The Connectivities of Trunk Graphs of 2-Connected Graphs

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Let G be a connected graph and $\mathcal{V}^*$ set of all the spanning trees except stars in G. An edge in a spanning tree is called 'inner' if the edge is not incident to endvertices. Define an adjacency relation in $\mathcal{V}^*$ as follows; two spanning trees $\mathbf{t}_1$ and $\mathbf{t}_2 \in \mathcal{V}^*$ are called to be adjacent if there exist inner edges $e_i \in E(\mathbf{t}_i)$ such that $\mathbf{t}_1 - e_1 = \mathbf{t}_2 - e_2$. The resultant graph is a subgraph of the tree graph, and we call it simply a trunk graph. The purpose of this paper is to show that if a 2-connected graph with at least five vertices is k-edge connected, then its trunk graph is $(k-1)$-connected .

1. Introduction

Let $G = (V, E)$ be a connected simple graph. A *tree graph* of G is defined on the set $\mathcal{V}$ of all the spanning trees of G as follows; two spanning trees $\mathbf{t}_1$ and $\mathbf{t}_2$ are said to be adjacent if and only if there exist edges $e_i \in E(\mathbf{t}_i)$ such that

$$\mathbf{t}_1 - e_1 = \mathbf{t}_2 - e_2. \tag{1}$$

Cummings showed that a tree graph is Hamiltonian in [2]. Connectivities of a tree graph was shown by Liu [5].

Theorem 1 (Liu). *The tree graph of a connected graph $G = (V, E)$ is $2(|E| - |V| + 1)$-connected.*

We can consider two subgraphs of a tree graph which are determined by a kind of edges in the equation (1). If an edge is incident to endvertices in a spanning tree $\mathbf{t}$, then we call it an *outer edge.* An edge is not outer is called *inner.* It is a plain fact that, in the equation (1), the edge e_1 is an outer edge in $\mathbf{t}_1$ if and only if e_2 is also outer in $\mathbf{t}_2$. A *leaf graph* is defined as follows; $\mathbf{t}_1$ and $\mathbf{t}_2 \in \mathcal{V}$ are said to be adjacent if there exist outer edges $e_i \in E(\mathbf{t}_i)$ which satisfy the equation (1). The following fact is well-known.

Theorem 2. *The leaf graph of any 2-connected graph is connected.*

Broersma and Li Xueliang characterized the graphs of which the leaf graph is connected in [1]. If e_1 and e_2 are outer, then they have a common endvertex u. Therefore we can define a natural map from the edge set of the leaf graph to the vertex set of G by $\mu : \mathbf{t}_1\mathbf{t}_2 \longmapsto u$. A lemma found in an unpublished paper by Bondy and Lovász can be stated as follows.

Theorem 3 (Bondy and Lovász). *Let G be a 2-connected graph and u any vertex in G. Then the induced subgraph of $\mu^{-1}(V-u)$ of the leaf graph is connected.*

Furthermore Kaneko and Yoshimoto showed the following theorem in [4].

Theorem 4 (Kaneko and Yoshimoto). *Let G be a 2-connected graph of minimum degree δ. Then the leaf graph of G is $(2\delta-2)$-connected.*

They conjecture that if given a graph G is 2-connected graph with minimum degree three, then the leaf graph is Hamiltonian. A spanning tree does not contain always an inner edge. A *trunk graph* is defined on the set $\mathcal{V}^*$ of all the spanning trees except stars as follows; $\mathbf{t}_1$ and $\mathbf{t}_2 \in \mathcal{V}^*$ are called to be adjacent if there exist inner edges $e_i \in E(\mathbf{t}_i)$ which satisfy the equation (1). The connectivity of a trunk graph is shown in [7].

Theorem 5 (Nakamura and Yoshimoto). *If G is a 2-connected graph with at least five vertices, then the trunk graph of G is connected.*

They also characterized the graphs of which the trunk graph is connected. In this paper, we shall show the following theorem.

Theorem 6. *Let G be a 2-connected graph with at least five vertices. If G is k-edge connected, then the trunk graph of G is $(k-1)$-connected.*

The lower bound of the theorem is best possible. See Figure 1. It is a natural question to ask whether the trunk graph of a 2-connected graph has a Hamiltonian cycle or not. However, in the previous example, if there are exactly two edges between two complete graphs, then the trunk graph is not Hamiltonian. Thus the author conjectures that if a 2-connected graph G with $|V(G)| \geq 5$ is 3-edge connected, then the trunk graph is Hamiltonian.

Finally, we introduce notations used in the subsequent arguments. Let $\mathbf{t}$ be a spanning tree of a connected graph G. Because a spanning tree does not include a cycle, there exists exactly one path between any vertices u and $v \in V$. We denote the path by $P_{\mathbf{t}}(u,v)$ and the number of the edges in this path by $d_{\mathbf{t}}(u,v)$. The degree of u in $\mathbf{t}$ is denoted by $\deg_{\mathbf{t}}(u)$. The set of all the edges between subgraphs A and B is denoted by $E_G(A,B)$.

