f}(n)x^n$$

Furthermore, let $f_A(k) = \sum\{d : d|k \ , \ d \in A\}$. Then

$$np_{A,f}(n) = \sum_{k=1}^{n} f_A(k)p_{A,f}(n-k) \tag{9}$$

Remarks: (1) and (4) appear in [5]. (2) and (3) were proven in [4]. (5) is (13) in [3], except that a missing factor of $(-1)^{k-1}$ has been inserted. (9) is Theorem 14.8 on p. 322 of [1].

The Main Results

Theorem 1: If $n \geq 1$, then

$$n\tau(n+1) = -24\sum_{k=1}^{n}\sigma(k)\tau(n+1-k)$$

Proof: This follows from (9) and (1), letting $f(n) = -24n$ and $A = N$.

Corollary 1: If n is even, then $8|\tau(n)$.

Proof: This follows immediately from the hypothesis and from Theorem 1.

Remark: An earlier proof of Corollary 1 was given in [2].

Theorem 2:

$$\sum_{k=0}^{n}\tau(n+1-k)p(k) \equiv \begin{cases} (-1)^j \pmod{23} & \text{if } n = 23\omega(\pm j) \\ 0 \pmod{23} & \text{otherwise} \end{cases}$$

Proof: (1) implies

$$(\sum_{n=0}^{\infty}\tau(n)x^n)\prod_{n=1}^{\infty}(1-x^n)^{-1} = x\prod_{n=1}^{\infty}(1-x^n)^{23}$$

Now (6) implies

$$(\sum_{n=0}^{\infty}\tau(n)x^n)(\sum_{n=0}^{\infty}p(n)x^n) \equiv x\prod_{n=1}^{\infty}(1-x^{23n}) \pmod{23}$$

Invoking (8), we have

$$\sum_{n=0}^{\infty}(\sum_{k=0}^{n}\tau(n-k)p(k))x^n \equiv x\{1+\sum_{j=1}^{\infty}(-1)^j(x^{23\omega(j)}+x^{23\omega(-j)})\} \pmod{23}$$

Matching coefficients of like powers of x, we get

$$\sum_{k=0}^{n}\tau(n-k)p(k) \equiv \begin{cases} (-1)^j \pmod{23} \text{ if } n = 1+23\omega(\pm j) \\ 0 \pmod{23} \text{ otherwise} \end{cases}$$

Since $\tau(0) = 0$, this yields:

$$\sum_{k=0}^{n-1}\tau(n-k)p(k) \equiv \begin{cases} (-1)^j \pmod{23} \text{ if } n = 1+23\omega(\pm j) \\ 0 \pmod{23} \text{ otherwise} \end{cases}$$

The conclusion now follows by replacing n by $n+1$.

Remark: Other congruence properties of $\tau(n) \pmod{23}$ were given by Wilton [6].

Theorem 3

$$\tau(n)+\sum_{k=1}^{\infty}(-1)^{k-1}(\tau(n-\omega(k))+\tau(n-\omega(-k))) \equiv$$

$$\begin{cases} (-1)^j \pmod 5 \text{ if } n = 1+25\omega(\pm j) \\ 0 \pmod 5 \text{ otherwise} \end{cases}$$

Proof: If (1) is multiplied by $\prod_{n=1}^{\infty}(1-x^n)$, we get

$$(\sum_{n=1}^{\infty}\tau(n)x^n)\prod_{n=1}^{\infty}(1-x^n) = x\prod_{n=1}^{\infty}(1-x^n)^{25}$$

Now (8) implies

$$\sum_{n=1}^{\infty}(\tau(n)+\sum_{k=1}^{\infty}(-1)^k(\tau(n-\omega(k))+\tau(n-\omega(-k)))x^n \equiv$$

$$x\prod_{n=1}^{\infty}(1-x^{25n}) \pmod 5$$

hence

$$\sum_{n=1}^{\infty}(\tau(n)+\sum_{k=1}^{\infty}(-1)^k(\tau(n-\omega(k))+\tau(n-\omega(-k)))x^n \equiv$$

$$x(1+\sum_{j=1}^{\infty}(x^{25\omega(j)}+x^{25\omega(-j)}) \pmod 5$$

The conclusion now follows by matching coefficients of like powers of x.

Remarks: It is known that if $n \equiv 0 \pmod 5$, then $\tau(n) \equiv 0 \pmod 5$. A weaker version of Theorem 2 follows from (5), namely:

Theorem 3a:

$$\tau(n)+\sum_{k=1}^{\infty}(-1)^{k-1}(\tau(n-\omega(k))+\tau(n-\omega(-k))) \equiv$$

$$0 \pmod 5 \;\; if \;\; n \not\equiv 1 \pmod{25}$$

Theorem 4: $\tau(n) \equiv b_{25}(n-1) \pmod 5$

Proof: (1) and (7) imply

$$\sum_{n=1}^{\infty}\tau(n)x^n \equiv x\prod_{n=1}^{\infty}\frac{(1-x^n)^{25}}{1-x^n} \equiv$$

$$x\prod_{n=1}^{\infty}\frac{1-x^{25n}}{1-x^n} \equiv x\sum_{n=0}^{\infty}b_{25}(n)x^n \pmod 5$$

The conclusion now follows by matching coefficients of like powers of x.

The following theorem permits the convenient calculation of the function $b_r(n)$.

Theorem 5: If $r \geq 2$, then

$$b_r(n) = p(n) + \sum_{k=1}^{\infty}(-1)^k(p(n - r\omega(k)) + p(n - r\omega(-k)))$$

Proof: (7) and (6) imply

$$\sum_{n=0}^{\infty} b_r(n)x^n = \prod_{n=1}^{\infty}(1 - x^n)^{-1} \prod_{n=1}^{\infty}(1 - x^{rn}) =$$

$$(\sum_{n=0}^{\infty} p(n)x^n)(1 + \sum_{k=1}^{\infty}(-1)^k(x^{r\omega(k)} + x^{r\omega(-k)}) =$$

$$\sum_{n=0}^{\infty}(p(n) + \sum_{k=1}^{\infty}(-1)^k(p(n - r\omega(k)) + p(n - r\omega(-k)))x^n$$

The conclusion now follows by matching coefficients of like powers of x.

References:

1. T. M. Apostol *Introduction to Analytic Number Theory* (1976) Springer-Verlag

2. R. P. Bambah, S. Chowla, and H. Gupta A congruence property of Ramanujan's function $\tau(n)$ *Bull. Amer. Math. Soc.* 53 (1947) 766-767

3. D. H. Lehmer Ramanujan's function $\tau(n)$ *Duke Math. J.* 10 (1943) 483-492

4. L. J. Mordell On Mr. Ramanujan's empirical expansions of modular functions *Proc. Cambridge Philosophical Soc.* 19 (1917) 117-124

5. S. Ramanujan On certain arithmetical functions *Proc. London Math. Soc.* v.22 (1916) 159-184

6. J. R. Wilton Congruence properties of Ramanujan's function $\tau(n)$ *Proc. London Math. Soc.* v.51 (ser.2) (1930) 1-10