On the Toughness of the Middle Graph of a Graph

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ABSTRACT. The toughness $t(G)$ of a noncomplete graph G is defined as

$$t(G) = \min\{|S|/\omega(G-S) \mid S \subset V(G), \omega(G-S) \geq 2\},$$

where $\omega(G-S)$ is the number of components of $G-S$. We also define $t(K_n) = +\infty$ for every n.

The middle graph $M(G)$ of a graph G is the graph obtained from G by inserting a new vertex into every edge of G and by joining by edges those pairs of these new vertices which lie on adjacent edges of G.

In this article, we give the toughness of the middle graph of a graph, and using this result we also give a sufficient condition for the middle graph to have k-factor.

1 Introduction and Preliminaries

In this article, all graphs are finite, undirected, without loops or multiple edges. The toughness of a graph is an invariant first introduced by Chvátal [1]. He observed some relationships between this parameter and the existence of hamiltonian cycles or k-factors. The toughness is an interesting invariant in graph theory.

Let G be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges, respectively.

We denote the order of G by $|G|$ and the number of connected components of G by $\omega(G)$. If S is a subset of G with $\omega(G-S) \geq 2$, we call it a cutset of G. If $S \subset V(G)$, $\langle S \rangle$ is the subgraph of G induced by S. We write $G-S$ for $\langle V(G)-S \rangle$.

A graph G is t-tough if the implication

$$\omega(G-S)>1 \rightarrow |S| \geq t\cdot\omega(G-S)$$

holds for any $S \subset V(G)$.

A complete graph is t-tough for any real number t. If G is not complete, there exists the largest t such that G is t-tough. This number is denoted by $t(G)$ and is called the toughness of G. We define $t(K_n)=+\infty$ for every n. If G is not complete,

$$t(G)=\min\{|S|/\omega(G-S) \mid S\subset V(G), \omega(G-S)\geq 2\}.$$

The middle graph $M(G)$ of a graph G is the graph obtained from G by inserting a new vertex into every edge of G and by joining by edges those pairs of these new vertices which lie on adjacent edges of G. Terms not defined here can be found in [2].

The main purpose of this article is to give the toughness of the middle graph of a graph.

Here, in order to prove our results in the section 2, we describe some well-known results. We first give the definition of the endline graph of a graph. Let G be a graph and $V(G)=\{v_1,v_2,\ldots,v_n\}$. We add to G n new vertices and n edges $\{u_i,v_i\}$ $(i=1,2,\ldots,n)$, where u_i are different from any vertex of G and from each other. Then we obtain a new graph G' with $2n$ vertices, called the endline graph of G.

Let us denote the line graph of a graph G by $L(G)$. Then, from the definition of the endline graph and the middle graph of a graph G, we have, $L(G')\cong M(G)$, which is proven in [3]. This fact will be used for the proof our Theorem 1. Moreover, we need the following theorems for the proofs of Theorem 1, Corollaries 1 and 2 in the next section.

Theorem A. [3]. *If a graph G is n-edge connected, then the middle graph $M(G)$ is n-connected.*

Theorem B. [3]. *Let G be a graph, then the middle $M(G)$ of G is hamiltonian if and only if G contains a closed spanning trail.*

Theorem C. [4]. *Let G be graph. If G is k-tough, $|G|\geq k+1$, and k $|G|$ is even, then G has a k-factor.*

2 Results

We proved in [5] that if a connected graph has a bridge, then $t(M(G))=1/2$. In general, the following theorem holds;

Theorem 1. *Let G be a graph. Then $t(M(G))=\lambda(G)/2$, where $\lambda(G)$ is the edge-connectivity of G.*

Proof: Let λ be the edge-connectivity of graph G. If $\lambda = 0$, there is nothing to show. Hence we may assume that λ is a positive integer. Then there exists an edge-set $F \subset E(G)$ such that $|F| = \lambda$ and $G - F$ is disconnected.

Let $F = \{e_1, e_2, \ldots, e_\lambda\}$ and let v_i be a new vertex inserted into an edge e_i of G. Then $S = \{v_1, v_2, \ldots, v_\lambda\}$ would be a cutset of $M(G)$. Let us set $\omega(M(G) - S) = k$, then we have

$$t(M(G)) \leq |S|/k \leq \lambda/2. \tag{1}$$

Hence we may prove that $t(M(G)) \geq \lambda/2$.

Let S be a cutset such that $t(M(G)) = |S|/\omega(M(G) - S)$, and let $C_1, C_2, \ldots, C_k$ be the components of $M(G) - S$.

From the assumption that the edge-connectivity of G is λ, G becomes a λ-edge-connected graph. Hence, we see that $M(G)$ is a λ-connected graph from Theorem A.

Therefore there exist λ disjoint paths from $u_i \in C_i$ to $u_j \in C_j$ $(i \neq j)$. Each of these paths contain a vertex of S. Hence for each i there are at least λ edges coming from C_i to distinct vertices of S. Hence in all there are at least $k\ \lambda$ edges from $M(G) - S$ to S.

By the way, Since $L(G') \cong M(G)$ and $L(G')$ is a $K_{1,3}$-free graph, $M(G)$ becomes a $K_{1,3}$-free graph.

Therefore any vertex of S is adjacent to at most two components of $M(G) - S$. Hence there are at most $2\ |S|$ edges coming from $M(G) - S$ to vertices of S. This implies that $k\ \lambda \leq 2|S|$. Therefore we have

$$\lambda/2 \leq |S|/k = t(M(G)). \tag{2}$$

Combining (1) with (2), we obtain that $t(M(G)) = \lambda(G)/2$. This completes the proof. □

Corollary 1. *Let G be a $2k$-edge-connected graph and $|V(G)| + |E(G)|$ is even, then $M(G)$ has a k-factor.*

Proof: From Theorem 1 we have that $t(M(G)) = \lambda(G)/2 \geq k$. Hence $M(G)$ is a k-tough graph. Obviously $|V(M(G))| = |V(G)| + |E(G)| > 2k + 1 > k + 1$. Hence, from Theorem C, we see that $M(G)$ has a k-factor. □

Corollary 2. *Let G be a 3-edge-connected graph which does not contain a closed spanning trail, then the middle graph $M(G)$ is a 3/2-tough graph which is not Hamiltonian.*

Proof: Let G be a 3-edge-connected graph, then $t(M(G)) = \lambda(G)/2 \geq 3/2$. On the other hand, from Theorem B, $M(G)$ is not Hamiltonian. This completes the proof. □

For example, since the Petersen graph O_3 is 3-edge-connected graph which does not contain a closed spanning, from Corollary 2 we see that the middle graph $M(O_3)$ is toughness 3/2 and not Hamiltonian.

References

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