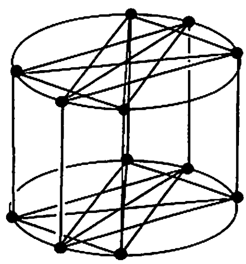

FIGURE 1

2. THE PROOF OF THE THEOREM

If $k = 2$, then the statement holds by Theorem 5. Therefore we suppose $k \geq 3$. Assume for contradiction that there exists a cut set $\mathcal{S}$ of the trunk graph $\mathcal{T}$ which contains at most $k-2$ vertices, and let $\mathcal{D}$ and $\mathcal{D}'$ be the connected components of $\mathcal{T} - \mathcal{S}$. Let $\bar{\mathbf{t}}$ be a vertex in $\mathcal{S}$ and $\mathbf{t} \in \mathcal{D}$ and $\mathbf{t}' \in \mathcal{D}'$ be adjacent to $\bar{\mathbf{t}}$ in $\mathcal{T}$. By the definition of a trunk graph, there exist inner edges $e \in E(\mathbf{t})$ and $e' \in E(\bar{\mathbf{t}})$ such that $\mathbf{t} - e + e' = \bar{\mathbf{t}}$. Similarly there are inner edges $f \in E(\bar{\mathbf{t}})$ and $f' \in E(\mathbf{t}')$ such that $\bar{\mathbf{t}} - f + f' = \mathbf{t}'$. If the edge f coincides with e', then it holds that $\mathbf{t} - e + e' - f + f' = \mathbf{t} - e + f' = \mathbf{t}'$. Because this is contrary to the hypothesis, we have that $f \in E(\mathbf{t})$ and the subgraph $\mathbf{t} - e - f$ contains three connected components. Let A, B, C be these components such that $e \in E_G(A, B)$ and $f \in E_G(B, C)$. In the following, we shall find out $k-1$ internally disjoint paths from $\mathbf{t}$ to $\mathbf{t}'$ in the trunk graph. Since the edge e is an inner edge in $\mathbf{t}$, the connected component A contains at least two vertices.

Now we divide the argument into the following three cases.

Case 1. B and C contain at least two vertices.

Case 2. B is one vertex w.

Case 3. C is one vertex v.

Case 1. B and C contain at least two vertices.
Let us denote $E_G(A, B) = \{e_i\}$ and $E_G(B, C) = \{f_j\}$ and $E_G(A, C) = \{g_l\}$. Without loosing generality we may label the index of edges so that $e = e_1$ and $f = f_1$ and

- if $e' \in E_G(A, B)$, then $e' = e_{|E_G(A,B)|}$ and
- if $e' \in E_G(A, C)$, then $e' = g_{|E_G(A,C)|}$.

Let us find out fundamental paths. The edge f is an inner edge in $\mathbf{t}$, two spanning trees $\mathbf{t}$ and $\mathbf{p}_i = \mathbf{t} - f + f_i$ are adjacent in the trunk graph.

Similarly, since e is an inner edge in $\mathbf{p}_i$, this spanning tree is adjacent to $\mathbf{p}'_i = \mathbf{p}_i - e + e'$. Because $\mathbf{p}'_i - f_i = \mathbf{t}' - f'$, we found out the path

$$\mathcal{P}_i = (\mathbf{t}, \mathbf{p}_i = \mathbf{t} - f + f_i, \mathbf{p}'_i = \mathbf{p}_i - e + e', \mathbf{t}' = \mathbf{p}'_i - f_i + f')$$

between $\mathbf{t}$ and $\mathbf{t}'$ for any edge $f_i \in E_G(B, C)$. Especially we have $\mathcal{P}_1 = (\mathbf{t}, \bar{\mathbf{t}}, \mathbf{t}')$. It is a plain fact that $\mathcal{P}_i$ and $\mathcal{P}_{i'}$ are internally disjoint if $i \neq i'$. Similarly if $f' \notin E_G(A, B)$, then there is the path

$$\mathcal{Q}_j = (\mathbf{t}, \mathbf{q}_j = \mathbf{t} - e + e_j, \mathbf{q}'_j = \mathbf{q}_j - f + f', \mathbf{t}' = \mathbf{q}'_j - e_j + e')$$

for any $e_j \in E_G(A, B)$. Furthermore if $f' \notin E_G(A, C)$, then we can find out the path

$$\mathcal{R}_l = (\mathbf{t}, \mathbf{r}_l = \mathbf{t} - e + g_l, \mathbf{r}'_l = \mathbf{r}_l - f + f', \mathbf{t}' = \mathbf{r}'_l - g_l + e')$$

for any $g_l \in E_G(A, C)$ in the trunk graph.

At first we consider the case when $e' \in E_G(A, B)$ and $f' \in E_G(A, C)$. Then there exist the paths $\mathcal{P}_i$ and $\mathcal{Q}_j$ for any $i \leq |E_G(A, B)|$ and $j \leq |E_G(B, C)|$ since $f' \notin E_G(A, B)$. The paths $\mathcal{P}_i$ and $\mathcal{Q}_j$ are not internally disjoint if and only if $\mathbf{p}'_i = \mathbf{q}_j$. Thus we have $\mathbf{p}'_1 = \bar{\mathbf{t}} = \mathbf{q}_{|E_G(A,B)|}$ when this happens. Now we found out $|E_G(A, B)| + |E_G(A, C)| - 1$ internally disjoint paths

$$\{\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_{|E_G(B,C)|}, \mathcal{Q}_1, \mathcal{Q}_2, \cdots, \mathcal{Q}_{|E_G(A,B)|} \mid \mathcal{P}_1 = \mathcal{Q}_{|E_G(A,B)|}\}$$

between $\mathbf{t}$ and $\mathbf{t}'$. Since G is k-edge connected, it holds that $|E_G(B, C)| + |E_G(A, B)| \geq k$. Thus there are more paths than is allowed by our hypothesis. A contradiction.

By symmetry, the case when $e' \in E_G(A, C)$ and $f' \in E_G(B, C)$ is same as the previous case. If $e' \in E_G(A, B)$ and $f' \in E_G(B, C)$, then there exist the paths $\{\mathcal{Q}_1, \mathcal{Q}_2, \cdots, \mathcal{Q}_{|E_G(A,B)|}, \mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_{|E_G(A,C)|}\}$. Since $g_l \in E_G(A, C)$, these paths are internally disjoint from each other, and this case is shown. If $e' \in E_G(A, C)$ and $f' \in E_G(A, B)$, then there exist the paths $\mathcal{R}_l$ and $\mathcal{P}_i$. The paths $\mathcal{R}_l$ and $\mathcal{P}_i$ are not internally disjoint if and only if $\mathbf{r}_l = \mathbf{p}'_i$. Because we have $\mathbf{r}_{|E_G(A,C)|} = \mathbf{p}_1$ when this happens, there are at least $k - 1$ internally disjoint paths in

$$\{\mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_{|E_G(A,C)|}, \mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_{|E_G(B,C)|} \mid \mathcal{R}_{|E_G(A,C)|} = \mathcal{P}_1\}.$$

Now Case 1 is shown.

Case 2. B is one vertex w.

Because the edge f is an inner edge in $\bar{\mathbf{t}}$, the edge e' is contained in $E_G(A, B)$ and the connected component C contains at least two vertices. Furthermore since $f' \notin E_G(A, B)$, there is the path $\mathcal{P}_i$ for any $i \leq l = |E_G(B, C)|$. If $l \geq k - 1$, then there are more paths than is allowed by our hypothesis.

Therefore we assume that $l \leq k-2$, and we shall find out $k-l-1$ paths between $\mathbf{t}$ and $\mathbf{t}'$ which is internally disjoint from $\mathcal{P}_i$. In the following, let $e = uw$ and $e' = u'w$ and $f = vw$. Notice that $|E_G(A, C)| \geq k-l$, and let $g = \overline{u}\overline{v}$ be any edge in $E_G(A, C)$.

1. At first, we shall find out a path when $\deg_{\mathbf{t}}(u) \geq 3$ and $\overline{u} \neq u$. If $P_{\mathbf{t}}(u, u') \cap P_{\mathbf{t}}(u, \overline{u}) \neq \emptyset$, then, for an edge $h \in P_{\mathbf{t}}(u, u') \cap P_{\mathbf{t}}(u, \overline{u})$, there is the path

$$\mathcal{G}_g^1 = (\mathbf{t}, \mathbf{g}_1^1 = \mathbf{t} - f + g, \mathbf{g}_2^1 = \mathbf{g}_1^1 - h + e', \mathbf{g}_3^1 = \mathbf{g}_2^1 - e + h, \mathbf{t}' = \mathbf{g}_3^1 - g + f').$$

See Figure 2. If $P_{\mathbf{t}}(u, u') \cap P_{\mathbf{t}}(u, \overline{u}) = \emptyset$, then there exists the path

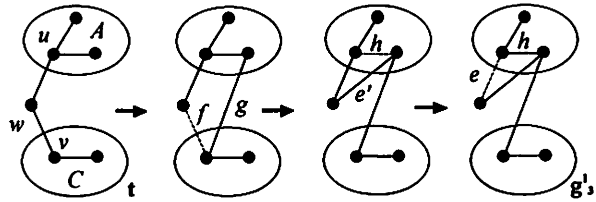

FIGURE 2

$$\mathcal{G}_g^2 = (\mathbf{t}, \mathbf{g}_1^2 = \mathbf{t} - f + g, \mathbf{g}_2^2 = \mathbf{g}_1^2 - h + f, \mathbf{g}_3^2 = \mathbf{g}_2^2 - e + e',$$
$$\mathbf{g}_4^2 = \mathbf{g}_3^2 - f + h, \mathbf{t}' = \mathbf{g}_4^2 - g + f')$$

for an edge $h \in P_{\mathbf{t}}(u, \overline{u})$. See Figure 3. Conversely, if $\deg_{\mathbf{t}}(v) \geq 3$ and

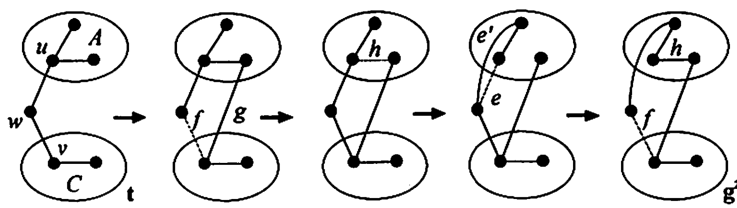

FIGURE 3

$\overline{v} \neq v$, then there exists the path

$$\mathcal{H}_g = (\mathbf{t}, \mathbf{h}_1 = \mathbf{t} - e + g, \mathbf{h}_2 = \mathbf{h}_1 - h + e', \mathbf{h}_3 = \mathbf{h}_2 - f + h, \mathbf{t}' = \mathbf{h}_3 - g + f')$$

for an edge $h \in P_{\mathbf{t}}(v, \overline{v})$. See Figure 4. Since $\mathbf{g}_{i'}^i$ or $\mathbf{h}_{i'}$ contains the edge g, the paths $\mathcal{G}_g^i$ and $\mathcal{H}_g$ are internally disjoint from $\mathcal{P}_j$. Thus, if $\deg_{\mathbf{t}}(u) \geq 3$ and $\deg_{\mathbf{t}}(v) \geq 3$, then we can find out $k-l-1$ internally disjoint paths in $\{\mathcal{G}_g^1, \mathcal{G}_g^2, \mathcal{H}_g \mid g \in E_G(A, C) - uv\}$. See Table 1.

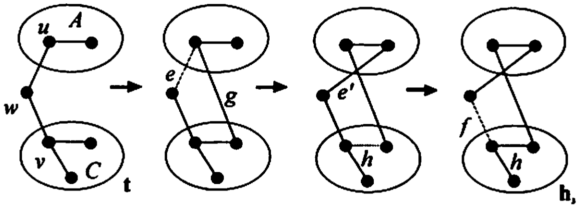

FIGURE 4

$g = \overline{u}\overline{v} \in E(A,C)$	$\overline{v} = v$	$\overline{v} \neq v$
$\overline{u} = u$		$\mathcal{H}_g$
$\overline{u} \neq u$	$\mathcal{G}_g^1, \mathcal{G}_g^2$	$\mathcal{G}_g^1, \mathcal{G}_g^2$

TABLE 1

2. Next we shall find out a path when $\deg_{\mathbf{t}}(u) = 2$ and $\overline{u}$ is neither u nor u'. If $P_{\mathbf{t}}(u,u') \cap P_{\mathbf{t}}(u',\overline{u}) = \emptyset$, then there exists the path

$$\mathcal{L}_g^1 = (\mathbf{t}, \mathbf{l}_1^1 = \mathbf{t} - f + g, \mathbf{l}_2^1 = \mathbf{l}^1 - h + f, \mathbf{l}_3^1 = \mathbf{l}_2^1 - e + e',$$

$$\mathbf{l}_4^1 = \mathbf{l}_3^1 - f + h, \mathbf{t}' = \mathbf{l}_4^1 - g + f')$$

for an edge $h \in P_{\mathbf{t}}(u',\overline{u})$. See Figure 5. Similarly, if $P_{\mathbf{t}}(u,u') \cap P_{\mathbf{t}}(u',\overline{u}) \neq \emptyset$,

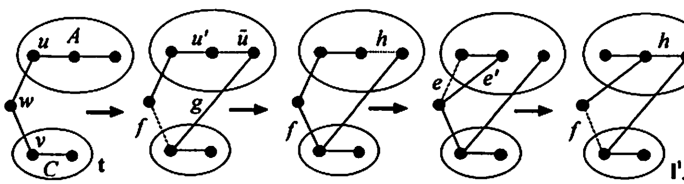

FIGURE 5

then there is the path

$$\mathcal{L}_g^2 = (\mathbf{t}, \mathbf{l}_1^2 = \mathbf{t} - f + g, \mathbf{l}_2^2 = \mathbf{l}^2 - h + e', \mathbf{l}_3^2 = \mathbf{l}_2^2 - h' + h,$$

$$\mathbf{l}_4^2 = \mathbf{l}_3^2 - e + h', \mathbf{t}' = \mathbf{l}_4^2 - g + f')$$

for edges $h \in P_{\mathbf{t}}(u,\overline{u}) \cap P_{\mathbf{t}}(u,u')$ and $h' \in P_{\mathbf{t}}(u',\overline{u}) \cap P_{\mathbf{t}}(u',u)$. See Figure 6. Since $\mathcal{L}_g^i$ is internally disjoint from $\mathcal{P}_j$, there are $k - l - 1$ internally disjoint paths in $\{\mathcal{H}_g, \mathcal{L}_g^1, \mathcal{L}_g^2 \mid g \in E_G(A,C) - \{uv, u'v\}\}$ if $\deg_{\mathbf{t}}(u) = 2$ and $\deg_{\mathbf{t}}(v) \geq 3$ and G does not contain either uv or $u'v$. See Table 2. If G

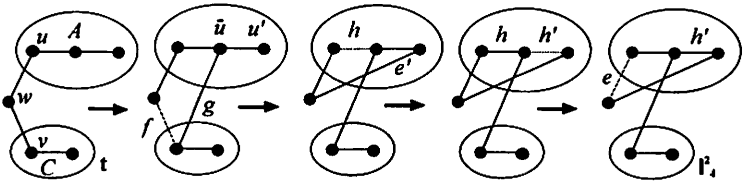

FIGURE 6

$g = \overline{u}\overline{v} \in E(A,C)$	$\overline{v} = v$	$\overline{v} \neq v$
$\overline{u} = u$		$\mathcal{H}_g$
$\overline{u} = u'$		$\mathcal{H}_g$
$\overline{u} \neq u, u'$	$\mathcal{L}_g^1, \mathcal{L}_g^2$	$\mathcal{L}_g^1, \mathcal{L}_g^2$

TABLE 2

contains both of the edges, then one path is not enough. But there exists the following path for $h \in P_{\mathbf{t}}(u, u')$.

$$\mathcal{M}^1 = (\mathbf{t}, \mathbf{m}_1^1 = \mathbf{t} - f + u'v, \mathbf{m}_2^1 = \mathbf{m}_1^1 - h + e', \mathbf{m}_3^1 = \mathbf{m}_2^1 - u'v + uv,$$
$$\mathbf{m}_4^1 = \mathbf{m}_3^1 - e + h, \mathbf{t}' = \mathbf{m}_4^1 - uv + f').$$

See Figure 7. Since $\mathbf{m}_i^1$ contains the edges $u'v$ or uv, the path $\mathcal{M}^1$ is

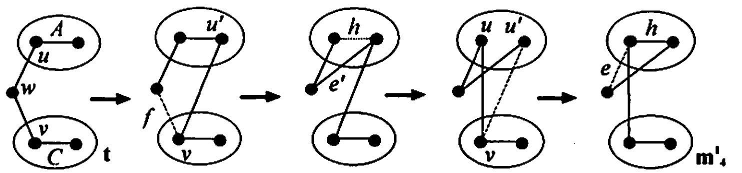

FIGURE 7

internally disjoint from $\mathcal{P}_j, \mathcal{H}_g$ and $\mathcal{L}_g^i$. Thus we supplied a deficiency.

Conversely, if $\deg_{\mathbf{t}}(v) = 2$ and $\overline{v}$ is not v and $\overline{u} \neq u'$, then there exists the path

$$\mathcal{M}_g = (\mathbf{t}, \mathbf{m}_1 = \mathbf{t} - e + g, \mathbf{m}_2 = \mathbf{m}_1 - h + e', \mathbf{m}_3 = \mathbf{m}_2 - h' + h,$$
$$\mathbf{m}_4 = \mathbf{m}_3 - f + h', \mathbf{t}' = \mathbf{m}_4 - g + f')$$

for edges $h \in P_{\mathbf{t}}(v, \overline{v})$ and $h' \in P_{\mathbf{t}}(\overline{u}, u')$. See Figure 8. Since $\mathcal{M}_g$ is internally disjoint from $\mathcal{P}_j$, there are $k - l - 1$ internally disjoint paths in $\{\mathcal{G}_g^1, \mathcal{G}_g^2, \mathcal{M}_g \mid g \in E_G(A, C) - uv\}$ when $\deg_{\mathbf{t}}(u) \geq 3$ and $\deg_{\mathbf{t}}(v) = 2$.

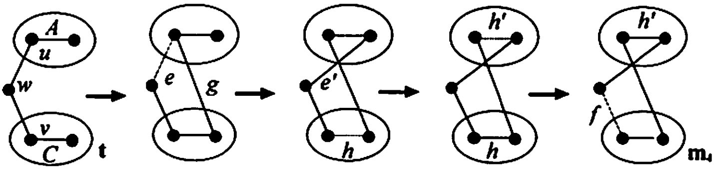

FIGURE 8

3. Assume that $\deg_{\mathbf{t}}(u) = 2$ and $\deg_{\mathbf{t}}(v) = 2$. Let $v' \in V(C)$ such that $d_{\mathbf{t}}(v, v') = 1$. If $d_{\mathbf{t}}(u, u') \geq 2$, then, for any $g \in E(A, C)$ which is incident to u', there exists the path

$\mathcal{N}_g^1 = (\mathbf{t}, \mathbf{n}_1^1 = \mathbf{t} - f + g, \mathbf{n}_2^1 = \mathbf{n}_1^1 - h + e', \mathbf{n}_3^1 = \mathbf{n}_2^1 - e + h, \mathbf{t}' = \mathbf{n}_3^1 - g + f')$

for an edge $h \in P_{\mathbf{t}}(u, u')$ which is not incident to u. See Figure 9. Thus

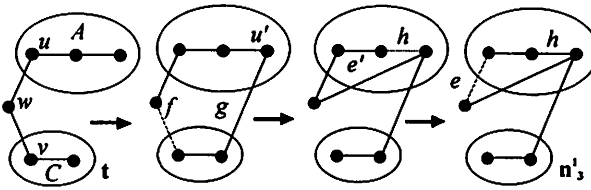

FIGURE 9

we can find out $k - l - 1$ internally disjoint paths in $\{\mathcal{L}_g^1, \mathcal{L}_g^2, \mathcal{M}_g, \mathcal{N}_g^1 \mid g \in E_G(A, C) - uv\}$.

Suppose that $d_{\mathbf{t}}(u, u') = 1$. If $d_{\mathbf{t}}(v, \overline{v}) \geq 2$, then there is the path

$\mathcal{N}_g^2 = (\mathbf{t}, \mathbf{n}_1^2 = \mathbf{t} - e + g, \mathbf{n}_2^2 = \mathbf{n}_1^2 - h + e', \mathbf{n}_3^2 = \mathbf{n}_2^2 - f + h, \mathbf{t}' = \mathbf{n}_2^2 - g + f')$

for an edge $h \in P_{\mathbf{t}}(v, \overline{v})$ which is not incident to v. See Figure 10. Thus if G

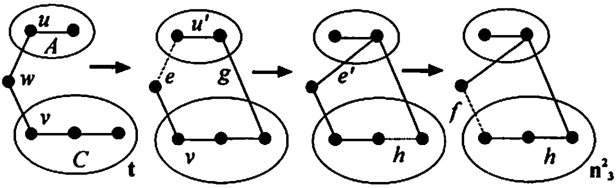

FIGURE 10

contains at most one edge in $\{uv, u'v, u'v'\}$, then there are $k-l-1$ internally disjoint paths in $\{\mathcal{L}_g^1, \mathcal{L}_g^2, \mathcal{M}_g, \mathcal{N}_g^2 \mid g \in E(A, C) - \{uv, u'v, u'v'\})\}$. See

$g = \overline{u}\overline{v} \in E(A,C)$	$\overline{v} = v$	$\overline{v} = v'$	$d_t(\overline{v}, v) \geq 2$
$\overline{u} = u$		$\mathcal{M}_g$	$\mathcal{M}_g$
$\overline{u} = u'$			$\mathcal{N}_g^2$
$\overline{u} \neq u, u'$	$\mathcal{L}_g^1, \mathcal{L}_g^2$	$\mathcal{L}_g^1, \mathcal{L}_g^2$	$\mathcal{L}_g^1, \mathcal{L}_g^2$

TABLE 3

Table 3. If G contains both of the edges uv and $u'v$, then there exists the path $\mathcal{M}^1$. Similarly if G contains both of the edges uv and $u'v'$, then there is the path

$$\mathcal{M}^2 = (\mathbf{t}, \mathbf{m}_1^2 = \mathbf{t} - f + u'v', \mathbf{m}_2^2 = \mathbf{m}_1^2 - uu' + e', \mathbf{m}_3^2 = \mathbf{m}_2^2 - u'v' + uv,$$
$$\mathbf{m}_4^2 = \mathbf{m}_3^2 - e + uu', \mathbf{t}' = \mathbf{m}_4^2 - uv + f').$$

See Figure 11. Because $\mathcal{M}^2$ is internally disjoint from $\mathcal{P}_j, \mathcal{L}_g^i, \mathcal{M}_g$ and

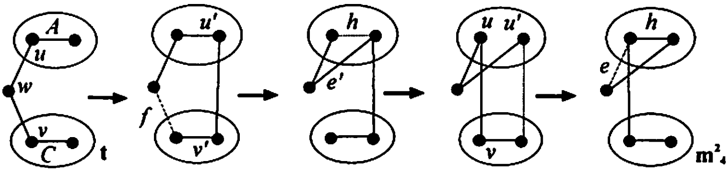

FIGURE 11

$\mathcal{N}_g^2$, we supplied a deficiency when G does not contain either $u'v$ or $u'v'$. Suppose that G contains both of the edges. If there is an edge h in $E_G(v, C - v) - vv'$, then there exists the path

$$\mathcal{N}^3 = (\mathbf{t}, \mathbf{n}_1^3 = \mathbf{t} - vv' + h, \mathbf{n}_2^3 = \mathbf{n}_1^3 - e + e', \mathbf{n}_3^3 = \mathbf{n}_2^3 - f + u'v,$$
$$\mathbf{n}_4^3 = \mathbf{n}_3^3 - h + vv', \mathbf{t}' = \mathbf{n}_4^3 - u'v + f').$$

See Figure 12. Because $\mathbf{n}_i^3$ contains h or $u'v$, the path $\mathcal{N}^3$ is internally

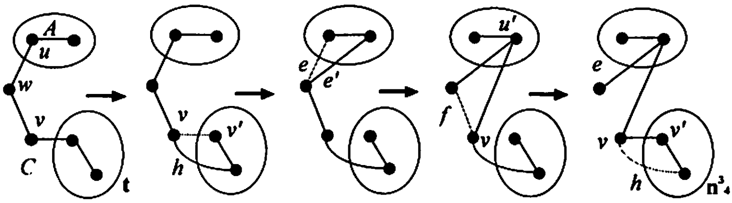

FIGURE 12

disjoint from $\mathcal{P}_j, \mathcal{L}_g^i, \mathcal{M}_g, \mathcal{M}^2$ and $\mathcal{N}_g^2$, and we supplied a deficiency. If

$E_G(v, C-v) = \{vv'\}$, then $|E_G(A \cup w, C-v) - u'v'| \geq k-2$ and there are $k-2$ internally disjoint paths in $\{\mathcal{P}_j, \mathcal{L}_g^1, \mathcal{L}_g^2, \mathcal{M}_g, \mathcal{N}_g^2 \mid 2 \leq j \leq |E(B,C)|, g \in E_G(A, C-v) - u'v'\}$. Since $\mathcal{P}_1$ is internally disjoint from the above paths, there are more paths than is allowed by our hypothesis. We showed Case 2.

Case 3. C is one vertex v.
In this case, we have that $e' \in E_G(A, C)$ and $f' \notin E_G(A, C)$. Because the case when $f' \in E_G(B, C)$ is same as Case 2 by symmetry, we assume that $f' \in E_G(A, B)$. In the following, let $e = e_1$ and $f = f_1 = vw$ and $e' = g_1 = u'v$. For any $i \leq l = \min\{|E_G(B, C)|, |E_G(A, C)|\}$, there exists the path

$$\mathcal{X}_i = (\mathbf{t}, \mathbf{x}_1 = \mathbf{t} - e + g_i, \mathbf{x}_2 = \mathbf{x}_1 - f + f_i, \mathbf{x}_3 = \mathbf{x}_2 - g_i + e', \mathbf{t}' = \mathbf{x}_3 - f_i + f')$$

for any $g_i \in E_G(A, C)$ in the trunk graph. Especially, we have $\mathcal{X}_1 = (\mathbf{t}, \overline{\mathbf{t}}, \mathbf{t}')$. Because there are more paths than is allowed by our hypothesis if $l \geq k-1$, we suppose that $l \leq k-2$ and shall find out $k-l-1$ paths between $\mathbf{t}$ and $\mathbf{t}'$ which is internally disjoint from $\mathcal{X}_i$.

For any edge $\overline{e} = \overline{u}\overline{w}$ in $E_G(A, B)$, the subgraph $\overline{\mathbf{t}} + \overline{e}$ contains exactly one cycle $C_{\overline{e}}$. We shall find out a path when $C_{\overline{e}}$ contains at least five edges. If $C_{\overline{e}} \cap A$ contains at least two edges, then there exists the following path for an edge $h \in P_{\mathbf{t}}(u', \overline{u})$ which is incident to $\overline{u}$.

$$\mathcal{Y}_{\overline{e}}^A = (\mathbf{t}, \mathbf{y}_1^A = \mathbf{t} - e + \overline{e}, \mathbf{y}_2^A = \mathbf{y}_1^A - h + e', \mathbf{y}_3^A = \mathbf{y}_2^A - f + h, \mathbf{t}' = \mathbf{y}_3^A - \overline{e} + f').$$

Similarly if $C_{\overline{e}} \cap B$ contains at least two edges, then there is the path

$$\mathcal{Y}_{\overline{e}}^B = (\mathbf{t}, \mathbf{y}_1^B = \mathbf{t} - e + \overline{e}, \mathbf{y}_2^B = \mathbf{y}_1^B - h + e', \mathbf{y}_3^B = \mathbf{y}_2^B - f + h, \mathbf{t}' = \mathbf{y}_3^B - \overline{e} + f')$$

for an edge $h \in P_{\mathbf{t}}(w, \overline{w})$ which is incident to $\overline{u}$. Otherwise, there exists the following path for the edges $h = C_{\overline{e}} \cap B$ and $h' = C_{\overline{e}} \cap A$.

$$\mathcal{Y}_{\overline{e}}^0 = (\mathbf{t}, \mathbf{y}_1^0 = \mathbf{t} - e + \overline{e}, \mathbf{y}_2^0 = \mathbf{y}_1^B - h + e', \mathbf{y}_3^0 = \mathbf{y}_2^0 - h' + h,$$
$$\mathbf{y}_4^0 = \mathbf{y}_3^0 - f + h', \mathbf{t}' = \mathbf{y}_4^0 - \overline{e} + f').$$

Since $\overline{e} \in E_G(A, B)$, the above paths are internally disjoint from $\mathcal{X}_i$. Assume that $C_{\overline{u}}$ contains four edges. If $\deg_{\overline{\mathbf{t}}}(u') \geq 3$ and $d_{\overline{\mathbf{t}}}(u', \overline{u}) = 1$, then there exists the path $\mathcal{Y}_{\overline{e}}^A$. Similarly, there is the path $\mathcal{Y}_{\overline{e}}^B$ when $\deg_{\overline{\mathbf{t}}}(w) \geq 3$ and $d_{\overline{\mathbf{t}}}(w, \overline{w}) = 1$.

1. If $\deg_{\overline{\mathbf{t}}}(u') \geq 3$ and $\deg_{\overline{\mathbf{t}}}(w) \geq 3$, then there are $k-l-1$ internally disjoint paths in $\{\mathcal{Y}_{\overline{e}}^A, \mathcal{Y}_{\overline{e}}^B, \mathcal{Y}_{\overline{e}}^0 \mid \overline{e} \in E_G(A, B) - u'w\}$. See Table 4.

2. Suppose that $\deg_{\overline{\mathbf{t}}}(u') = 2$ and $\deg_{\overline{\mathbf{t}}}(w) \geq 3$. Let $u'' \in V(A)$ be the vertex such that $d_{\overline{\mathbf{t}}}(u', u'') = 1$. If $A = u'u''$, then it holds that $|E_G(A, B)| \geq 2k-4$. If $k \geq 4$, then $|E_G(A, B) - \{u'w, u''w\}| \geq k-2$ and there exist $k-2$ internally disjoint paths in $\{\mathcal{Y}_{\overline{e}}^B, \mathcal{Y}_{\overline{e}}^0 \mid \overline{e} \in E_G(A, B) -$

$\overline{e} = \overline{u}\overline{w} \in E_G(A,B)$	$\overline{w} = w$	$d_{\overline{t}}(\overline{w}, w) = 1$	$d_{\overline{t}}(\overline{w}, w) \geq 2$
$\overline{u} = u'$		$\mathcal{Y}^B_{\overline{e}}$	$\mathcal{Y}^B_{\overline{e}}$
$d_{\overline{t}}(\overline{u}, u') = 1$	$\mathcal{Y}^A_{\overline{e}}$	$\mathcal{Y}^0_{\overline{e}}$	$\mathcal{Y}^B_{\overline{e}}$
$d_{\overline{t}}(\overline{u}, u') \geq 2$	$\mathcal{Y}^A_{\overline{e}}$	$\mathcal{Y}^A_{\overline{e}}$	$\mathcal{Y}^A_{\overline{e}}$

TABLE 4

$\{u'w, u''w\}\}$. Assume that $k = 3$. If $E_G(A, B - w)$ contains an edge $\overline{e}$, then there is $\mathcal{Y}^B_{\overline{e}}$ or $\mathcal{Y}^0_{\overline{e}}$ for this edge. Otherwise there exists an edge f_2 joining $B - w$ and v because G is 2-connected. Furthermore since $\deg_G(u'') \geq 3$, the vertex u'' is adjacent to w. Thus there is the path $\mathcal{X}_2$ for f_2 and $g_2 = u''w$ which is internally disjoint from $\mathcal{X}_1$.

Suppose that A contains at least three vertices. If G contains at most one edge in $\{u'w, u''w\}$, then there exist $k - l - 1$ internally disjoint paths in $\{\mathcal{Y}^A_{\overline{e}}, \mathcal{Y}^B_{\overline{e}}, \mathcal{Y}^0_{\overline{e}} \mid \overline{e} \in E_G(A, B) - \{u'w, u''w\}\}$. If G contains both of the edges, then one path is not enough. If there is an edge $h \in E_G(A - u', u') - u'u''$ or $h' \in E_G(A - u', v)$, then there exists the path

$$\mathcal{Z}^1 = (\mathbf{t}, \mathbf{z}^1_1 = \mathbf{t} - e + u'w, \mathbf{z}^1_2 = \mathbf{z}^1_1 - u'u'' + h, \mathbf{z}^1_3 = \mathbf{z}^1_2 - u'w + e',$$
$$\mathbf{z}^1_4 = \mathbf{z}^1_3 - f + u'w, \mathbf{z}^1_5 = \mathbf{z}^1_4 - h + u'u'', \mathbf{t}' = \mathbf{z}^1_5 - u'w + f')$$

or

$$\mathcal{Z}^2 = (\mathbf{t}, \mathbf{z}^2_1 = \mathbf{t} - e + u'w, \mathbf{z}^2_2 = \mathbf{z}^2_1 - u'u'' + h', \mathbf{z}^2_3 = \mathbf{z}^2_2 - f + e',$$
$$\mathbf{z}^2_4 = \mathbf{z}^2_3 - h' + u'u'', \mathbf{t}' = \mathbf{z}^2_4 - u'w + f'),$$

respectively. See Figure 13. Because $\mathbf{z}^j_{j'}$ contains the edge $u'w$ or h, the path $\mathcal{Z}^j$ is internally disjoint from $\mathcal{X}_i, \mathcal{Y}^A_{\overline{e}}, \mathcal{Y}^B_{\overline{e}}$ and $\mathcal{Y}^0_{\overline{e}}$. Thus we supplied a deficiency. If $E_G(A - u', u'v) = \{u''u'\}$, then $|E_G(A - u', B) - u''w| \geq k - 2$ and there are $k - 2$ internally disjoint paths in $\{\mathcal{Y}^A_{\overline{e}}, \mathcal{Y}^B_{\overline{e}}, \mathcal{Y}^0_{\overline{e}} \mid \overline{e} \in E_G(A - u', B) - u''w\}$ which are more paths than is allowed by our hypothesis. The case when $\deg_{\overline{t}}(u') \geq 3$ and $\deg_{\overline{t}}(w) = 2$ is same as the previous case by symmetry.

3. Suppose that $\deg_{\overline{t}}(u') = 2$ and $\deg_{\overline{t}}(w) = 2$. Let w' be the vertex in B such that $d_{\overline{t}}(w, w') = 1$. Suppose that $A = u'u''$. If $k \geq 4$, then $|E_G(A, G - A)| \geq k + 2$. If $E_G(A, C) = \{u'v\}$, then $|E_G(A, B) - \{u'w, u'w', u''w\}| \geq k - 2$ and there exist $k - 2$ internally disjoint paths in $\{\mathcal{Y}^B_{\overline{e}}, \mathcal{Y}^0_{\overline{e}} \mid \overline{e} \in E_G(A, B) - \{u'w, u'w', u''w\}\}$. Assume that $E_G(A, C) = \{u'v, u''v\}$. Since $E_G(B, C)$ contains at least two edges because $\deg_G(v) \geq 4$, there exist two paths $\mathcal{X}_1$ and $\mathcal{X}_2$ which supply a deficiency. Suppose $k = 3$, and we shall find out a path which is internally disjoint from $\mathcal{X}_1$. If $E_G(u', B - ww') \cup E_G(u'', B - w)$ contains an edge $\overline{e}$, then there exists $\mathcal{Y}^B_{\overline{e}}$ or $\mathcal{Y}^0_{\overline{e}}$. Otherwise $E_G(B - w, C)$

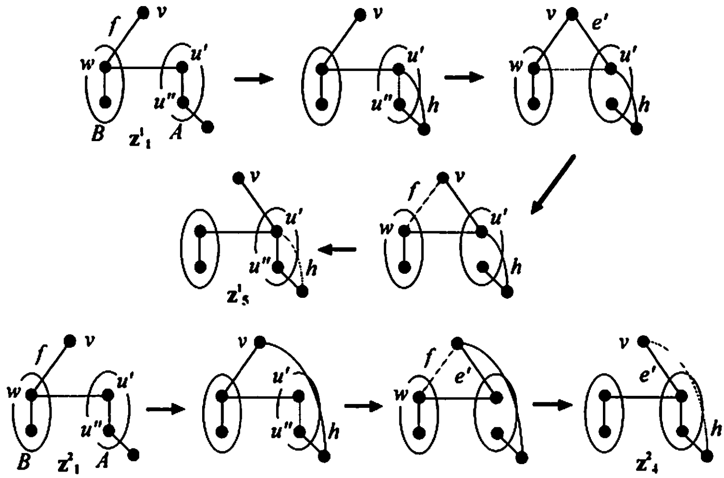

FIGURE 13

contains an edge because G is 3-edge connected. Since $\deg_G(u'') \geq 3$, the vertex u'' is adjacent to v. Therefore there are the paths $\mathcal{X}_1$ and $\mathcal{X}_2$ for these edges.The case when B contains two vertices is same as the previous case by symmetry.

Suppose that A and B contains at least three vertices. If G contains at most one edge in $\{u'w, u'w', u''w\}$, then there are $k - l - 1$ internally disjoint paths in $\{\mathcal{Y}_{\overline{e}}^A, \mathcal{Y}_{\overline{e}}^B, \mathcal{Y}_{\overline{e}}^0 \mid \overline{e} \in E_G(A, B) - \{u'w, u'w', u''w\}\}$. If $E_G(A-u', u'v) = \{u'u''\}$, then $|E_G(A-u', B-w)| \geq k-2$ and there are $k-2$ internally disjoint paths in $\{\mathcal{Y}_{\overline{e}}^A, \mathcal{Y}_{\overline{e}}^B, \mathcal{Y}_{\overline{e}}^0 \mid \overline{e} \in E_G(A-u', B-w)\}$ which are internally disjoint from $\mathcal{X}_1$. Thus we assume that $E_G(A-u', u'v) - u'u'' \neq \emptyset$. If $u'w \in E$, then there exists the path $\mathcal{Z}^1$ or $\mathcal{Z}^2$. Therefore we supplied a deficiency when $E_G(A, B)$ does not contain either $u'w'$ or $u''w$. If G contains both of the edges, then there exists the following path

$$\mathcal{Z}^3 = (\mathbf{t}, \mathbf{z}_1^3 = \mathbf{t} - e + u'w', \mathbf{z}_2^3 = \mathbf{z}_1^3 - ww' + u''w, \mathbf{z}_3^3 = \mathbf{z}_2^3 - u'u'' + e',$$

$$\mathbf{z}_4^3 = \mathbf{z}_3^3 - f + ww', \mathbf{z}_5^3 = \mathbf{z}_4^3 - u'w' + u'u'', \mathbf{t}' = \mathbf{z}_5^3 - u''w + f').$$

See Figure 14. Since $\mathbf{z}_j^3$ contains $u'w'$ or $u''w$, the path $\mathcal{Z}^3$ is internally disjoint from $\mathcal{X}_i, \mathcal{Y}_{\overline{e}}^A, \mathcal{Y}_{\overline{e}}^B, \mathcal{Y}_{\overline{e}}^0, \mathcal{Z}^1$ and $\mathcal{Z}^2$. Now we find out the paths which are more path than is allowed by our hypothesis, and the proof is complete.

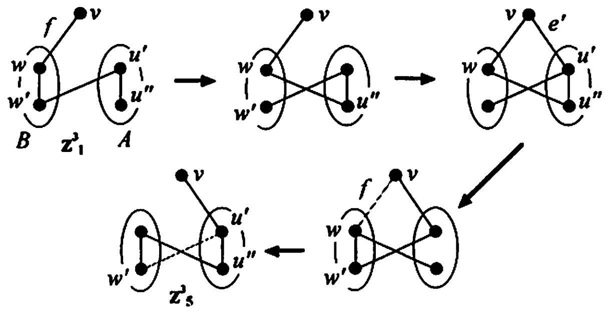

FIGURE 14

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